Delay Analysis of the Approximate Maximum Weight Scheduling in Wireless Networks

Cem Boyacı and Ye Xia
Computer and Information Science and Engineering Department
University of Florida, Gainesville, Florida, USA
Emails: {cboyaci, yx1}@cise.ufl.edu

***Abstract*—We provide bounds for the expectations of the stationary delay and the sum of the stationary queue sizes under the approximate maximum weight scheduling (MWS) policy and under the longest-queue-first (LQF) policy in one-hop wireless networks. For MWS, our results improves the previously provided bounds; for LQF, the results are new. In the derivation of the new bounds, a connection with certain graph theory quantities has been established. We also derive bounds for the sum of the second moments of the stationary queue sizes, which are also bounds for the sum of the queue-size variances. The method is general and it can be used to derive bounds for the sum of even higher moments of the stationary queue sizes. However, for the approximate MWS policy, the bounds for the second or higher moments are only valid when the average arrival rate vector is confined to a subset of the capacity region that depends on the order of the moment.**

***Index Terms*—wireless link scheduling; maximum weight schedule; longest-queue-first schedule; stability; queue size; delay**

I. Introduction

Performance of wireless communication systems relies on efficient utilization of the shared transmission medium. Simultaneous link transmissions may cause interference and degrade network efficiency or user-perceived quality of communication. Researchers address the wireless medium contention problem by devising link scheduling algorithms (policies) that optimize performance objectives of choice. Throughput has been considered as one of the most important performance metrics. Recently, there is a growing interest on the delay performance of wireless scheduling algorithms [1] [2]. This paper studies the delay bounds and queue size bounds under the approximate maximum weight scheduling (k-MWS) policy and the longest queue first (LQF) policy. The two policies are very well known and useful.

In different families of wireless communication technologies, link interference may have different consequences. One of the important cases is known as the *protocol interference model*, which is the focus of this paper. In the protocol model, the transmission on a link (i.e., a transmitter-receiver pair) can be successful only when there is no interference from other link transmissions.

In [3], Tassiulas et al. proposed a throughput-optimal scheduling policy that, when applied to the protocol model, activates a weighted maximum independent set (WMIS) of the interference graph on each time slot, where the weight of a node is the queue length of the corresponding wireless link. It is also known as the maximum weight scheduling (MWS) policy. However, the policy may not be applicable to large networks since the WMIS problem is NP-hard in general. Later, Tassiulas provided a probabilistic algorithm that achieves throughput optimality [4]. Loosely speaking, the probabilistic algorithm tries to find a WMIS on each time slot using randomized search [5]. However, with the probabilistic algorithm, the queue sizes may grow very large, leading to exorbitant memory usage and delay. Accordingly, recent research focuses on the queue size bounds (which can also be understood as the delay bounds) of well-known scheduling algorithms [1] [2]. To the best of our knowledge, there is little progress in deriving bounds for the transient queue sizes. Researchers are concentrating on bounds for the expected steady-state queue sizes. Previous research on network-switch scheduling provided many results that are applicable to wireless link scheduling with appropriate modifications. In [6], the authors investigated bounds for the expected queue size at stationarity for virtual-output-queued switches. Their object of study corresponds to a special case of the wireless link scheduling problem, which is the node exclusive interference model on a complete bipartite network graph. The authors of [1] provided a generalization for arbitrary networks and interference relationship under the MWS algorithm. The latest effort in this domain introduced a variant of MWS to achieve tight bounds for the expected stationary queue size [2].

To counter the difficulty of finding a WMIS on each time slot, a stream of research concentrates on devising simple algorithms that have high performance in practice. The k-MWS algorithm is one such example; it can be much simpler than the MWS algorithm for a fixed network. Another example is the LQF algorithm, which is known for its simplicity, high performance, and aptness for distributed implementation [7], [8], [9]. The focus of attention in these studies has been the throughput performance. The queue-size/delay characteristics of the two algorithms have not been sufficiently addressed.

The following is a summary of the main results and contributions of this paper.

- We provide bounds for the expectations of the stationary delay and the sum of the stationary queue sizes under the k-MWS policy and the LQF policy. For k-MWS, our

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results improve the previous bounds in [1]. They are also more general, applicable to all the cases where $0 < k \leq 1$. The bounds for LQF are new.

- We also derive bounds for the sum of the second moments of the stationary queue sizes, which are also bounds for the sum of the queue-size variances. The method is general and it can be used to derive bounds for the sum of even higher moments of the stationary queue sizes. However, for the k-MWS policy, the bounds for the second or higher moments are only valid when the average arrival rate vector is confined to a subset of the capacity region. The subset reduces as the order of the moment increases.
- For the LQF policy, all the bounds are valid for any arrival rate vector in $\frac{1}{\delta}\Lambda^o$, which is a well-known stability region for LQF [8].
- All the bounds for the n-th moments of the stationary queue sizes, where $n > 1$, are new results.
- In the derivation of the new bounds, a connection with certain graph theory quantities has been established.

This paper is structured as follows. In Section II, we provide the system model. Graph theoretical results and supporting lemmas are provided in Section III. In Section IV, we derive bounds for the expected queue-size sum and the expected delay under the k-MWS policy and the LQF policy. In Section V, we provide bounds for the sum of the queue-size squares. The conclusions are in Section VI.

II. System Model

For simplicity of analysis, we assume the following widely used model. We consider a wireless network with one-hop traffic. That is, any data is transmitted only once, and after that, it leaves the network. Let L be the set of the wireless links. The system is time-slotted, the packet sizes are identical, and the capacity of every link is one packet per time slot. We model the interference relations of the wireless link set L with an interference graph, $G = (V, E)$. Each wireless link $l \in L$ in the physical network is represented by a node $v \in V$. Two nodes $v_1, v_2 \in V$ are connected in G if and only if the corresponding links in the network interfere with each other. We assume symmetric interference relation; thus G is undirected. A *feasible schedule* is defined to be a set of non-interfering nodes in G. A *maximal schedule* is a feasible schedule that cannot include any more nodes without causing interference. A feasible schedule corresponds to an independent set in G. We denote the set of all maximal schedules by M_L. When applicable, M_L is regarded as a $|V| \times |M_L|$ 0-1 matrix. Each column of the matrix is a 0-1 vector representation of a maximal independent set of G, with 1 indicating that the corresponding node (in G) is selected (and the corresponding wireless link is activated in the schedule) and 0 otherwise. Throughout the paper, we use the term *schedule* to refer to a maximal schedule, unless mentioned otherwise.

The packets arriving for a wireless link are queued at the transmitter (sender) of the link. There is one queue for each link, or equivalently, there is one queue associated with each node in the interference graph. Each arrival process is i.i.d. in time; and the arrival processes for all the links are mutually independent. Let $A_i(t)$ denote the number of packets arriving at link i at time t. Since the distribution of $A_i(t)$ is time invariant, we use $E(A_i^k)$ to denote its k-th moment, if it exists. We assume the second moments $\mathrm{E}(A_i^2)$ are bounded by C. Let $A(t)$ denote the vector $(A_i(t))_{i\in V}$. Let $Q(t)$ be the queue length vector, i.e., $Q(t) = (Q_i(t))_{i\in V}$, where $Q_i(t)$ is the queue length at the transmitter of link i. A link can be activated on a time slot only if its queue is non-empty. At each queue, at most one packet can be served on any time slot. For each link i, let $D_i(t)$ indicate whether or not link i receives service on time slot t. Note that

$$D_i(t) = \begin{cases} 1, & \text{if } Q_i(t) \geq 1 \text{ and } i \text{ is scheduled.} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Since the capacity of each link is one packet per time slot, $D_i(t)$ is also the number of departures from queue i at time t. Let $D(t) = (D_i(t))_{i\in V}$. The evolution of the queues is as follows:

$$Q(t+1) = Q(t) + A(t) - D(t). \quad (2)$$

We adopt the conventional definition of the capacity region Λ (see [3]).

$$\Lambda = \{\lambda \mid 0 \leq \lambda \leq \mu \text{ for some } \mu \in Co(M_L)\},$$

where $Co(M_L)$ stands for the convex hull of the column vectors in M_L. The interior of the capacity region, denoted by Λ^o, is defined as:

$$\Lambda^o = \{\lambda \mid 0 \leq \lambda < \mu \text{ for some } \mu \in Co(M_L)\}.$$

Stability of the queueing system is defined as: The irreducible discrete-time Markov chain (DTMC) that represents the system is positive recurrent. It has been shown in [3] that the stability region is Λ^o. A scheduling algorithm is said to be *throughput-optimal* if keeps the queues stable under any arrival rate vector in Λ^o.

In this paper, we will evaluate the delay performance of two scheduling algorithms. The first is called the k-MWS policy ($k \in (0, 1]$), in which the selected schedule on each time slot t, denoted by $m(t)$, has at least the fraction k times of the maximum schedule weight of the current time slot, i.e., $m(t)^T Q(t) \geq k \max_{m\in M_L} m^T Q(t)$. Here, the superscript T denotes vector transpose. It is known that the k-MWS policy achieves network stability in the region $k\Lambda^o$ [10]. The second algorithm is the LQF policy, which schedules the links in non-increasing order of the queue lengths subject to schedule feasibility.

In general, for any vector x, let x_i denote its i-th component. Let e denote the column vector with the value 1 in every entry.

III. Graph Theoretical Results

The focus of this section is to present supporting lemmas for the performance evaluation of the two scheduling algorithms.

Under the protocol interference model, the capacity region can be characterized using the concept of fractional coloring. Let $w \in \mathbb{Q}_+^{|V|}$ be a (component-wise) non-negative weight vector. Consider the following optimization problem, where $\alpha = (\alpha_m)_{m \in M_L}$ is the decision variable.

$$\chi_f(G, w) \triangleq \min \sum_{m \in M_L} \alpha_m, \text{ subject to } M_L \alpha \geq w, \ \alpha \geq 0.$$

The optimal value $\chi_f(G, w)$ is called the *weighted fractional coloring/chromatic number*. Computing $\chi_f(G, w)$ is NP-hard in general. We denote the weighted independence number, i.e., the total weight of the WMIS of a graph, by $\alpha(G, w)$, where $w \in \mathbb{Q}_+^{|V|}$ is the weight vector. We denote $\chi_f(G, e)$ and $\alpha(G, e)$ by $\chi_f(G)$ and $\alpha(G)$, respectively.

The following has been shown in [11].

Lemma 1. *For each $k \in (0, 1]$, $\lambda \in k\Lambda^o$ if and only if $\chi_f(G, \lambda) < k$.*

Let $\gamma(G, Q, \pi)$ be the queue length sum of a LQF schedule, where Q is the vector of the queue lengths. The additional parameter π represents a linear order of the nodes in G, and it provides the tie-breaking ability so that, for a fixed π, the LQF schedule is unique. Given π, the nodes are assigned IDs 1, 2, ..., $|V|$ according to π. When two queues are identical in size, the one with a smaller ID is given a higher priority to be selected in the schedule. We will derive results independent of π. In order to eliminate π, we define a quantity, $\gamma(G, \mu)$, which is the minimum queue sum under the LQF policy over the set of all possible linear orders of the nodes. Such a set is denoted by Π.

Definition 1. $\gamma(G, Q) \triangleq \min_{\pi \in \Pi} \gamma(G, Q, \pi)$,

In [8] and [7], the concept of *interference degree* is used in characterizing the performance of LQF. Let $N[v]$ denote the closed neighborhood of a node $v \in V$, i.e., all the neighbors of v and v itself. For a subset of the nodes, $S \subseteq V$, let $G[S]$ denote the node-induced subgraph of G, induced by the nodes in S [12]. The following is the definition of the interference degree:

Definition 2. $\delta \triangleq \max_{v \in V} \alpha(G[N[v]])$.

In words, the interference degree of the graph G is the maximum number of nodes that could be activated simultaneously in the neighborhood of some node. In [8], it is shown that, when an average arrival rate vector is within $\frac{1}{\delta}\Lambda^o$, LQF is able to stabilize the network. The following lemma relates the LQF scheduling to WMIS and justifies the performance bound shown later.

Lemma 2. $\alpha(G, Q) \leq \delta\gamma(G, Q), \qquad \forall Q \in \mathbb{R}_+^{|V|}$.

Proof: Omitted for brevity. ∎

Let $\hat{m}$ be an LQF schedule under the queue size vector Q.

Corollary 1. $\alpha(G, Q) \leq \delta \hat{m}^T Q$.

Hence, the greedy schedule of LQF provides a $1/\delta$-approximation to the problem of finding a WMIS. Another interpretation is that the weight of any LQF schedule is lower-bounded by $\alpha(G, Q)/\delta$.

The next lemma and its corollary demonstrate that any LQF schedule under the queue weights Q is a $1/\delta$-approximation to the WMIS problem under the new weights $Q^n \triangleq (Q_i^n)$, where $n \geq 0$.

Lemma 3. *For $n > 0$, $\gamma(G, Q^n) \leq \hat{m}^T Q^n$.*

Proof: Notice that since each component of Q is non-decreasing in n when $Q \in \mathbb{R}_+^{|V|}$, $\hat{m}$ is still an LQF schedule under the weights Q^n. From this fact and the definition of $\gamma(G, Q^n)$, the lemma follows. ∎

Corollary 2. *For $n > 0$, $\frac{1}{\delta}\alpha(G, Q^n) \leq \hat{m}^T Q^n$.*

Given a k-MWS schedule $\tilde{m}_{(k)}$ under the queue vector Q, we provide a lower bound in the next lemma.

Lemma 4. *For $n \geq 1$, $\dfrac{k^n}{(\alpha(G))^{n-1}}\alpha(G, Q^n) \leq \tilde{m}_{(k)}^T Q^n$.*

Proof:

$$\begin{aligned}
\alpha(G, Q^n) &\leq (\alpha(G, Q))^n && (3)\\
&\leq (\frac{1}{k}\tilde{m}_{(k)}^T Q)^n && (4)\\
&\leq \frac{1}{k^n}(\tilde{m}_{(k)}^T Q)^n &&\\
&\leq \frac{1}{k^n}(\alpha(G))^{n-1}\tilde{m}_{(k)}^T Q^n. && (5)
\end{aligned}$$

To see inequality (3), let m be a schedule that achieves $\alpha(G, Q^n)$. Then, $\alpha(G, Q^n) = m^T Q^n \leq (m^T Q)^n$. Since m is a feasible schedule, we have $(m^T Q)^n \leq (\alpha(G, Q))^n$. Inequality (4) is by the definition of the k-MWS policy and the fact that $\tilde{m}_{(k)}$ is a schedule under such a policy. Inequality (5) follows from two facts. First, $(\sum_{i=1}^l a_i)^n \leq l^{n-1} \sum_{i=1}^l (a_i)^n$, where $a_i \in \mathbb{R}_+$ for each i, $l \in \mathbb{N}$ and $n \geq 1$[1]. Second, the cardinality of any schedule is upper-bounded by $\alpha(G)$. ∎

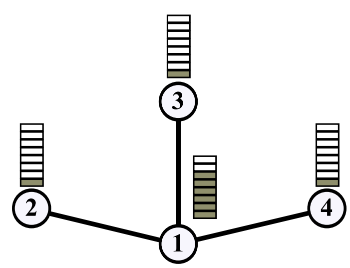

Fig. 1. Illustration on tightness of Lemma 4. Suppose that $Q_1 = 6$ and $Q_2 = Q_3 = Q_4 = 1$. For $k = \frac{1}{2}$, the schedule $\tilde{m}_{(k)} = (0, 1, 1, 1)^T$ is one possible schedule under the k-MWS policy. On the other hand, the MWS schedule is $m = (1, 0, 0, 0)^T$ under any Q^n, $n \geq 1$. Hence, $\alpha(G, Q^n) = 6^n$. The interference degree, δ, for the given graph is 3, and $\alpha(G) = \delta$. Readers can verify for $n = 1, 2, \ldots$, the bound given in Lemma 4 is tight.

The bound provided in Lemma 4 can be tight as illustrated in Fig. 1. Note that the approximation ratio in Lemma 4 depends on n, whereas the approximation ratio in Corollary 2, which is for the LQF schedule, does not.

[1]This inequality is a special case of Hölder's inequality.

Analyzing the performance of MWS via Lyapunov functions typically requires relating two inner products: $A(t)^T Q(t)$ and $D(t)^T Q(t)$. The next lemma bounds the inner product of two non-negative vectors by the associated graph theoretical quantities.

Lemma 5. $\lambda^T \mu \leq \chi_f(G,\lambda)\alpha(G,\mu), \qquad \forall \mu, \lambda \in \mathbb{R}_+^{|V|}.$

Proof: Let $m^i \in M_L$ be the i-th maximal schedule in M_L, where $i = 1, \cdots, |M_L|$. Let $\nu \in \mathbb{R}_+^{|M_L|}$ be an optimal solution to the optimization problem for finding $\chi_f(G,\lambda)$, i.e., $\sum_{i=1}^{|M_L|} \nu_i m^i \geq \lambda$ and $\sum_{i=1}^{|M_L|} \nu_i = \chi_f(G,\lambda)$. Then, we have

$$\begin{aligned}\lambda^T \mu &\leq (\sum_{i=1}^{|M_L|} \nu_i m^i)^T \mu = \sum_{i=1}^{|M_L|} \nu_i (m^i)^T \mu \\ &\leq \sum_{i=1}^{|M_L|} \nu_i \alpha(G,\mu) = \chi_f(G,\lambda)\alpha(G,\mu).\end{aligned}$$

■

Lemma 5 can be interpreted as an extension of Corollary 7.5.3 in [13] to arbitrary weight vectors. In the following sections, we will see the central role of Lemma 5 in the analysis.

IV. Queue and Delay Bounds

With the help of Lemma 5, we derive bounds for the expected queue-length sum under the stationary distribution. A delay bound is then derived using Little's law. Our derivation follows [1]; however, we improve the bounds and make a connection with the fractional coloring numbers.

Consider the quadratic Lyapunov function $\Phi : \mathbb{R}^{|V|} \to \mathbb{R}$ defined by $\Phi(Q) = \sum_{i=1}^{|V|} Q_i^2$. The drift of the Lyapunov function satisfies the following:

$$\begin{aligned}&\Phi(Q(t+1)) - \Phi(Q(t)) \\ =\ & (Q(t+1)-Q(t))^T (Q(t+1)+Q(t)) \\ =\ & (A(t)-D(t))^T (2Q(t)+A(t)-D(t)) \\ =\ & 2(A(t)-D(t))^T Q(t) + (A(t)-D(t))^T (A(t)-D(t)) \\ =\ & 2(A(t)-D(t))^T Q(t) \\ & + A(t)^T A(t) + D(t)^T D(t) - 2A(t)^T D(t). \end{aligned} \tag{6}$$

If the network is stable under a given policy, $(Q(t), A(t), D(t))$ forms a positive recurrent DTMC [3] and a stationary distribution exists. Under the stationary distribution, for a lower-bounded kth-degree polynomial function $h : \mathbb{R}^{|V|} \to \mathbb{R}$, we have $\mathrm{E}[h(Q(t+1)) - h(Q(t))] = 0$, if the kth moments exists [14] [15]. Since Φ satisfies the condition, under the stationary distribution,

$$\begin{aligned}0 &= \mathrm{E}[\Phi(Q(t+1)) - \Phi(Q(t))] \\ &= 2\,\mathrm{E}[(A(t)-D(t))^T Q(t)] + \mathrm{E}[A(t)^T A(t)] \\ &\quad + \mathrm{E}[D(t)^T D(t)] - 2\,\mathrm{E}[A(t)^T D(t)].\end{aligned} \tag{7}$$

Next, we bound the terms in (7). Applying Lemma 5, we get

$$\mathrm{E}[A(t)^T Q(t) | Q(t)] = \lambda^T Q(t) \leq \chi_f(G,\lambda)\alpha(G,Q(t)).$$

By taking expectation on both sides of the above, we have

$$\mathrm{E}[A(t)^T Q(t)] \leq \chi_f(G,\lambda)\,\mathrm{E}[\alpha(G,Q(t))]. \tag{8}$$

Since $D_i(t) \in \{0,1\}$ for each i, we have $D_i^2(t) = D_i(t)$. At stationarity, we also have $\mathrm{E}[A(t)] = \mathrm{E}[D(t)]$. Thus,

$$\mathrm{E}[D(t)^T D(t)] = \mathrm{E}[\sum_{i=1}^{|V|} D_i(t)] = \sum_{i=1}^{|V|} \lambda_i. \tag{9}$$

Since $A_i(t)$ is independent of $D_i(t)$, we have

$$\mathrm{E}[A(t)^T D(t)] = \mathrm{E}[A(t)]^T\,\mathrm{E}[D(t)] = \sum_{i=1}^{|V|} \lambda_i^2. \tag{10}$$

The next lemma provides a bound for the expected queue-length sum under the stationary distribution for k-MWS.

Lemma 6. *Under the k-MWS policy, for any arrival rate vector $\lambda \in k\Lambda^o$, the following holds under the stationary distribution.*

$$\mathrm{E}[\sum_{i=1}^{|V|} Q_i(t)] \leq \frac{\chi_f(G)}{2(k-\chi_f(G,\lambda))} B_1, \tag{11}$$

where $B_1 = \sum_{i=1}^{|V|} (\lambda_i + \mathrm{E}[A_i^2] - 2\lambda_i^2)$.

Proof: Under the k-MWS policy, we have[2]

$$\mathrm{E}[D(t)^T Q(t)] \geq k\,\mathrm{E}[\alpha(G,Q(t))]. \tag{12}$$

Combining with (8), we get

$$\begin{aligned}&2\,\mathrm{E}[A(t)^T Q(t) - D(t)^T Q(t)] \\ &\leq 2(\chi_f(G,\lambda) - k)\,\mathrm{E}[\alpha(G,Q(t))].\end{aligned} \tag{13}$$

We obtain the next result by using (9), (10) and (13) in (7).

$$\begin{aligned}0 &\leq 2(\chi_f(G,\lambda) - k)\,\mathrm{E}[\alpha(G,Q(t))] \\ &\quad + \sum_{i=1}^{|V|} \lambda_i + \sum_{i=1}^{|V|} \mathrm{E}[A_i^2] - 2\sum_{i=1}^{|V|} \lambda_i^2.\end{aligned}$$

Since $\lambda \in k\Lambda^o$, we have $\chi_f(G,\lambda) < k$ by Lemma 1. By rearranging, we obtain a bound for the expected total weight of the WMIS:

$$\mathrm{E}[\alpha(G,Q(t))] \leq \frac{B_1}{2(k-\chi_f(G,\lambda))}. \tag{14}$$

By applying Lemma 5 again, we get

$$\begin{aligned}\mathrm{E}[\sum_{i=1}^{|V|} Q_i(t)] &= \mathrm{E}[e^T Q(t)] \leq \mathrm{E}[\chi_f(G)\alpha(G,Q(t))] \\ &= \chi_f(G)\,\mathrm{E}[\alpha(G,Q(t))] \leq \frac{\chi_f(G)}{2(k-\chi_f(G,\lambda))} B_1.\end{aligned}$$

■

[2] By the definition of the k-MWS policy, the schedule selected at time t, denoted by $m(t)$, satisfies $m(t)^T Q(t) \geq k\alpha(G,Q(t))$. If $Q_i(t) = 0$, then $D_i(t)Q_i(t) = m_i(t)Q_i(t) = 0$. By (1), if $Q_i(t) \geq 1$ and $m_i(t) = 1$, then $D_i(t) = 1$; if $Q_i(t) \geq 1$ and $m_i(t) = 0$, then $D_i(t) = 0$. In all these cases, we have $D_i(t)Q_i(t) = m_i(t)Q_i(t)$. Hence, (12) holds.

By applying Little's law, which says the product of the average arrival rate and the average delay experienced by an arrival is equal to the average queue size, the following bound on the expected delay, $\bar{d}$, is obtained.

Corollary 3. *Under k-MWS, for $\lambda \in k\Lambda^o$,*

$$\bar{d} \leq \frac{\chi_f(G)}{2(\sum_{i=1}^{|V|} \lambda_i)(k - \chi_f(G,\lambda))} B_1. \tag{15}$$

Our result improves the bound in [1] by a factor of $\chi(G)/\chi_f(G)$, where $\chi(G)$ is the integral coloring number of G. In [16], the authors prove that the difference between $\chi(G)$ and $\chi_f(G)$ can be arbitrarily large, as stated in the next lemma.

Lemma 7. *For any integer $n \geq 2$, there exits a uniquely colorable, vertex transitive graph G, such that $\chi(G) - \chi_f(G) > n - 2$.*

The above result counts on the growing size of the graph considered. Lovász in [17] provides inequalities to bound the integral coloring number as follows:

Lemma 8. $\chi_f(G) \leq \chi(G) < (1 + \ln(\alpha(G)))\chi_f(G)$.

Hence, by Lemma 8, the improvement factor of our bound over the bound in [1] is at most $1 + \ln(\alpha(G))$.

For LQF, by repeating the steps till (12) and by applying Corollary 1, we have,

$$\mathrm{E}[D(t)^T Q(t)] \geq \frac{1}{\delta} \mathrm{E}[\alpha(G, Q(t))]. \tag{16}$$

Then, (13) is replaced by

$$\begin{aligned} &2\,\mathrm{E}[A(t)^T Q(t) - D(t)^T Q(t)] \\ &\leq 2(\chi_f(G,\lambda) - 1/\delta)\,\mathrm{E}[\alpha(G, Q(t))]. \end{aligned}$$

Similarly, we replace k by $1/\delta$ in the rest of the steps. We get

Lemma 9. *Under the LQF policy, for any arrival rate vector $\lambda \in \frac{1}{\delta}\Lambda^o$, the following hold under the stationary distribution.*

$$\mathrm{E}[\sum_{i=1}^{|V|} Q_i(t)] \leq \frac{\chi_f(G)}{2(1/\delta - \chi_f(G,\lambda))} B_1 \tag{17}$$

$$\bar{d} \leq \frac{\chi_f(G)}{2(\sum_{i=1}^{|V|} \lambda_i)(1/\delta - \chi_f(G,\lambda))} B_1, \tag{18}$$

where $B_1 = \sum_{i=1}^{|V|} (\lambda_i + \mathrm{E}[A_i^2] - 2\lambda_i^2)$.

V. Bounds on Higher Moments

In this section, we first derive a bound for $\mathrm{E}[\sum_{i=1}^{|V|} Q_i^2(t)]$ when the k-MWS policy is utilized and when the average arrival rate vector is confined in $\frac{k^2}{\alpha(G)}\Lambda^o$. It also serves as a bound for the sum of the queue-size variances. The method can be used to derive bounds for the sum of even higher moments of the queue sizes. Additionally, when the LQF scheduling policy is employed, we can compute the upper bounds for the second or higher moments of the stationary queue sizes for all arrival rate vectors in $\frac{1}{\delta}\Lambda^o$, which is a well-known stability region for LQF [8].

Consider the cubic Lyapunov function $\Psi : \mathbb{R}^{|V|} \to \mathbb{R}$ defined by $\Psi(Q(t)) = \sum_{i=1}^{|V|} Q_i^3(t)$. Under the stationary distribution, we have

$$\begin{aligned} 0 &= \mathrm{E}[\Psi(Q(t+1)) - \Psi(Q(t))] \\ &= 3\,\mathrm{E}[(A(t) - D(t))^T Q^2(t)] \\ &\quad + 3\,\mathrm{E}[\sum_{i=1}^{|V|}((A_i(t) - D_i(t))^2 Q_i(t)] \\ &\quad + \mathrm{E}[\sum_{i=1}^{|V|}(A_i(t) - D_i(t))^3]. \end{aligned} \tag{19}$$

Next, we bound the terms in (19). Define $B_2 = \sum_{i=1}^{|V|}(\mathrm{E}[A_i^3] - 3\lambda_i\,\mathrm{E}[A_i^2] + 3\lambda_i^2 - \lambda_i)$. It is easy to see

$$\mathrm{E}[\sum_{i=1}^{|V|}(A_i(t) - D_i(t))^3] = B_2. \tag{20}$$

Recalling that the second moments of the arrival processes are bounded by the constant C and applying Lemma 6, we have,

$$\begin{aligned} \mathrm{E}[\sum_{i=1}^{|V|}((A_i(t) - D_i(t))^2 Q_i(t)] &\leq \mathrm{E}[(C+1)\sum_{i=1}^{|V|} Q_i(t)] \\ &\leq \frac{(C+1)\chi_f(G)}{2(k - \chi_f(G,\lambda))} B_1. \end{aligned} \tag{21}$$

Similar to the derivation of (8), by Lemma 5,

$$\begin{aligned} \mathrm{E}[\sum_{i=1}^{|V|}(A_i(t) Q_i^2(t) | Q(t))] &= \lambda^T Q^2(t) \\ &\leq \chi_f(G,\lambda)\alpha(G, Q^2(t)). \end{aligned}$$

Under the k-MWS policy, by Lemma 4, we have

$$\mathrm{E}[\sum_{i=1}^{|V|} D_i(t) Q_i^2(t) | Q(t)] \geq \frac{k^2}{\alpha(G)} \alpha(G, Q^2(t)). \tag{22}$$

Thus, we obtain

$$\begin{aligned} &\mathrm{E}[\sum_{i=1}^{|V|}(A_i(t) - D_i(t)) Q_i^2(t)] \\ &\leq \mathrm{E}[\alpha(G, Q^2(t))](\chi_f(G,\lambda) - \frac{k^2}{\alpha(G)}). \end{aligned} \tag{23}$$

Combining (20), (21), (23) into (19), we have

$$\begin{aligned} &\mathrm{E}[\alpha(G, Q^2(t))]3(\frac{k^2}{\alpha(G)} - \chi_f(G,\lambda)) \\ &\leq \frac{3}{2}\frac{(C+1)\chi_f(G)}{(k - \chi_f(G,\lambda))} B_1 + B_2. \end{aligned}$$

When $\lambda \in \frac{k^2}{\alpha(G)}\Lambda^o$,

$$\mathrm{E}[\alpha(G, Q^2(t))] \leq \frac{\frac{(C+1)\chi_f(G)B_1}{2} + \frac{B_2(k - \chi_f(G,\lambda))}{3}}{(\frac{k^2}{\alpha(G)} - \chi_f(G,\lambda))(k - \chi_f(G,\lambda))}.$$

By applying Lemma 5 again, we finally get

$$\mathrm{E}[\sum_{i=1}^{|V|} Q_i^2(t)] = \mathrm{E}[e^T Q^2(t)] \leq \chi_f(G)\,\mathrm{E}[\alpha(G, Q^2(t))]$$

$$\leq \chi_f(G) \frac{\frac{(C+1)\chi_f(G)B_1}{2} + \frac{B_2(k - \chi_f(G,\lambda))}{3}}{(\frac{k^2}{\alpha(G)} - \chi_f(G,\lambda))(k - \chi_f(G,\lambda))}.$$

Next, we provide a bound for $\mathrm{E}(\sum_{i=1}^{|V|} Q_i^2(t))$ under LQF for any arrival rate vector $\lambda \in \frac{1}{\delta}\Lambda^o$. Let $\hat{m}(t)$ be an LQF schedule under the queue-length vector $Q(t)$. Consider Corollary 2 with $n = 2$ and we have

$$\frac{1}{\delta}\alpha(G, Q^2(t)) \leq \hat{m}(t)^T Q^2(t). \tag{24}$$

Hence,

$$\mathrm{E}[D(t)^T Q^2(t) | Q(t)] \geq \frac{1}{\delta}\alpha(G, Q^2(t)). \tag{25}$$

Now, suppose $\lambda \in \frac{1}{\delta}\Lambda^o$ and suppose the LQF policy is used. With (25) replacing the role of (22) and by repeating the steps after (22), we can get

$$\mathrm{E}[\sum_{i=1}^{|V|} Q_i^2(t)]$$

$$\leq \chi_f(G) \frac{\frac{(C+1)\chi_f(G)B_1}{2} + \frac{B_2(\frac{1}{\delta} - \chi_f(G,\lambda))}{3}}{(\frac{1}{\delta} - \chi_f(G,\lambda))^2}.$$

The above framework can be extended to derive bounds for even higher moments of the queue sizes, provided the corresponding moments of the arrival processes are bounded. In the case of the k-MWS policy, the subset on which a bound is valid depends on the order of the moment in question. In contrast, for the LQF policy, the bounds are valid on the constant region $\frac{1}{\delta}\Lambda^o$. When $\alpha(G)$ grows with the size of G, the $1/\alpha(G)$ factor also restricts the applicability of the bound in the k-MWS case for large graphs.

VI. Conclusions

In this paper, we provide improved bounds for the stationary delay and queue-size sum under the k-MWS policy and the LQF policy in wireless networks. The framework is also used to derive bounds for sums of higher moments of the queue sizes. For the LQF case, all the bounds are valid for any arrival rate vector in $\frac{1}{\delta}\Lambda^o$.

The future work may consist of extending the presented framework and deriving bounds for higher moments of the queue sizes on the whole stability region of the k-MWS policy. It is known that all moments of the stationary queue sizes are finite under the right conditions [14]. It may be interesting to find appropriate Lyapunov functions and exploit Lemma 5 for deriving bounds on higher moments. Also, LQF is known to be throughput optimal when the underlying interference graph satisfies the *local pooling* condition [9]. An open question is whether, for local pooling graphs and under LQF, our framework can generate bounds for higher moments on the whole capacity region. If it can be done, the derived bounds can characterize the performance of LQF more precisely. Finally, the connection between wireless scheduling and the aspects of graph theory reported in the paper appears interesting and deserves further study.

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