Generating a Good Quality Point Set for the Meshless Methods

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Abstract: Meshless methods raises an important geometric problem that is how to locate the integration points and their patches so that the numerical problem is simulated accurately. This paper formalizes the criteria to define a good point set for the meshless methods. A sphere packing based advancing front algorithm is suggested to provide a provably good solution for the formalized problem. Practical variations of the method is also discussed.

keyword: Meshless methods, sphere packing, advancing front, point placement.

1 Introduction

For simulation of numerical problems, meshless methods have emerged as an alternative to mesh based methods and become popular due to several reasons. Primarily, mesh generation is a difficult problem, Also these methods have generated promising results in the accuracy of the simulations. Similar to mesh generation problems, meshless methods also induce challenging geometric problems. Partition of Unity Method (PUM) is one of the meshless methods that is based on the definition of overlapping patches covering the domain. Some overlap among the patches is required. These patches should overlap to cover the domain, however, it should not be to the extend that deteriorates the accuracy. Also any single point in the domain should not be covered by more than a certain number of patches. We formalize this criteria to define a good point set for meshless methods in the second section of the paper.

Earlier implementations of the meshless methods employed arbitrary point set generation which could not support required structure of PUM. The alternative "mesh free" approach uses a mesh to obtain the point set. This leads to an ironic classification of the meshless methods as truly meshless methods and mesh based meshless methods.

Biting method is an advancing front technique that generates a good sphere packing (9). The centers of the spheres, a well-spaced point set, can in turn be used to generate a provably good simplicial mesh. In this study, we use this approach to generate a good point set for meshless methods. We prove that the biting method can be used to generate a good point set for the meshless methods.

1.1 Meshless Methods

Due to the difficulty of the mesh generation problem, alternative approaches for simulation of the numerical problems are sought. Meshless methods eliminates the need for a mesh structure, attempts to solve the numerical problems by constructing the approximation on the node based patches covering the domain. During the last decade, several meshless methods are proposed including the Partition of Unity Method (2) and the element-free Galerkin method (3). See the survey paper (4) by Belytschko, Krongauz, Organ, Fleming and Krysl for an excellent overview of the meshless methods.

Meshless methods raises an important geometric problem; how to locate the nodes and their patches so that the numerical solution has higher accuracy. We call this problem as the point set generation problem. Unlike the mesh generation problem, the point set does not need to be topologically connected. This might imply that it is an easier problem to solve than the mesh generation problem. It, however, has its own challenges as discussed in the next section keeping it away from being a trivial problem to solve.

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1.2 Previous Approaches

Despite the variety of the meshless methods, point set generation problem did not take enough consideration. Indeed many of the implementations employ mesh generation packages, and use the point set of a mesh. Recent studies to generate point set for the meshless methods include octree based approach by Klaas and Shephard (8). They use a level one adjusted octree, i.e. the level difference between terminal octants and their neighbors is no more than one, partition of the domain. Corner nodes of the octants defines the point set together with the patches defined by the sum of all octants sharing a particular corner node. Another approach by Choi and Kim (6) uses Voronoi diagram and weighted bubble packing. Our approach is similar to the second one in using spherical elements to define the patches.

1.3 Biting Method

Introduced by Li, Teng, and Üngör, Biting Method is a provably good simplicial mesh generation algorithm (9; 10; 11). It uses an advancing front strategy to generate a good sphere packing. The centers of the spheres, a well-spaced point set, can in turn be used to generate a provably good tetrahedral mesh. In this study, we use this approach to generate a good point set for meshless methods. However, instead of relating the radii of the spheres with the mesh spacing function, size of the spheres (patches) is defined by a sizing function.

1.4 Outline

The following section defines the problem formally and also introduces a set of criteria to define a good point set. Next an algorithm is given to generate a good point set for the meshless methods. Finally, the quality of the point set is proved.

2 Problem Definition

Quality of the elements is very critical for numerical methods since it has big impact on the accuracy of the simulation. The shape and size criteria are well studied for mesh based methods. However, there is no widely agreed criteria on the quality of a point set for the meshless methods. Various geometric shapes (including spheres, rectangles, simple polygons) are used as the patches attached to each point describing the domain of influence for that point. The size and the shape of the patches together with the amount of overlap between the patches effect the accuracy of the numerical simulation.

Some of the meshless methods use more than a single integration point for each patch. Based on the work of Chen *et al.* (5) we limit ourselves using a single point (center) of each patch (sphere). Once the patches are located one can actually modify the method and problem formulation so that more than one integration point could be used for each patch.

2.1 Sizing function

For proximity reasons, the domain of influence shape (patch shape) is chosen to be a sphere. For flow problems, where anisotropy in the domain is important, this definition can be modified to use ellipse shaped patches. The size of the domain of influence differs through out the domain due to the numerical accuracy or the domain geometry. We use a sizing function to denote how big a patch should be at a particular point in the domain. This is analogous to the global spacing function used in the meshing version of the biting algorithm (9).

Definition [Sizing Function] Let x be a point in the domain Ω, sizing function, $f(x)$, denotes the size of the domain of influence at x.

The following assumption about how fast this function changes from point to point in the domain provides the theoretical base for our claims.

Definition [Lipschitz Property] A function $f()$ is α-*Lipschitz* if for any two points x, y in the domain Ω,

$$|f(x) - f(y)| \leq \alpha||x - y|| \quad (1)$$

where α is a constant number.

We assume that given sizing function is α-Lipschitz.

2.2 Good Point Set

Although there has been numerous research in meshless methods, a good point set has never been formally defined. The quality of a point set is critical for the success of the meshless methods. Following definition formalizes the criteria for a good point set.

Definition [Good Point Set for Meshless Methods] Given a domain Ω, we call a set of points P, *Good Point Set for Meshless Methods* if it satisfies the following criteria:

1. Each point $x \in P$ is associated with a sphere centered at x which has a radius defined by the sizing function at x. Let S be the set of spheres corresponding to the points in P.
2. The spheres in S cover the whole domain Ω, i.e. any point x in the domain Ω is contained in at least one sphere in S.
3. The size of the intersection between two overlapping spheres in S is bounded from above by a constant factor of the size of the smaller sphere.
4. The number of spheres covering a single point in the domain Ω is bounded from above by a constant number t.

Each of the above criteria that defines a good point set is motivated by various research results in meshless methods. Most common shape for patches in the meshless method literature is a circle (sphere). The first criterion that uses sphere shaped patches can easily be modified with no change in the other criteria to accommodate other type of patches (square, rectangle, triangle, etc.). The second criterion states that every point in the domain is considered for the numerical solution of the problem. The third criterion implies that no one patch is completely inside another one. It avoids redundancy and decreases the input size for the numerical solver. Fourth criteria prevents a single point in the domain effecting the solution more than necessary.

2.3 Point Placement Problem

Based on the definitions on the previous subsections, we formalize the point placement for meshless methods problem as follows

Problem 2.1 *Given a domain Ω and a sizing function $f()$ with Lipschitz property, generate good point set for meshless methods.*

3 Algorithm

This section adapts the *Biting Method* to generate a good point set for the meshless methods. Originally biting method is designed to generate a sphere packing, which in turn is used to generate a well-shaped mesh, i.e. the circumradius-to-shortest-edge ratio of every element in the mesh is bounded from above by a constant. It uses two types of spheres, biting spheres and packing spheres. They are centered at the same points x, and have different radii: biting sphere has radius $C_b f(x)$, where C_b is the biting constant specified by the user of the algorithm. Packing sphere at x has radius $C_p f(x)$, where $C_p = \frac{C_b}{2+\alpha C_b}$ (Figure 1). The reader is referred to (9; 10; 11) for detailed discussion of these two types of spheres. For mesh generation purposes, biting spheres are used to cover the domain thus for generating good sphere packing, packing spheres are used to provide the theoretical quality guarantee. In the actual biting method implementation, packing spheres are never created. For the meshless methods, the biting spheres centered on the generated point set are used to define the patches.

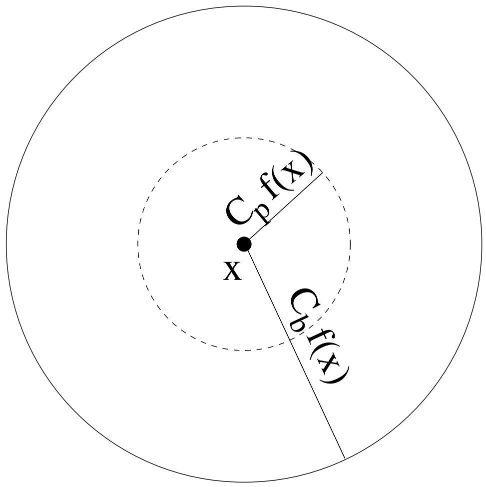
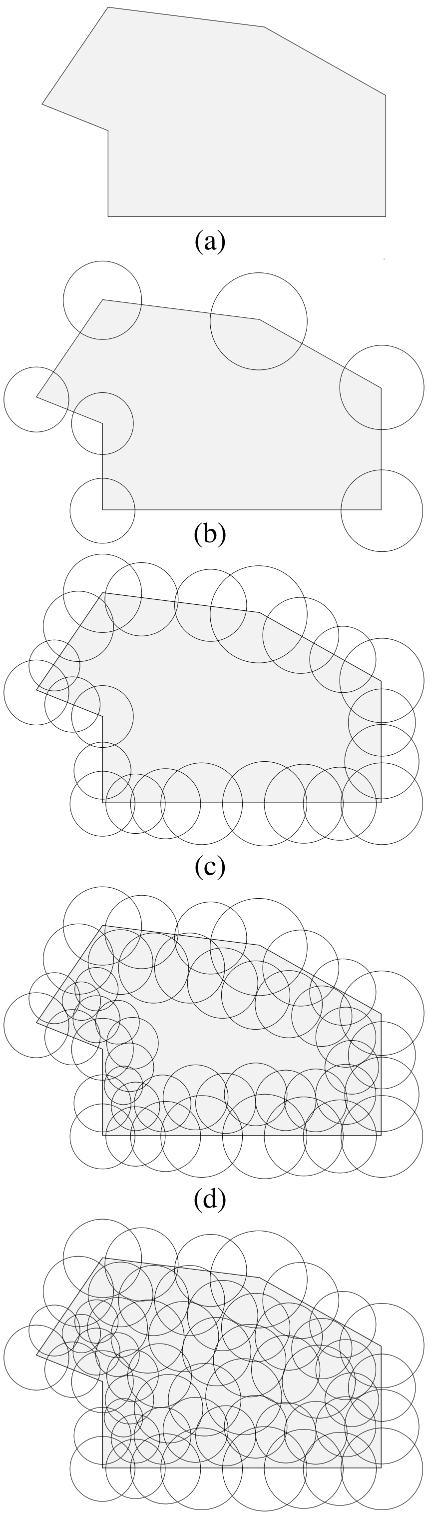


Figure 1 : Biting sphere (solid line) and packing sphere (dashed line) centered at a point $x \in \Omega$. Biting constant C_b and packing constant C_p determine the relation of the sizing function with the radius of the biting and packing spheres respectively.

The following algorithm was introduced for point generation of the meshing algorithms now suggested to be used for point placement of the meshless methods.



(a)

(b)

(c)

(d)

(e)

Figure 2 : Illustration of the *biting* scheme: (a) initial PSLG domain; (b) bites only on the vertices; (c) bites along the boundary; (d) bite and advance; (e) whole domain is covered by the biting spheres.

Algorithm `Biting`

1. Let the boundary of the domain be the initial front, see Figure 2 (a);
2. **[Points on the original Vertices]**: Bite all the input vertices by removing their biting spheres from the interior of the domain, see Figure 2 (b);

   Modify the front which becomes a set of segments and arcs. Segments are represented by the endpoints and arcs are represented by the center of the biting sphere.
3. **[Points on the Edges]**: Bite spheres centered on the input boundary: choose a vertex x on the front and remove its biting sphere. Whenever possible, we choose x on the intersection of some bitten spheres with the initial boundary, see Figure 2 (c).

   Modify the front by introducing the arc of the new biting spheres and removing the intersection of it with the front.

   Repeat until all initial input boundaries are bitten;
4. **[Points in the Interior of the Domain]**: Choose a vertex x on the front and remove its biting sphere, see Figure 2 (d) and (e);

   Modify the front by introducing the arc of the new biting sphere and removing the intersection of it with the front.

   Repeat until the advancing front is empty.

If the input is 3 dimensional domain, we need add one step after the third step to generate points on the faces of the domain boundary: keep biting all biting spheres centered on the initial input boundary faces, and modify the advancing front.

4 Guarantee for a Good Point Set

To provide the quality guarantee for the point set, we will discuss the achievement of the algorithm in terms of the quality criteria given in the second section. Due to the choice of the sizing function, the first criterion about the relation of the patches (spheres) with the point set is satisfied. Also it is very clear from the flow of the algorithm that there is no point in the domain that is not covered by at least one patch. For simplicity, let $B(x,r)$ be the sphere centered at x with radius r. The following lemma states that the third and the fourth criteria are also assured by the algorithm.

Lemma 4.1 (Size of the Overlap) *The size of the overlapping region between any two spheres in S is bounded from above by a constant factor of the size of the smaller sphere.*

Proof: We prove the lemma by proving that for any two overlapped biting spheres centered at points x and y, there is a constant factor of one sphere which is not covered by the other. For any two points x and y generated by the biting method, with no loss of generality suppose $B(x,C_b f(x))$ is bitten before $B(y,C_b f(y))$. This means that the point y is not in the interior of the ball $B(x,C_b f(x))$. In other words,

$$||x-y|| \geq C_b f(x) \tag{2}$$

In order to prove the lemma, it is sufficient to consider the following two cases. If $B(y,C_b f(y))$ is not the larger of the two biting spheres, that is if $C_b f(x) \geq C_b f(y)$, then the proof is immediate with a constant of $1/2$ since the intersection area would be less than half the size of either sphere (see Figure 3 (a)).

In the other case, if $C_b f(x) < C_b f(y)$, we use Lipschitz property of the sizing function, to prove a lower bound on the area of $B(x,C_b f(x))$ not covered by $B(y,C_b f(y))$. This implies an upper bound on the overlapping area and completes the proof.

Let uv be the diameter of $B(x,C_b f(x))$ on line xy, u being on the opposite side of y. Multiplying the Lipschitz inequality by the constant C_b, we get

$$C_b f(y) \leq C_b f(x) + \alpha C_b ||x-y|| \tag{3}$$

Rearranging the inequality gives

$$C_b f(x) - C_b f(y) \geq -\alpha C_b ||x-y|| \tag{4}$$

The length of uv not covered by sphere $B(y,c_b f(y))$ is the sum of the difference between the radii of the

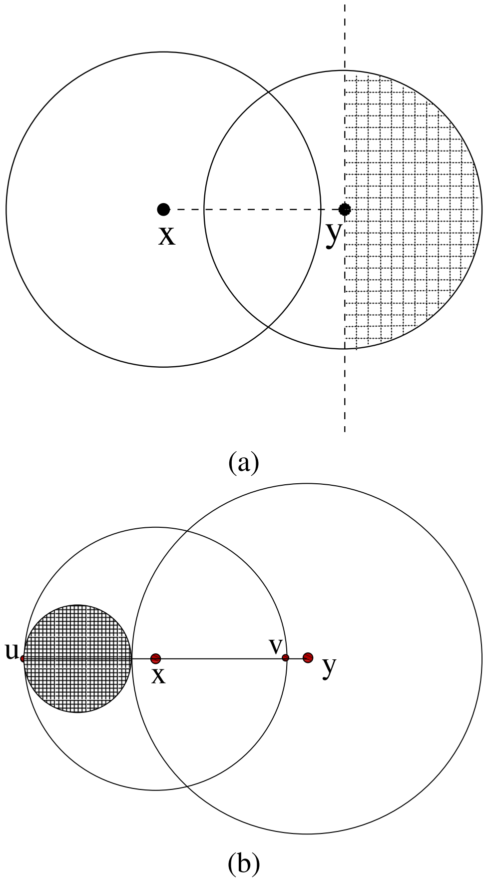


Figure 3 : Proof of Lemma 1 in two cases (a) If $C_b f(x) \geq C_b f(y)$ the shaded region shows that at least half of the smaller sphere is not overlapping (b) If $C_b f(x) < C_b f(y)$ then a bounded fraction of the area (shaded) of the smaller sphere is not overlapping.

two spheres and the distance between their centers, i.e. $||x-y|| + C_b f(x) - C_b f(y)$. Using Lipschitz property we could derive the following bound on this sum.

$$\begin{aligned} ||x-y|| + C_b(f(x) - f(y)) &\geq ||x-y|| - \alpha C_b ||x-y|| && (5)\\ &\geq (1-\alpha C_b)||x-y|| && (6)\\ &\geq (1-\alpha C_b) C_b f(x) && (7)\end{aligned}$$

Since the diameter of the ball $B(x,C_b f(x))$ is $2C_b f(x)$, at least $(1-\alpha C_b)/2$ of it is outside the ball $B(y,C_b f(y))$. Hence, the shaded region in Figure 3 (b) which is at least $(\frac{1-\alpha C_b}{2})^2$ factor of the area of $B(x,C_b f(x))$ is not covered by $B(y,C_b f(y))$. ■

Lemma 4.2 (Constant Ply) *Algorithm* `Biting` *generates a sphere set S, that has the constant ply property. In other words, there exists a constant t such that for every point $y \in \Omega$, there are at most t biting balls that covers y.*

Proof: We use the notion of packing spheres in order to prove this lemma. Packing spheres do not overlap and hence, there are only a constant number of packing spheres in a neighborhood of a point in the domain. We consider any $y \in \Omega$, and let $B(x_1, C_b f(x_1)), B(x_2, C_b f(x_2)), \cdots, B(x_t, C_b f(x_t))$ be the biting balls that cover y (see Figure 4). With no loss of generality, we assume that the ball $B(x_i, C_b f(x_i))$ is bitten before the ball $B(x_{i+1}, C_b f(x_{i+1}))$, for all $1 \le i < t$.

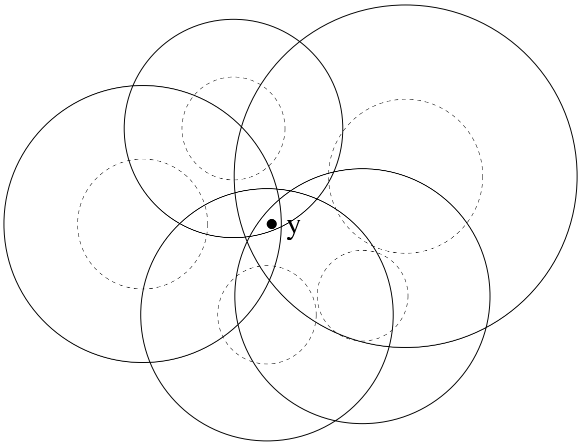


Figure 4 : Biting spheres (solid line) that cover a point $y \in \Omega$ and the packing spheres (dashed line) inside them.

As $y \in B(x_i, C_b f(x_i))$, we have

$$||x_i - y|| \le C_b f(x_i) \tag{8}$$

Also, $f()$ is α-Lipschitz implies that

$$\begin{aligned} -\alpha C_b f(x_i) &\le -\alpha||x_i - y|| && (9)\\ &\le f(x_i) - f(y) && (10)\\ &\le \alpha||x_i - y|| && (11)\\ &\le \alpha C_b f(x_i) && (12)\end{aligned}$$

Then we have for all $1 \le i \le t$,

$$\begin{aligned} \frac{f(y)}{1+\alpha C_b} &\le f(x_i) && (13)\\ &\le \frac{f(y)}{1-\alpha C_b} && (14)\end{aligned}$$

which implies that

$$||x_i - y|| \le \frac{C_b}{1-\alpha C_b} f(y) \tag{15}$$

In other words, for all $1 \le i \le t$, x_i is inside the ball $B(y, \frac{C_b}{1-\alpha C_b} f(y))$.
As $B(x_i, C_b f(x_i))$ is bitten before the ball $B(x_j, C_b f(x_j))$, for all $1 \le i < j \le t$. We have

$$||x_i - x_j|| \ge C_b f(x_i) \tag{16}$$

for all $1 \le i < j \le t$.
Also $f()$ is α-Lipschitz implies that

$$f(x_j) \le f(x_i) + \alpha||x_i - x_j|| \tag{17}$$

Let $C_p = \frac{C_b}{2+\alpha C_b}$, we have

$$\begin{aligned} C_p f(x_i) + C_p f(x_j) &\le 2C_p f(x_i) + \alpha C_p||x_i - x_j|| && (18)\\ &= ||x_i - x_j||. && (19)\end{aligned}$$

In other words, for all $1 \le i < j \le t$, the interior of balls $B(x_i, C_p f(x_i))$ and $B(x_j, C_p f(x_j))$ do not overlap.
Notice that for all $1 \le i \le t$, the ball $B(x_i, C_p f(x_i))$ is inside the ball $B(y, \frac{C_b}{1-\alpha C_b} f(y) + C_p f(x_i))$. As $f(x_i) \le \frac{f(y)}{1-\alpha C_b}$, we know that $B(y, \frac{C_b+C_p}{1-\alpha C_b} f(y))$ also covers ball $B(x_i, C_p f(x_i))$ for all $1 \le i \le t$.
As the interior of balls $B(x_i, C_p f(x_i))$, $1 \le i \le t$ do not overlap, and $f(x_i) \ge \frac{f(y)}{1+\alpha C_b}$, we have

$$\begin{aligned} \pi(\frac{C_b+C_p}{1-\alpha C_b} f(y))^2 &\ge \sum_{1 \le i \le t} \pi(C_p f(x_i))^2 && (20)\\ &\ge t\pi(\frac{C_p}{1+\alpha C_b} f(y))^2 && (21)\end{aligned}$$

It implies that

$$t \le (\frac{(1+\alpha C_b)(C_b+C_p)}{C_p(1-\alpha C_b)})^2. \tag{22}$$

■

If the input is three dimensional domain, then we have the similar proof by using the volume argument. For the correctness of the lemma proofs above C_b must be chosen such that $\alpha C_b < 1$.

Following from the above lemmas, we have the main theorem of our paper.

Theorem 4.3 (Good Point Set Guarantee) *Algorithm* Biting *generates a good point set for the meshless methods.*

5 Conclusion

A set of geometric criteria to define a good point set for the meshless methods is given. An algorithm, *Biting method*, is proposed to generate such a point set. Next step is to test the performance of our theoretically good algorithm, coupling it with a meshless method.

5.1 A Practical Variation.

Maintaining an advancing front of arcs and computing the intersections of them can be time consuming. As a practical version of the algorithm, one can use squares or rectangles as the patches.

5.2 Adaptation.

Adaptivity is a lot easier with the meshless methods, as one does not need to maintain the topology of the mesh. However, maintaining the quality of the point set is still an issue which can be achieved by maintaining a sphere packing. When the points are moved and bigger gaps are introduced, the domain can be oversampled and then the points, whose spheres overlap too much, can be eliminated.

5.3 Anisotropy.

Anisotropy becomes important when there is a multiple length-scale phenomenon involved in the problems, such as shear banding strain localization or composite materials. Definition of a good point set given in the second section can easily modified using ellipse shaped patches to cover these types of problems. Point set generation of such problems can employ ellipse packing biting method (11).

Acknowledgement: The work reported here was supported, in part, by the Center for Process Simulation and Design (NSF DMS 98-73945) and the Center for Simulation of Advanced Rockets (DOE LLNL B341494). We would like to thank J.S. Chen of University of Iowa for his valuable comments on our definition of good point set for meshless methods.

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