# Tiling 3D Euclidean Space with Acute Tetrahedra

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**Abstract.** We show that it is possible to tile the three dimensional space using only acute dihedral angle tetrahedra. Several constructions to achieve this objective are presented. Some necessary and/or sufficient conditions for acuteness of a tetrahedron are given. We also show that an acute angle triangulation is a Delaunay triangulation in two dimensions but not necessarily in three dimensions.

## 1 Introduction

**Definition 1** *A tetrahedra is* acute *if all of its dihedral angles are less than* $90°$. *A triangulation is* acute *if all of its simplices are acute.*

**Problem 1** *Is it possible to tile three dimensional Euclidean space with only acute tetrahedra?*

The two dimensional analog of this problem has a trivial answer – equilateral triangle tiles 2D Euclidean space. However, this does not extend to 3D since regular tetrahedron does not tile the space. All the other tetrahedra known to tile 3D space have right angles, as further discussed in Section 2.

Our motivation to address this problem is due to a space-time meshing method, called the *tent-pitching* algorithm [10]. The algorithm is designed to discretize space-time domains into meshes that obey a cone constraint, one which requires mesh faces to have smaller slopes than the cones that define the domain of influence imposed by the numerical (engineering) problem. Such meshes enable the use of very efficient element-by-element methods (e.g. space-time discontinuous Galerkin) to solve a wide variety of numerical problems (e.g. elastodynamics analysis). The tent-pitching algorithm starts with a space mesh and constructs the space-time mesh using an advancing front approach. Termination guarantee of the algorithm requires the initial space mesh to be an acute triangulation. For 3D×Time domains, it is required that the 3D space mesh have only acute dihedral angles.

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We note that, beyond our motivation, constructions that tile the space with only acute tetrahedra can be used to generate good quality meshes of models in 3D. Naylor [6] and Fuchs [4] study this idea using tilings that involve $90°$ dihedral angle tetrahedra, e.g. Sommerville constructions and subdividing cubes using five or six tetrahedra.

## 2 Background

It was believed for centuries that the regular tetrahedron, in addition to the cube, was one of the platonic solids that tile the space [7]. This, of course, is not true because a tiling of the space with all regular tetrahedra requires an edge of the tiling to be incident to $360°/70.53° = 5.1043$ tetrahedra, which is impossible (see Figure 1).

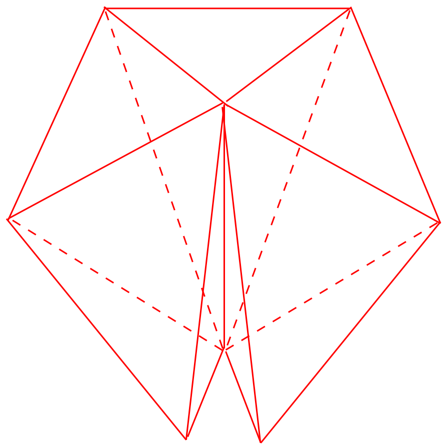

Figure 1: Regular tetrahedron does not tile the space.

There are tetrahedra whose congruent copies fill the space. Sommerville [8] gave four different tetrahedra each of which tiles the space. If a few different type of tetrahedra is allowed, then there are many constructions to fill the space, e.g. subdividing the cube into five or six tetrahedra [6]. All these constructions involve $90°$ or larger dihedral angles.

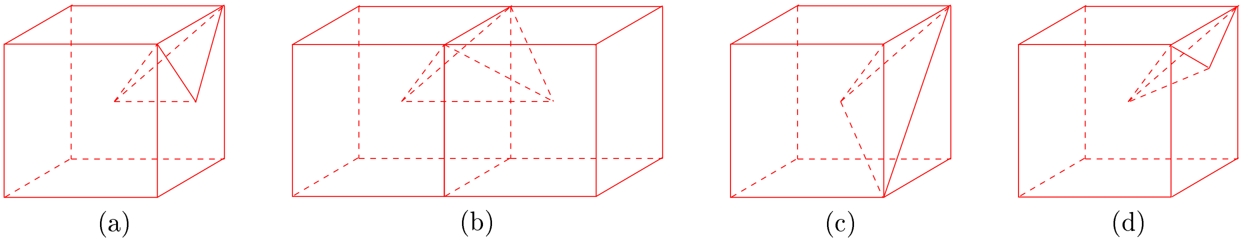

Figure 2: Sommerville constructions.

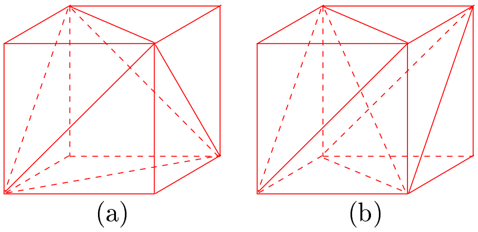

Figure 3: Cube subdivided into (a) 5; (b) 6 tetrahedra.

# 3 Acute Angle Tetrahedra

An acute tetrahedron is not necessarily a good quality one. In the classification of bad type of tetrahedra (see [2]), three (spire, splinter and wedge) out of nine may have all acute dihedral angles. This implies, in order to construct a tiling of the space, we are willing to sacrifice the quality of the tetrahedra. However, as discussed later, the quality of the tetrahedra in our constructions is acceptable for mesh generation and engineering purposes.

## 3.1 Acuteness test

The straightforward way to check whether a tetrahedron is acute is to check if all dihedral angles are smaller than 90°. The following lemma states some necessary and/or sufficient conditions for acuteness.

**Lemma 1** *A tetrahedron is acute,*

*(i) if and only if orthogonal projection of each apex onto its base plane is in the interior of the base triangle;*

*(ii) if and only if for each edge $e$, the diametral circle of the projection of the opposite edge of $e$ onto the normal plane of $e$ excludes the projection of $e$ on the same plane, which is a point;*

*(iii) only if all the face angles are acute. However, this is not a sufficient condition, e.g. sliver with all acute face triangles.*

**Proof (Sketch).** The statements (i) and (ii) are immediate because they essentially define the dihedral angles. The statement (iii) can be shown by proving the contra-positive: if a tetrahedron has a non-acute face angle then it also has a non-acute dihedral angle. Let $bcd$ be the non-acute face with $b$ the non-acute vertex and $a$ be the apex of tetrahedron $abcd$. If the projection of $a$ on the base is not in the interior of $bcd$ then the tetrahedron is not acute by (i) (the dihedral angle on $bc$, $cd$, or $db$ is not acute). Otherwise, the dihedral angle on the edge $ab$ is larger than the face angle $cbd$ making $abcd$ non-acute. □

## 3.2 Acuteness and Delaunay

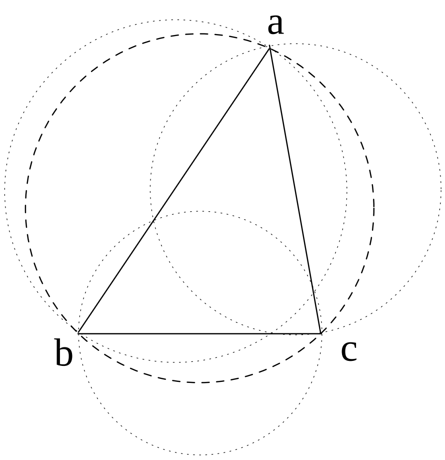


Figure 4: Acute triangles in 2D are Delaunay.

**Lemma 2** *A two dimensional triangulation $T$ is a Delaunay triangulation if $T$ is an acute triangulation.*

**Proof (Sketch).** Since T is acute, the diametral circles of all the edges in T are empty of other points. This implies the circumcircle of every simplex in T are empty of other points (see Figure 4). □

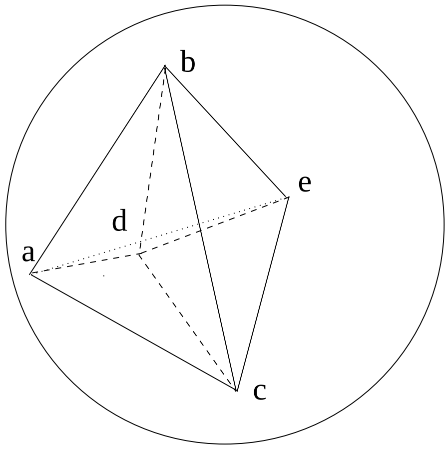


Figure 5: An example where Delaunay and acute triangulations are different.

**Lemma 3** *A three dimensional triangulation $T$ is not necessarily a Delaunay triangulation even if $T$ is an acute triangulation.*

**Proof.** We provide an example of such a case. Consider the five points $a = (-\epsilon, -\epsilon, -\epsilon)$, $b = (1, 0, 0)$, $c = (0, 1, 0)$, $d = (0, 0, 1)$, $e = (2/3+\epsilon, 2/3+\epsilon, 2/3+\epsilon)$. where $\epsilon$ is a very small positive number (see Figure 5). Delaunay triangulation of these points, consists of three tetrahedra (*abce*, *acde*, and *abde*), where the three dihedral angles incident to the diagonal *ae* are each 120°. It is possible to construct an acute triangulation of these five points by performing a 3-to-2 flip on the Delaunay construction. The two tetrahedra *abcd* and *bcde* are acute. □

This lemma implies that Delaunay triangulation in three dimensions does not minimize the maximum dihedral angle.

## 4 Constructions

**Foams.** Constructions come as a dual of foam structures described in [9]. Each one of foam structures is defined by a point set, whose Voronoi regions are polytopes with pentagonal and hexagonal faces. The duals of these structures are likely to give acute triangulations because each edge has five or six faces incident to it. Here we describe a few acute angle triangulation of the 3D space inspired by the foams.

We first tile the 2D space with equilateral triangle and squares. Then points defined on this tiling are projected into three dimensional space at different heights. Figures 6 and 7 show several tilings of the 2D space. There are three kinds of points, black (filled circles) white (empty circles) and yellow points (end-points of square and triangle edges). Suppose the length of the sides of the square and triangle is $4k$. Then, the yellow points are at height $2k$, the black points are at height $4k - 1$ and the white points are at height $4k + 1$. Points inside the square are $k$ away from the sides. Three dimensional triangulations based on these constructions are shown in Figure 9 (a), (b) and (c).

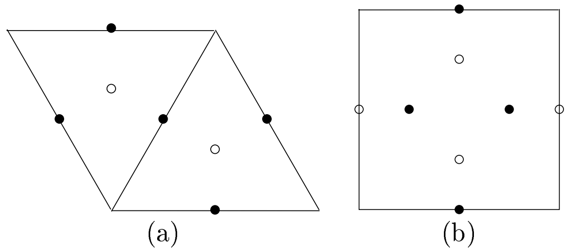


Figure 6: Foam constructions using only (a) triangle; (b) square.

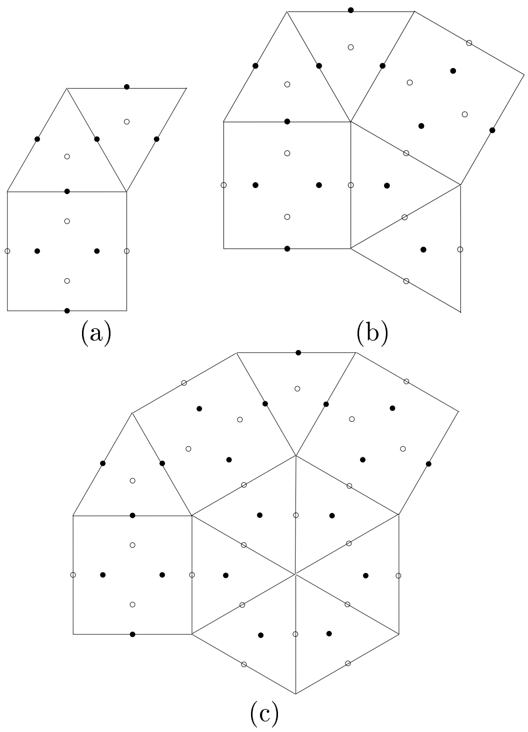


Figure 7: Three of many possible foam constructions using both triangles and squares.

**Icosahedron.** The next construction is inspired by the work of Field [3]. His tiling involves right angle tetrahedra. By being selective in the point set we have a tiling with only acute tetrahedra. We use icosahedron as the main gadget, which does not tile the space by itself. Icosahedron, however, can be subdivided into acute tetrahedra simply by inserting its center-point. A top view of the icosahedron construction is shown in Figure 8. Only faces that are shown as large triangles in this figure are shared by two neighbor icosahedra. The rest of them are also a face of the gaps which can be filled with four different kind of acute tetrahedra. There are two type of gap regions. The deeper gaps are defined by the points $a, b, c, d, e, f, g$, and $g'$ (opposite to $g$) and can be filled with two type of tetrahedra, e.g. $bdfg$ and $abfg$. The shallower gaps are defined by the points $d, f, h, i, j, k, e$, and $e'$ (opposite of $e$) and can be filled with two type of tetrahedra, e.g. $ee'df$ and $ee'fh$. Delaunay triangulation of the point set is acute for this construction and shown in Figure 9 (d).

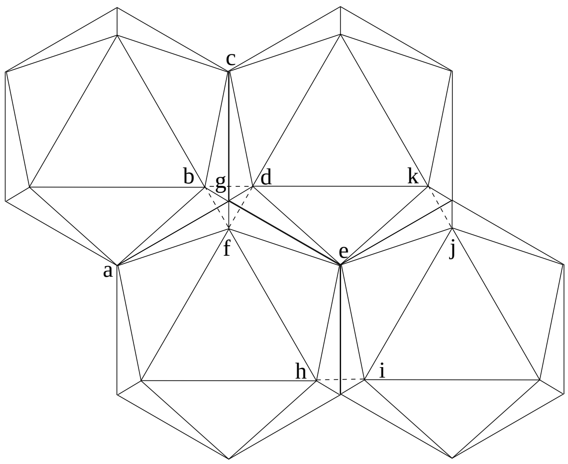


Figure 8: Icosahedron construction.

These constructions lead to the main result of this paper.

**Theorem 1** *It is possible to tile the three dimensional Euclidean space with acute tetrahedra.*

# 5 Conclusions

To address acute dihedral angle assumption of tent-pitching algorithm for 3D×Time, we ask if it is possible to fill the space with all acute dihedral angle tetrahedra. This is an important geometry problem and relates to tiling theory. We gave a positive answer to this fundamental question. The quality of each construction, summarized in Table 1, is very promising for mesh generation use. There are still some challenges left for making use of these constructions as real-life meshing techniques. A strategy is required to fit the tilings into a 2D projection of the 3D domain. Malkevitch studies this problem in [5]. Also, our constructions have dimples (cavity on the surface) making them not suitable to mesh the domains with planar surfaces. Related open problems include

- Given a point set that has an acute triangulation, how to compute it?
- Are these constructions the best possible, i.e. which tiling of the space with tetrahedra minimizes the maximum dihedral angle?
- Is it possible to triangulate a cube with all acute tetrahedra?
- Is it possible to make use of these constructions to extend the quad-tree based algorithm [1] for computing acute triangulations of point sets?

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# References

[1] M. Bern, D. Eppstein and J. Gilbert. *Provably good mesh generation.* Journal of Computer and System Sciences, (48), 384-409, 1994.

[2] S.-W. Cheng, T.K. Dey, H. Edelsbrunner, M.A. Facello, and S.-T. Teng. *Sliver exudation.* Proc. of $15^{th}$ ACM Symp. Comp. Geometry, 1-13, 1999.

[3] D.A. Field. *Implementing Watson's algorithm in three dimensions.* Proc. of $2^{nd}$ ACM Symp. Comp. Geometry, 246-259, 1986.

[4] A. Fuchs. *Automatic Grid Generation with Almost Regular Delaunay Tetrahedra.* Proc. of $7^{th}$ International Meshing Roundtable, 133-147, 1998.

[5] J. Malkevitch. *Tiling Convex Polygons with Equilateral Triangles and Squares.* Discrete Geometry and Convexity, ed. Goodman *et al.* New York Academy of Sciences, New York, 299-303, 1985.

[6] D.J. Naylor. *Filling space with tetrahedra.* Int. J. Numer. Meth. Engng., (44), 1383-1395, 1999.

[7] M. Senechal. *Which tetrahedra fill space?* Mathematics Magazine, (54) 227-243, 1981.

[8] D.M.Y. Sommerville. *Space-filling tetrahedra in Euclidean space.* Proc. of the Edinburgh Mathematical Society, (41), 49-57, 1923.

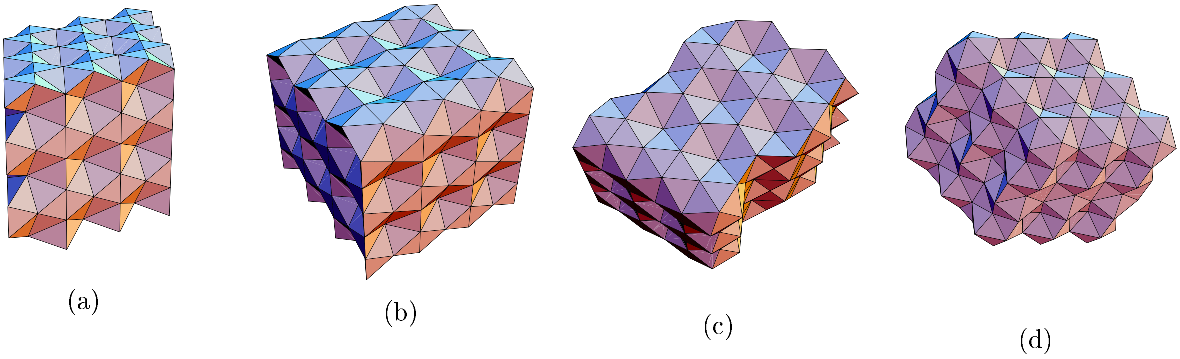


Figure 9: Acute triangulations filling the space. (a) Triangle tiling construction of Figure 6 (a); (b) Square tiling construction of Figure 6 (b); (c) Triangle/Square tiling construction of Figure 7 (a); (d) Icosahedron construction of Figure 8.

| | radius-to-shortest-edge | | minimum dihedral angle | | maximum dihedral angle | |
|---|---|---|---|---|---|---|
| Construction | minimum | maximum | minimum | maximum | minimum | maximum |
| Triangle tiling | .651 | .737 | 53.13 | 67.37 | 73.89 | 77.07 |
| Square tiling | .645 | .707 | 53.13 | 67.79 | 73.39 | 78.46 |
| Triangle/Square | .645 | .737 | 53.13 | 67.79 | 73.39 | 78.46 |
| Icosahedron | .629 | 1.000 | 41.81 | 69.09 | 71.99 | 83.62 |
| Regular Tetrahedron | .612 | .612 | 70.53 | 70.53 | 70.53 | 70.53 |

Table 1: Quality of the constructions and regular tetrahedron in terms of the radius-to-shortest-edge ratio and the dihedral angles.


[9] J. M. Sullivan. *The geometry of bubbles and foams.* Foams and Emulsions (NATO ASI volume E 354), 379-402, Kluwer, 1999.

[10] A. Üngör and A. Sheffer. *Pitching tents in space-time: mesh generation for discontinuous Galerkin method.* Proc. of 9th International Meshing Roundtable, 111-122, New Orleans, LA, 2000.