COT5405: ANALYSIS OF ALGORITHMS

Homework # 1

Due date: Jan 31, 2006, Tuesday (**beginning of the class**)

Your solutions should be concise, but complete, and typed or handwritten clearly. Feel free to consult textbooks, articles, internet, and also each other, but write the solutions yourself and cite your sources. Answer **only five** of the following six questions. Each problem is worth 20 points (no bonus points). Use a separate single sheet of paper for each problem and write your name on each of the five pages. Attach (staple, clip) your papers together. Late submissions will not be accepted.

1. *The Kingdom of TooManyCinderellas* has a tradition when it comes to choosing a princess for their esteemed prince to marry. All young ladies of the kingdom attend a ball and they drop their shoes at the stairs of the palace as they leave the ball. The girl with the median shoe size should be chosen to marry the prince and named the princess. The sizes of the shoes (and the feet of the ladies) are so similar that you cannot compare two pairs of shoes (or the feet of two ladies) to see which is larger. You can, however, check whether a pair of shoe is too tight, too large or perfect fit for a lady.

   (a) Design and analyze an efficient algorithm that identifies the princess. How many shoe-feet comparisons does your algorithm perform in the worst case?

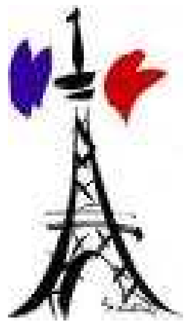
   (b) Design and analyze an efficient randomized algorithm that identifies the princess.

2. Give two arrays $A[1 \dots n]$ and $B[1 \dots k]$ which are sorted respectively in a non-decreasing order, i.e., $A[1] \le A[2] \le \dots \le A[n]$ and $B[1] \le B[2] \le \dots \le B[k]$. The problem is to merge B into A (to become a single sorted array) assuming A can be expanded dynamically (to accommodate k additional elements). Suppose k is "sufficiently small" compared to n.

   (a) Describe such an algorithm which, in the worst case, uses $\lfloor (n+k)/2 \rfloor$ array element comparisons, and at most $O(n+k)$ moves, i.e., copying an element of either array to a location). You may use an auxiliary array of size $O(k)$ for the algorithm. (Hint: Compare $B[1]$ with $A(\lfloor (n+1)/2 \rfloor)$ as the first step.)

(b) How small does k have to be (in terms of n) in order to achieve the time bound for the number of comparisons required in Part (a)?

3. Say you work for *NoMoreBrokenBeerBottlesOnTheStreets Co.* who wants to design the world's most durable beer bottle. The company made an agreement with the French officials to allow you run some tests at the Eiffel Tower which has n stairsteps. Your test study involves dropping a number of copies of a particular bottle design and find the highest stairstep one can drop the bottle without breaking it. They let you to break at most k bottles both because developing the bottle prototypes is expensive and the Parisians cannot stand more than k broken bottles. Also, they want you to design a strategy with asymtotically sublinear worst-case number of drops. Note that if you were allowed to break $\log n$ bottles, you could simply do a binary search. On the other hand if you were allowed to perform $O(n)$ drops you could incrementally go from bottom to top and find the solution breaking only one bottle. However, due to the rules described above, both these options are out.



(a) For $k = 2$, design an strategy for finding the highest safe stairstep that requires you to drop a bottle at most $f(n)$ times, for some function $f(n)$ that grows slower than linearly, i.e., $\lim_{n\to\infty} f(n)/n = 0$.

(b) Now suppose you are allowed to break $k > 2$, bottles, for some given k. Design a strategy for finding the highest safe stairstep using at most k bottles. If $f_k(n)$ denotes the number of times you need to drop a bottle according to your strategy, then the functions $f_1(n), f_2(n), \ldots$ should have the property that each grows asymptotically slower than the previous one, i.e., $\lim_{n\to\infty} f_k(n)/f_{k-1}(n) = 0$ for each k.

4. You are given two functions f and g such that $f(n)$ is $O(g(n))$. Give a proof or a counterexample for each of the following statements to show that the statement is true or false.

(a) $\lg f(n)$ is $O(\lg g(n))$.

(b) $2^{f(n)}$ is $O(2^{g(n)})$.

(c) $f(n)^2$ is $O(g(n)^2)$.

5. Sort the following functions from asymtotically smallest to largest indicating the ties:

$n!$	n^3	$\lg n$	$\lg(n \lg n)$	$(\lg n)^n$
2^n	$(1+\frac{1}{1000})^n$	$(1-\frac{1}{1000})^n$	$\lg^* n$	$\lg^* \lg n$
$\lg \lg^* n$	$n^{\lg n}$	$(\lg n)^{\lg n}$	$\log_{1000} n$	$\lg^{1000} n$
$2^{\lg^2 n}$	$\lg^n 1000$	$n^{n/\lg n}$	$2^{\lg n/1000}$	$n^{(1/1000)}$
e^n	1000	$\lg^{(1000)} n$	$(n+1)!$	n

(For simplicity of the notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example the functions n^3, $\binom{n}{2}$, n, and n^2 could be sorted as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.)

6. You went to a casino where the dealer has a special deck of fifty-two cards: all hearts numbered from 1 to 52. The dealer shuffles the deck until each of the 52! possible orderings of the cards is equally likely. He then takes cards one at a time from the top of the deck and gives them to you stopping as soon as he gives you the three of hearts.

   (a) On average, how many cards do you receive from the dealer?

   (b) What is the expected value of the smallest-numbered card that you receive?

   (c) What is the expected value of the largest-numbered card that you receive?

   Prove that your answers are correct. You might find it easier to solve it for an n-card deck and then set n to 52. Your solutions for (a), (b) and (c) will help the casino to setup three different games: (a) invest a fixed amount k_a dollars and the casino will pay you one dollar for every card you received; (b) invest a fixed amount k_b dollars and you will be paid by the dollar amount of the value on the smallest card you received; (c) invest a fixed amount k_c dollars and you will be paid by the dollar amount of the value on the largest card you received. How much profit should the casino expect out of these three games?