Unextendibility of Mutually Unbiased Bases

P. Oscar Boykin[1] *, Meera Sitharam[2], Mohamad Tarifi[2], Pawel Wocjan[3]

1. Dept. of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611

2. Computer and Information Science and Engineering, University of Florida, Gainesville, FL 32611

3. School of Electrical Engineering and Computer Science University of Central Florida, Orlando, FL

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1 Questions and Motivation

We consider the following questions about Mutually Unbiased Bases (MUBs).

- (Problem 1) When is a given set of MUBs non-extendible? I.e., characterize sets $M_1, M_2, .., M_k$ of MUBs for which there is no basis M unbiased to $M_1, .., M_k$?
- (Problem 2) Can one characterize families of bases where non-extendibility of a set of MUBs imply maximality, and are there natural families of bases with this property? I.e, within such a family of bases, a greedy method for constructing MUBs would guarantee a maximal cardinality MUB collection.

Besides being of independent interest in understanding the structure of (partial) MUB collections and their conjugacy classes, satisfactory answers to these questions will help the following investigations:
– the search for lower bounds on the number of MUBs and specifically for *efficient* constructions;
– the search for upper bounds on the number of MUBs even for small non-prime-power dimensions, either using automated refutation methods, or using computer programs that exhaustively generate all (conjugacy classes of) MUB collections.

2 Results and Conjectures

Result 1. We have shown that Latin MUBs i.e., MUBs constructed from mutually orthogonal Latin squares (Beth and Wocjan, 2005) are non-extendible, even for prime power dimensions d where it is known that Latin MUBs are not maximal collections. Since Latin MUBs are monomial MUBs and maximal monomial collections exist in those dimensions d, the family of bases in Problem (2) cannot, for instance, be chosen as monomial bases.

*corresponding author boykin@ece.ufl.edu

We conjecture that there is not even a single vector unbiased to the Latin MUBs in prime power dimensions. We additionally conjecture that a collection of *real* MUBs in $d = 2^n$ dimensions is maximal if and only if it is unextendible over the complex numbers.

Result 2. We additionally consider the simpler problem of characterizing non-extendible sets of mutually unbiased *lattice lines*.

A *lattice line* in $\mathbb{R}^d$ is a line passing through the origin and some point v in $\{-1, 1\}^d$. We denote the line by either of the vectors v or $-v$. Two lattice lines v and w are *mutually unbiased* if $|\langle v, w\rangle| = \frac{1}{\sqrt{d}}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. We denote by $L_{\mathbb{R}^d}$ the maximum number of mutually unbiased lattice lines that can be found in $\mathbb{R}^d$.

Mutually unbiased lattice lines are related to "extremally or uniformly distributed" collections of Euclidean line sets with arbitrary, prescribed angles that arise in diverse venues and have been studied by numerous researchers (see well-known survey by Calderbank, Cameron, Kantor, Seidel). Real MUBs consist of these lines.

We construct a set of 2^i lattice lines in dimension $d = 4^i s^2$ using the $2^i \times 2^i$ Sylvester Hadamard. We then show that this particular set cannot be extended by even one lattice line. If any set can be extended into an optimal set, then $L_{\mathbb{R}^d} = 2^i$ for $d = 4^i s^2$. However, we later give a construction that gives $L_{\mathbb{R}^d} \geq 2^i(2^i s^2 - s)$ for some cases of i, which shows that not every set can be extended into an optimal set.

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