Today Zia finished part 3 of his talk.

**Definition 1 (Depth Complexity).** For a function $f$, the depth complexity $\text{d}(f)$ is the minimum depth of a circuit computing $f$. The depth of a circuit $C$ is denoted by $\text{d}(C)$.

**Definition 2.** For a boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$ (i.e. the set of all $x$'s such that $f(x) = 1$) and $Y = f^{-1}(0)$, let $R_f \subseteq X \times Y \times \{1, \ldots, n\}$ consist of all triples $(x, y, i)$ such that $x_i \neq y_i$.

Notice that there is always an $i$ such that $(x, y, i)$ satisfies the relation $R_f$ since $f(x) = 1$ and $f(y) = 0$, and so $x \neq y$.

**Definition 3 (Communication Problem).** The communication problem for $R_f$ is the following: Alice is given $x \in X$, Bob is given $y \in Y$ and their task is to find some $i \in \{1, \ldots, n\}$ such that $(x, y, i) \in R_f$.

The definition of a protocol as was given in the previous lecture remains unchanged. Based upon these definitions we can define the communication complexity as follows:

**Definition 4 (Communication Complexity).** A protocol $P$ computes $R_f$ if for every input $(x, y) \in X \times Y = f^{-1}(1) \times f^{-1}(0)$, the protocol reaches a leaf labeled by some $i \in \{1, \ldots, n\}$ such that $(x, y, i) \in R_f$. The deterministic communication complexity of $R_f$ denoted $D(R_f)$, is the minimum cost of $P$ over all protocols $P$ that compute $R_f$.

**Definition 5 (Monochromatic Rectangle).** The set $A \times B \subseteq X \times Y = f^{-1}(1) \times f^{-1}(0)$ is an $R_f$-monochromatic rectangle if there exists an $i \in \{1, \ldots, n\}$ such that for every $(x, y) \in A \times B$, we have $(x, y, i) \in R_f$.

**Proposition 6.** Any depth $t$ protocol that computes the relation $R_f$ induces a partition $X \times Y = f^{-1}(1) \times f^{-1}(0)$ into at most $2^t$ $R_f$-monochromatic rectangles.

**Proof.** The same as the proof for functions given in the previous lecture, mutatis mutandis □

**Lemma 7.** For every circuit $C$ for $f$, there is a corresponding protocol $P$ for $R_f$ such that the depth of $P$ is at most $\text{d}(C)$, i.e., at most $\text{d}(C)$ bits are exchanged during the run of $P$
Proof. Given a circuit $C$ computing $f$, the idea of the protocol $\mathcal{P}$ for $R_f$ is the following: Alice knows $x \in f^{-1}(1)$ whereas Bob knows $y \in f^{-1}(0)$. Alice and Bob traverse the nodes of $C$, starting from the output node, and they continue toward the input nodes in such a way as to maintain an invariant, namely that the function $g$ computed by the current node satisfies $g(x) = 1$ and $g(y) = 0$. We will now show that Alice (who only knows $x$) and Bob (who only knows $y$) can indeed traverse $C$ in a way that maintains the above invariant.

Since $x \in X = f^{-1}(1)$ and $y \in Y = f^{-1}(0)$, the invariant is true at the output node of $C$. Now suppose that the current node reached by Alice and Bob is an $\lor$ gate computing a function $g$, and the invariant is true at this $\lor$ gate, i.e., that $g(x) = 1$ and $g(y) = 0$. Let $g_1$ and $g_2$ be the functions corresponding to the nodes of $C$ entering the current $\lor$ node. Then $g = g_1 \lor g_2$. Since $g(y) = 0$, we have $g_1(y) = g_2(y)$. And since $g(x) = 1$, either $g_1(x) = 1$ or $g_2(x) = 1$ (or both). Alice, who knows $x$ and $g_1$ and $g_2$ (since she obviously knows $C$), sends a single bit to Bob indicating for which $i \in \{1, 2\}$ the function $g_i(x) = 1$. In the case where both are 1 she chooses $g_1$. They then both proceed to the corresponding node, where the invariant clearly holds.

Symmetrically, if the current node is an $\land$ gate computing a function $g$ such that $g(x) = 1$ and $g(y) = 0$, and $g_1$ and $g_2$ are the functions corresponding to the nodes entering the current $\land$ node, then $g = g_1 \land g_2$, and so $g_1(x) = g_2(x) = 1$, while either $g_1(y) = 0$ or $g_2(y) = 0$ (or both). This time Bob sends a single bit indicating for which $i \in \{1, 2\}$ the function $g_i(y) = 0$, and the both proceed to the corresponding node.

Now suppose Alice and Bob have reached an input node of $C$. Assuming $f$ is a function of the variables $z_1, \ldots, z_n$, this input node is labeled with $z_i$ or $\neg z_i$ for some $i \in \{1, \ldots, n\}$. We claim that both players know that $i$ is a correct output, i.e., that $(x, y, i) \in R_f$. To see this, let $g$ be the function computed by this input node. If the node is labeled $z_i$ than by the invariant, we have $g(x) = 1$ and $g(y) = 0$. But $g(y) = y_i$. Hence $x_i \neq y_i$; and so $(x, y, i) \in R_f$. Similarly, if the input node is labeled $\neg z_i$, then by the invariant, $g(x) = 1$ and $g(y) = 0$ once again. But this time $g(x) = \neg x_i$ and $g(y) = \neg y_i$, and we hence $x_i = 0$ and $y_i = 1$. So in this case we have $x_i \neq y_i$ as well, and hence $(x, y, i) \in R_f$. \qed

Lemma 8. For every protocol $\mathcal{P}$ for $R_f$, there is a corresponding circuit $C$ for $f$ such that $d(C) = \text{depth of } \mathcal{P}$, i.e. the communication complexity $\mathcal{P}$

Proof. Given a protocol $\mathcal{P}$ for $R_f$, we will convert this binary tree $\mathcal{P}$ into a circuit $C$ as follows: Each internal node in which Alice speaks (i.e., a node labeled by a function with domain $X$) is labeled by $\lor$ and each internal node in which Bob speaks is labeled by $\land$. As for the leaves of $\mathcal{P}$, by proposition 6, each leaf is an $R_f$-monochromatic rectangle $A \times B$ with which an output $i$ is associated. Take any $x \in A$ and let $x_i = \psi$. Then since for all $y \in B$, the value $i$ is a legal output on $(x, y)$ for $\mathcal{P}$, we must have $y_i = \neg \psi$ for all $y \in B$. This in turn implies $x_i = \psi$ for every $x \in A$. Therefore either:

1. $\forall x \in A \forall y \in B (x_i = 1 \land y_i = 0)$
2. $\forall x \in A \forall y \in B (x_i = 0 \land y_i = 1)$

In the first case we label the leaf by $z_i$ whereas in the second case we label the leaf by $\neg z_i$.

Clearly the depth of $C$ equals the depth of $\mathcal{P}$. It remains to prove that $C$ computes $f$. We claim that for every node of $C$, the function $g$ corresponding
to that node satisfies:

$$\forall x \in A \forall z' \in B (g(z) = 1 \land g(z') = 0)$$

where $A \times B$ are the inputs that reach the corresponding node of $\mathcal{P}$. This immediately implies that the function computed by the output node of $C$ (i.e., the function computed by $C$) is 1 for all $z \in X = f^{-1}(1)$ and 0 for all $z \in Y = f^{-1}(0)$. Hence $C$ computes $f$.

The claim is proved by induction, starting from the input nodes of $C$ and moving toward the output node of $C$. The claim is true for the input nodes because of the way in which these input nodes were labeled. Now consider an internal node $w$ of $V$ computing a function $g$ such that the claim is true for its two children computing the functions $g_1$ and $g_2$ respectively. Let $A \times B$ be the inputs reaching this node $w$ in $\mathcal{P}$. Assume without loss of generality, that Alice speaks in this node. (The case for Bob is similar). That means $w$ is labeled by $\lor$ in $C$, i.e., $g = g_1 \lor g_2$. In $\mathcal{P}$, since Alice speaks at $w$, this entails a partitioning of $A$ into $A_1$ and $A_2$, where the inputs in $A_1$ travel to the left subtree of $\mathcal{P}$ and those in $A_2$ travel to the right subtree. By the induction hypothesis, for all $y \in B$ we have $g_1(y) = g_2(y) = 0$, and for all $x \in A_1$ we have $g_1(x) = 1$, while for all $x \in A_2$, we have $g_2(x) = 1$. Hence for all $y \in B, g(y) = g_1(y) \lor g_2(y) = 0$, and for all $x \in A = A_1 \cup A_2$, we have $g(x) = g_1(x) \lor g_2(x) = 1$. \hfill \square

**Theorem 9.** For ever $f : \{0, 1\}^n \rightarrow \{0, 1\}$, we have $d(f) = D(R_f)$.

**Proof.** Applying lemma 7 to the circuit $C^*$ with minimal depth that compute $f$, we see that there exists a protocol $\mathcal{P}$ for $R_f$ such that the depth of $\mathcal{P} \leq d(C^*) = d(f)$. Hence $D(R_f) \leq d(f)$. Apply lemma 8 to the protocol $\mathcal{P}^*$ with minimal depth for $R_f$, we see that there exists a circuit $C$ for $f$ such that $d(C) = \text{depth of } \mathcal{P}^* = D(R_f)$. Hence $d(f) \leq D(R_f)$. \hfill \square