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| **Recent advances in Complexity CIS 6930/CIS 4930** | September 29, 2011 |

Lecture 13-18

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# Contents



# 1 Global rigidity in 2D

**Definition 1** *A framework $G(p)$ is globally rigid in $\mathbb{R}^d$ if $\forall q$ s.t. $\rho(p)_G = \rho(q)_G$, we have $p = q$.*

The property global rigidity has only recently been proved to be a generic property in any $d$ [4]. We can say that *a graph is globally rigid in $\mathbb{R}^d$* if it has a unique realization $G(p)$ in $\mathbb{R}^d$.

In the following discussion, without loss of generality, we assume that the graph $G$ has more than $d$ vertices.

In 2D, global rigidity has full characterization. There are two conditions: 3-connected and redundantly rigid.

**Definition 2** *A graph $G = (V, E)$ is $k$-connected if $|V| > k$ and $G - X$ is connected for every set $X \subset V$ with $|X| < k$. In other words, no two vertices of $G$ are separated by fewer than $k$ other vertices.*

**Definition 3** *A graph $G = (V, E)$ is redundantly rigid in $\mathbb{R}^d$ if $\forall e \in E$, $G' = (V, E - e)$ is rigid in $\mathbb{R}^d$.*

**Theorem 1** *([1][2]) A graph $G = (V, E)$ with $|V| > 3$ is globally rigid in $\mathbb{R}^2$ iff:*

1. *$G$ is 3-connected;*
2. *$G$ is redundantly rigid.*

In the following sections, we are going to demonstrate the proof of Theorem 1.

# 2 Necessary condition for global rigidity (in any dimension)

The necessary condition in Theorem 1 holds for any dimension.

**Theorem 2** *If a graph $G = (V, E)$ with $|V| > d + 1$ has unique realization in $\mathbb{R}^d$, then $G$ is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

*If a graph $G = (V, E)$ with $|V| = d + 1$ has unique realization in $\mathbb{R}^d$, then $G$ is the clique $K_{d+1}$. [1]*

When discussing the proof of Theorem 2,we will only consider rigid graphs, since global rigidity obviously requires rigidity.

## 2.1 $(d + 1)$-connectivity

The necessity of $(d + 1)$-connectivity is easy to show.

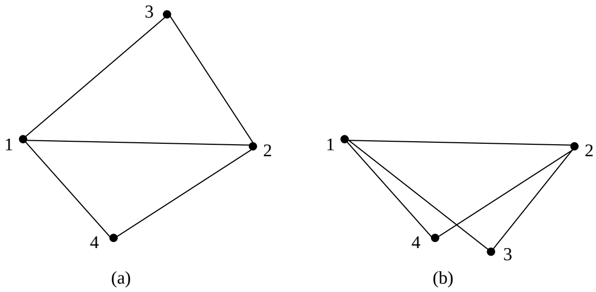


**Figure 1**: Partial reflection

**Theorem 3** *A generic framework $G(p)$ with $|G| > d + 1$ in $\mathbb{R}^d$ has a equivalent but non-congruent framework $G(q)$ obtained via partial reflection, if $G$ is not $(d + 1)$-connected.*

**Proof:** If $G$ is not $(d+1)$ connected, $\exists X \subset V$ such that $X$ separates $G$ into $G_1, G_2$, $|X| \leq d$. So for any generic framework $G(p)$, $|X|$ lies on a $d$-dimensional hyperplane in $\mathbb{R}^d$, and reflecting $G_2(p)$ about $X(p)$ will give us an equivalent framework $q$, $q \neq p$. ■

Refer to Figure 1. Vertices 2 and 3 form a 2-separator of $G$ in $\mathbb{R}^2$. From the framework in (a), reflecting vertex 1 about $2, 3$ gives us the framework in (b), which is equivalent but not congruent to (a).

## 2.2 Redundant rigidity

A graph can be $(d + 1)$-connected but still not globally rigid. See Figure 2.

In Figure 2(a), after removing the edge $(a, f)$, the graph becomes flexible. We can rotate it until we reach the position in (c) where the distance between $(a, f)$ is just the same as in (a). Adding back $(a, f)$ gives us an equivalent and non-congruent framework.

To see why redundant rigid is necessary, we will investigate the space of satisfying realizations for flexible graphs.

We restrict our consideration to those $G(p)$ where not all vertices lies in a hyperplane. This is a subset $T$ of $\mathbb{R}^{nd}$. The edge distance map $\rho_G$ can be defined as a composition $\rho_G = \pi_E \circ \rho$, where $\rho$ is the complete distance map and $\pi_E$ is just a projection onto the edge set. Let $|V| = n$, $|E| = m$, we obtain the upper half Figure 3:

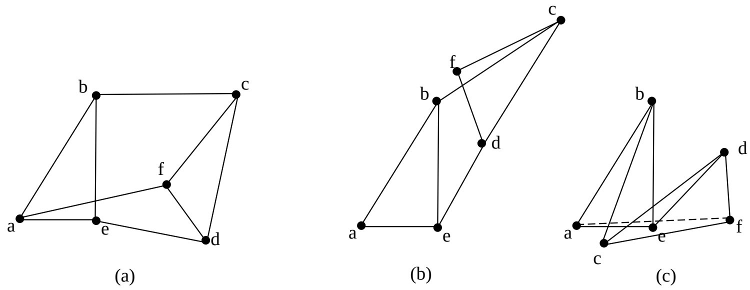


**Figure 2**: Partial reflection

$$\begin{tikzcd}
& \mathbb{R}^{n(n-1)/2} \arrow[dr, "\pi_E"] & \\
T \subset \mathbb{R}^{nd} \arrow[ur, "\rho"] \arrow[rr, "\rho_G"] \arrow[dr, "q"'] & & Z \subset \mathbb{R}^m \\
& \mathrm{X} \arrow[ur, "\bar{\rho}"'] &
\end{tikzcd}$$

**Figure 3**:

**Definition 4** *The realization set of $G(p)$ is $\pi_E^{-1}\rho(p)_G$.*

The realization set consists of all the possible completion of the partial Euclidean distance matrix given by $\rho(p)_G$. In other words, it contains all realizations of the linkage $(G, \rho(p)_G)$.

Our goal is to show that $\pi_E^{-1}\rho(p)_G$ contains more than one point.

**Definition 5** *A manifold is a topological space that on a small enough scale has a smooth bijective map to the Euclidean space $\mathbb{R}^d$. $d$ is called the dimension of the manifold.*

For example, take a small enough region around any point on the surface of a sphere, that region can be flattened to a region of the plane $\mathbb{R}^2$. So the sphere surface is an example of a 2-dimensional manifold.

**Definition 6** *A diffeomorphism is an invertible function that maps one differentiable manifold to another, such that both the function and its inverse are smooth.*

Since we treat congruent frameworks as the same framework, we want to rule out Euclidean transformations. Therefore we take the following procedure: Select any $d$ vertices $v_1, v_2, ..., v_d$ from $V$. Translate and rotate so that $v_1 = (0, 0, ..., 0)$ at the origin, $v_2 = (p_{11}, 0, ..., 0)$ on the first axis, $v_3 = (p_{21}, p_{22}, 0, ..., 0)$, etc. This procedure gives a smooth mapping that makes $d(d+1)/2$ coordinates zero. We call this procedure $q : T \to X$, where $X$ is a $nd - d(d+1)/2$ dimensional space. Let the mapping from $X$ to $Z$ be $\bar{\rho}$. The rank of the Jacobian of $\bar{\rho}(q)$ and $\rho_G$ is the same. We have a new mapping diagram, as in the lower half of Figure 3.

Since $X$ is just a projection of $\mathbb{R}^{nd}$, we have the following Lemma:

**Lemma 4** *$X$ is a smooth manifold of dimension $nd - d(d+1)/2$.*

We use $S(n, d) = nd - d(d+1)/2$ to indicate the dimension of $X$. We know that $S(n, d)$ is also the maximal rank of the rigidity matrix of a graph with $n$ vertices in $\mathbb{R}^d$.

The generic frameworks of $G$ correspond to regular values in differential topology.

**Definition 7** *Suppose we have $g : A \to B$, $A$ and $B$ are smooth manifolds, the largest rank that the Jacobian of $g$ can attain is $k$. $x \in A$ is a regular point if the $\mathbb{J}(g)$ has rank $k$ at $x$, otherwise it is a singular point. $y \in B$ is a regular value if all $x$ in its preimage $g^{-1}(y)$ are regular points, otherwise it is a singular value.*

For our distance function $\rho_G$, $k$ is the number of independent edges, and the singular points are not generic (w.r.t. infinitesimal rigidity).

**Lemma 5** *For generic $p$, $\rho(p)_G$ is a regular value.*

**Proof:** A well known result from differential topology is that the singular value set of $\rho(p)_G$ has $k$-meaure zero. The regular points of $\rho(p)_G$ can be covered by a countable number of open neighbourhoods s.t. rank of $\mathbb{J}(\rho(p)_G)$ is maximal within each neighbourhood. Consider one of these neighbourhoods $A$, let its image under $\rho(p)_G$ be $B$. On this neighbourhood $\rho_G$ is diffeomorphic to a projection from $\mathbb{R}^{nd}$ to $\mathbb{R}^k$. Suppose $B' \subset B$ has $k$-measure zero, its preimage must have $(nd)$-measure zero. Thus for almost every $p$, $\rho(p)_G \notin B'$. ■

Now we can conclude that the realization set of a generic framework is a manifold.

**Lemma 6** *For generic $p$, $\pi_E^{-1}\rho(p)_G$ restricted to $X$ is a $(S(n,d) - k)$-dimensional manifold.*

**Proof:** Generic $p$ maps to regular values of $\rho_G$ and $\bar{\rho}(q)$. $X$ has dimension $S(n,d)$, so by the implicit function theorem, the preimage of a regular value is a submanifold of $X$ of dimension $S(n,d) - k$. ■

This manifold describes the allowed flexing. Also since $G$ is connected, $\pi_E^{-1}\rho(p)_G$ is bounded and closed, so this is a compact manifold.

We also need to show that the flexings remain entirely in $X$.

**Lemma 7** *If $G$ has more than $S(n, d-1)$ independent edges, for generic $p$, $\pi_E^{-1}\rho(p)_G$ stays within $X$.*

**Proof:** Assume $\pi_E^{-1}\rho(p)_G$ has some point outside $X$. By definition of $X$ and $T$, such a point has all vertices lie in a $(d-1)$-dimensional hyperplane. The infinitesimal motions are within this hyperplane, so the rank of the rigidity matrix can be no larger than $S(n,(d-1))$. This means that $\rho(p)_G$ is a singular value, which is a contradiction by Lemma 5. ■

**Theorem 8** *If $G$ is connected, flexible with more than $d+1$ vertices, then for generic $p$, $\pi_E^{-1}\rho(p)_G$ contains a compact 1-dimensional submanifold (contains more than one point).*

**Proof:** Add edges to $G$ until there are more than $S(n,d) - 1$ independent edges, and call the resultant graph $G'$. $\pi_E^{-1}\rho(p)'_G \subset \pi_E^{-1}\rho(p)_G$. By Lemma 6 and 7, $\pi_E^{-1}\rho(p)'_G$ is a compact 1-dimensional manifold. ■

**Theorem 9** *If $G$ with more than $d+1$ vertices is not redundantly rigid in $\mathbb{R}^d$, then $G$ is not globally rigid in $\mathbb{R}^d$.*

**Proof:** W.l.o.g. assume $G$ is rigid in $\mathbb{R}^d$. $G$ has $S(n,d)$ independent edges, and $\exists e$ s.t. $G - e$ is flexible. By Theorem 8, for generic $p$, $\pi_E^{-1}\rho(p)_G$ contains a 1-dimensional compact submanifold $C$, which is diffeomorphic to a circle. $||e||$ changes continuously during flexing on $C$. Since $G$ is rigid and $p$ is generic, $x = \rho(p)_G$ is not a critical point in $C$. Therefore there must exists another

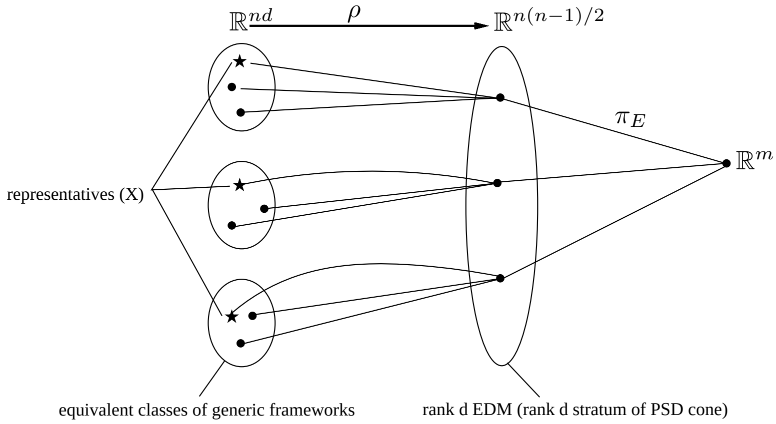


**Figure 4**:

point $y = \rho(q)$ in $C$ s.t. $||e||_x = ||e||_y$. So $p(G)$ and $q(G)$ are two equivalent but non-congruent realizations. ■

**Remark** The mapping $q$ picks a representative from each equivalent class of congruent frameworks in $\mathbb{R}^{nd}$. $\rho$ maps $\mathbb{R}^{nd}$ into the P.S.D. matrices of rank $d$. The image is the rank $d$ stratum of the convex P.S.D. cone. See Figure 4.

$\pi_E$ project the cone onto the edge set. By taking the preimage $\pi_E^{-1}$ of $\rho(p)_G$, we obtain a section of the cone. See Figure 5.

For regular frameworks, the section $\pi_E^{-1}\rho(p)_G$ is a compact manifold with one or more dimensions, and contains one or more equivalent realizations of $p$.

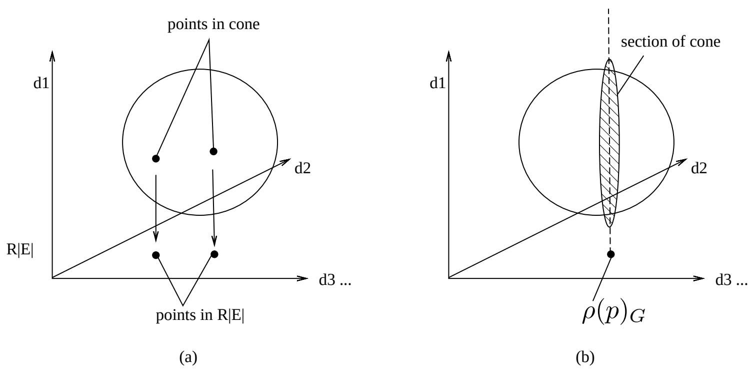


**Figure 5**: (a) the mapping $\pi_E$ from cone to $R^{|E|}$; (b) the mapping $\pi_E^{-1}\rho(p)_G$ yields a section of the cone

# 3 Sufficient condition for global rigidity (in 2D)

Although the necessity of the two conditions in Theorem 1 holds for any dimension, the sufficiency holds only for $d = 2$. One counter-example is $K_{5,5}$ for $d = 3$ [8].

**Theorem 10** *A graph $G$ is globally rigid in $\mathbb{R}^2$ if $G$ is 3-connected and redundantly rigid. [2].*

The proof of Theorem 10 is based on the rigidity matroid of the graph. We will first introduce several basic definitions of the rigidity matroid.

## 3.1 Rigidity matroid and M-connected

**Definition 8** *For a graph $G = (V, E)$, a edge set $F \subset E$ is independent in $\mathbb{R}^d$ if for generic framework $p$, the rows in the rigidity matrix $R(p)$ corresponding to edges in $F$ are linearly independent.*

In 2D, $F$ is independent if it satisfies the first part of Laman's condition.

**Definition 9** *For a graph $G = (V, E)$, the $d$-dimensional rigidity matroid $\mathcal{M}(G) = (E, \mathcal{I})$ is defined on the edge set $E$ with*

$$\mathcal{I} = \{F \subseteq E : F\, independent\, in\, G\, in\, \mathbb{R}^d\}$$

*For any $E' \subseteq E$, the maximal independent subsets of $E'$ are called the bases of $\mathcal{M}(G(E'))$.*

$\mathcal{M}(G)$ satisfies the following three axioms of matroid:

(M1) $\emptyset \in \mathcal{I}$,

(M2) if $D \subset F \in \mathcal{I}$ then $D \in \mathcal{I}$,

(M3) $\forall E' \subseteq E$, all bases of $E'$ have the same cardinality (the rank $r$).

In 2D, $G = (V, E)$ is rigid iff $r(E) = 2|V| - 3$ in $\mathcal{M}(G)$.

A M-circuit is a dependent graph with the minimum number of edges.

**Definition 10** *Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an M-circuit in $G$ if $C$ is dependent and $\forall e \in C$, $C - e$ is independent. In particular, $G$ itself is an M-circuit if $E$ is a circuit.*

**Observation 11** *The following three statements are equivalent in 2D:*

*(a) $G$ is an M-circuit.*

*(b) $|E| = 2|V| - 2$ 2and $G - e$ is minimally rigid $\forall e \in E$.*

*(c) $|E| = 2|V| - 2$ and $E(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$.*

*However for 3D, we cannot determine M-circuits by counting the edges. For example, double-banana (see Figure 6) is a M-circuit, but has $|E| = 3|V| - 6$, not $3|V| - 5$.*

$K_4$ is the minimum M-circuit in 2D.

**Definition 11** *Given a graph $G = (V, E)$ with rigidity matroid $\mathcal{M}$, for $e \neq f \in E$, if $\exists circuit C \in \mathcal{M}$ s.t. $e, f \in C$, we say that $e, f$ are in the same M-component of $\mathcal{M}$. If $\mathcal{M}(G)$ has only one component then $G$ is said to be M-connected.*

The proof of Theorem 10 uses two types of constructions: edge-addition and 1-extension.

**Definition 12** *Given a graph $G = (V, E)$, edge addition is adding a new edge $(u, v)$ where $(u, v) \notin E$. In $\mathbb{R}^d$, pick a vertex-set $U \in V$ s.t. $|U| = d + 1$, $\exists u, v \in U$ s.t. $(u, v) \in E$. 1-extension is removing the edge $(u, v)$, adding a new vertex $z$ and connecting $v$ to every vertex in $U$.*

1-extension in 2D is the same as Henneberg-II construction.

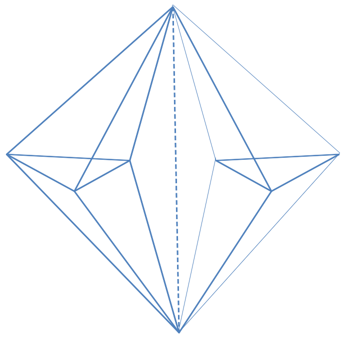

**Figure 6**: Double-banana

## 3.2 Outline of the proof

We use M-connected instead of redundantly rigid in this proof. We have the following main steps:

1. In Section 3.3, Lemma 12 and Theorem 14 show that when $G$ is 3-connected, M-connected is equivalent to redundantly rigid in 2D.
2. In Section 3.4, Theorem 3.4 shows that edge-addition and 1-extension preserves global rigidity.
3. In Section 3.5, Theorem 16 shows that in 2D, a 3-connected and M-connected graph $G$ can be constructed from $K_4$ using edge-addition and 1-extension.

Combining the three results gives us a proof for Theorem 10.

## 3.3 M-connected graphs

We call the maximal rigid subgraphs of $G$ its rigid components. Note that in 2D, every M-component of $G$ lies entirely inside one of its rigid components.

**Lemma 12** *If $G$ is M-connected, then $G$ is redundantly rigid in 2D.*

**Proof:** First notice that $G$ is rigid, since otherwise $G$ has at least two rigid components and hence at least two M-components. Since $G$ is M-connected, $\forall e \in E$ is contained in a circuit of $\mathcal{M}(G)$. Thus $G$ is redundantly rigid. ■

**Observation 13** *A counter-example of Lemma 12 in 3D is the double-banana (see Figure 6). It is a M-circuit, thus M-connected. However, it is clearly flexible.*

**Definition 13** *A graph $G$ is nearly 3-connected if $G$ can be made 3-connected by adding at most one new edge.*

**Theorem 14** *If $G$ is nearly $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$, Then $G$ is M-connected.*

**Proof:** For contradiction, suppose $G$ is not M-connected. Let $H_1, H_2, ..., H_q$ be the M-components of $G$, $q \geq 2$. Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices $H_i$ does not share with other M-components, and $Y_i = V(H_i) - X_i$ be the shared vertices of $H_i$. Let $n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$.

Clearly , $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^{q} x_i + |\cup_{i=1}^{q} Y_i|$. More over, since each vertex in $Y_i$ is shared by at least two components, we have $\sum_{i=1}^{q} y_i \geq 2|\cup_{i=1}^{q} Y_i|$.

Since $G$ is redundantly rigid, every edge of $G$ is in some M-circuit, so each $H_i$ contains a M-circuit. Since the minimum M-circuit in $\mathbb{R}^d$ has $d+1$ vertices, we have $n_i \geq d+2$. Since G is nearly $(d+1)$-connected, $n_i \geq d+2$, we have $y_i \geq d$ for all $H_i$, and $y_i \geq d+1$ for all but at most two $H_i$. So $\sum_{i=1}^{q} y_i \geq (d+1)q - 2$.

Choose a base $B_i$ in each rigidity matroid $\mathcal{M}(H_i)$. We have $\cup_{i=1}^{q}|B_i| = \sum_{i=1}^{q} B_i$, since $H_i$ are separate M-components. $\sum_{i=1}^{q} B_i = \sum_{i=1}^{q}(dn_i - \binom{d+1}{2})$ since each $H_i$ is rigid. Then

$$
\begin{aligned}
\sum_{i=1}^{q}(dn_i - \binom{d+1}{2}) &= d\sum_{i=1}^{q} n_i - \binom{d+1}{2}q \\
&= d\sum_{i=1}^{q} x_i + \frac{d}{2}\sum_{i=1}^{q} y_i + \frac{d}{2}\sum_{i=1}^{q} y_i - \frac{d(d+1)}{2}q \\
&\geq (d\sum_{i=1}^{q} x_i + d|\cup_{i=1}^{q} Y_i|) + \frac{d(d+1)}{2}q - d - \frac{d(d+1)}{2}q \\
&= d|V| - d \\
&> d|V| - \binom{d+1}{2}
\end{aligned}
$$

Since rank of $\mathcal{M}(G)$ is $d|V| - \binom{d+1}{2}$, this implies that $\cup_{i=1}^{q}|Bi|$ contains a circuit, contradiction. Therefore $G$ is M-connected.

■

## 3.4 1-extension and edge-addition

**Theorem 15** *If $H$ is globally rigid in 2D with $|V(H)| \geq 4$ and $G$ is obtained from $H$ by an edge-addition or a 1-extension, Then $G$ is globally rigid in 2D.*

Edge-addition obviously preserves global rigidity in any dimension. We will show the proof of 1-extension for 2D.

Let $(u,w)$ be the edge we subdivided in the 1-extension, and let the vertex added be $v$, where $v$ is adjacent to $\{u,w,t\}$. We want to prove that given a generic $p(G)$, $\forall q$ s.t. $\rho(q)_G = \rho(p)_G$, we must have $q = p$. Notice that since $H = G - v + (u,w)$ is globally rigid, we only need to prove that $\rho(p(u,w)) = \rho(q(u,w))$.

**Proof:** We first translate and rotate $p$ and $q$ s.t. $p = (0,0,0,p_4,p_5,...,p_{2n})$, $p(u) = (0,0)$, $p(w) = (0,p_4)$, $p(t) = (p_5,p_6)$, $p(v) = (p_{2n-1},p_{2n})$. By genericity, $\{p_4,p_5,...,p_{2n}\}$ is algebraically independent over $\mathbb{Q}$. Similarly $q(u) = (0,0)$, $q(w) = (0,q_4)$, $q(t) = (q_5,q_6)$, and $q(v) = (q_{2n-1},q_{2n})$.

To prove that $\rho(p(u,w)) = \rho(q(u,w))$, we only need to show that $p_4^2 - q_4^2 = 0$. By symmetry we may assume that $p_4^2 - q_4^2 \geq 0$.

From $\rho(p(u,v)) = \rho(q(u,v))$, $\rho(p(v,w)) = \rho(q(v,w))$, $\rho(p(v,t)) = \rho(q(v,t))$, we have:

$$
\begin{aligned}
q_{2n-1}^2 + q_{2n}^2 &= p_{2n-1}^2 + p_{2n}^2 && (1)\\
(q_{2n-1} - q_4)^2 + q_{2n}^2 &= (p_{2n-1} - p_4)^2 + p_{2n}^2 && (2)\\
(q_{2n-1} - q_5)^2 + (q_{2n} - q_6)^2 &= (p_{2n-1} - p_5)^2 + (p_{2n} - p_6)^2 && (3)
\end{aligned}
$$

Using these equations we can substitute $q_{2n-1}$ and $q_{2n}$ to obtain a polynomial of $p_{2n-1}$ and $p_{2n}$:

$$f = a_{11}p_{2n-1}^2 + a_{22}p_{2n}^2 + a_{12}p_{2n}p_{2n-1} + a_1 p_{2n-1} + a_2 p_{2n} + a_0 = 0,$$

where the coefficients are all in the rational closure $\tilde{K}$ of $\{p_4, p_5, ..., p_{2n}\}$. Since $\{p_4, p_5, ..., p_{2n}\}$ is algebraically independent over $\mathbb{Q}$, $\{p_{2n-1}, p_{2n}\}$ is algebraically independent over $\tilde{K}$. Thus in order to get $f = 0$, all coefficients of $f$ must be zero. In particular,

$$a_{11} = 4q_6^2(p_4^2 - q_4^2) + 4(p_4 q_5 - q_4 p_5)^2 = 0$$

Since $p_4^2 - q_4^2 \geq 0$, we have $p_4^2 - q_4^2 = 0$.
■

**Remark** In fact, 1-extension preserves global rigidity for any dimension [5]. That result requires a more complicated proof.

## 3.5 Construction of 3-connected and M-connected graphs

**Theorem 16** *$G = (V, E)$ is 3-connected and M-connected in 2D iff $G$ can be obtained from $K_4$ by 1- extensions and edge additions.*

The entire proof for Theorem 16 is rather long, based on lots of case analyses. Here we will only show several important results used in the proof.

**Definition 14** *Given $G = (V, E)$, let $V_{d+1} = \{v \in V : d(v) = d + 1\}$. $v \in V_{d+1}$ are called nodes of $G$. $G[V_{d+1}]$ is called the node-subgraph of $G$. A node of $G$ with degree at most one in $G[V_{d+1}]$ is called a leaf node. A node of $G$ with degree exactly two in $G[V_{d+1}]$ is called a series node.*

**Definition 15** *The inverse operation to 1-extension is called splitting. For example, in 2D it chooses a node $v$ in $G$, deletes $v$ and adds a new edge connecting two non-adjacent neighbours of $v$.*

*A splitting on a M-connected graph $G$ is admissible if the resultant graph is still M-connected.*

**Definition 16** *A node $v$ in an M-connected graph is an admissible node if it has an admissible splitting.*

*An edge $e$ in an M-connected graph $G$ is an admissible edge if $G - e$ is M-connected.*

The following theorem shows that an M-circuit must have an admissible node.

**Theorem 17** *Let $G$ be a 3-connected M-circuit in 2D with $|V| > 4$. Then either $G$ has three non-adjacent admissible nodes or $G$ has four admissible nodes. [6]*

**Observation 18** *Extension of Theorem 17 does not hold for 3D. $K_{5,5}$ is a counter-example.*

For the case that $G$ is not a M-circuit, we use a concept called ear-decomposition.

**Definition 17** *For $G = (V, E)$, let $H_1$, $H_2$, ..., $H_t$ be a non-empty sequence of M-circuits of $G$. Let $G_j = H_1 \cup H_2 \cup ... \cup H_j$ for $1 \leq j \leq t$. $H_1$, $H_2$, ..., $H_t$ is an M-ear decomposition of $G$ if $G_t = G$, and for all $2 \leq i \leq t$,*

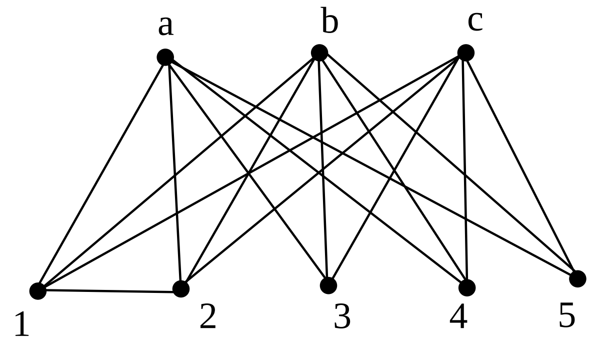


**Figure 7**: M-ear decomposition: $H_1 = G - 1$, $H_2 = G - 2$ and $H_3 = G - \{4, 5\}$

1. *$E(H_i) \cap E(G_{i-1}) \neq \emptyset \neq E(H_i) - E(G_{i-1})$, and*
2. *No M-circuit $H$ of $G_i$ which satisfies 1 has $E(H) - E(G_{i-1})$ properly contained in $E(H_i) - E(G_{i-1})$.*

An example of an 2D M-ear decomposition is given in Figure 7.
The following Lemma is a general result for ear-decomposition:

**Lemma 19** *$G$ is M-connected iff $G$ has an M-ear decomposition (for any dimension). [7]*

**Lemma 20** *Given $G = (V, E)$ M-connected in 2D and $H_1, H_2, ..., H_t$ be the M-ear decomposition of $G$. Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$, $X = V(H_t) - Y$, then:*

1. *Either $Y = \emptyset$ and $|E(H_t) - E(G_{t-1})| = 1$, or $Y \neq \emptyset$ and every $e \in E(H_t) - E(G_{t-1})$ is incident to $Y$.*
2. *$|E(H_t) - E(G_{t-1})| = 2|Y| + 1$.*
3. *If $Y \neq \emptyset$ then $|E(X)| = 2|X| - 3$ in $H_t$ .*
4. *$G[Y]$ is connected.*
5. *If $G$ is 3-connected then $|X| \geq 3$.*

The following theorem shows that 2D 3-connected M-connected $G$ that are not M-circuit either have an admissible node or an admissible edge.

**Theorem 21** *Let $G$ be a 2D 3-connected M-connected graph and $H_1, H_2, ..., H_t$ be an M-ear decomposition of $G$ with $t \geq 2$. Suppose that $G-e$ is not M-connected for all $e \in E(H_t) - \cup_{i=1}^{t-1} E(H_i)$ and for all but at most two edges $e \in E(H_t)$. Then there exists an admissible node $v \in V(H_t) - \cup_{i=1}^{t-1} V(H_i)$.*

**Sketch of Proof:** The proof is by contradiction and case analysis. Let $Y = V(H_t) - \cup_{i=1}^{t-1} V(H_i)$, $X = V(H_t) - Y$, $L = \cup_{i=1}^{t-1} V(H_i)$. By Lemma 20, $Y \neq \emptyset$, every $e \in E(H_t) - E(G_{t-1})$ is incident to $Y$, $|X| \geq 3$.

Choose a node $v$ of $G$ in $Y$ such that $v$ is a leaf node in $G[Y \cap V_3] = H_t[Y \cap V_3]$. We assume the number of neighbours of $v$ in $X$ is 3, 2, 1 or 0, and arrive at a contradiction in each case. ■

Combining Theorem 17 and 21, we know that every 3-connected M-connected graph with more than 4 vertices can be obtained from a smaller M-connected graph by an edge addition or a 1-extension. We still need to show that there exists such a smaller M-connected graph which is also 3-connected.

The proof is by contradiction. Suppose $G$ is a smallest counterexample. If $G' = G - e$ or the splitted graph $G' = G - w + (x, y)$ is not 3-connected, we can choose a 2-separator $\{u, v\}$ of $G'$, and choose the smallest component $F$ of $G' - \{u, v\}$. Let $H$ be the graph $G(F \cup \{u, v\}) + (u, v)$, which is obtained from $F$ by adding the vertices $u$ and $v$, all edges of $G$ between $F$ and $\{u, v\}$, and a new edge $(u, v)$.

Consider all admissible edge-removal and splitting, choose one s.t. we get the smallest $H$. We can show that $H$ is 3-connected and M-connected, and then applying Theorems 17 and 21 to $H$ in each possible case to find an admissible edge or node in $H$, which contradicts the minimality of $H$.

**Problem 1** *The proof of Theorem 16 is long-winded mainly because we have to go through edge-addition and 1-extension. If we can find out more operations that preserve global rigidity, the proof could be simplified.*

*Moreover, one of the reasons for hardness of characterizing rigidity in higher dimension is that the known widgets in generation of rigid graphs are very few. If we can find more operations generating rigid graphs, it may also help us to characterize rigidity in higher dimension.*

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