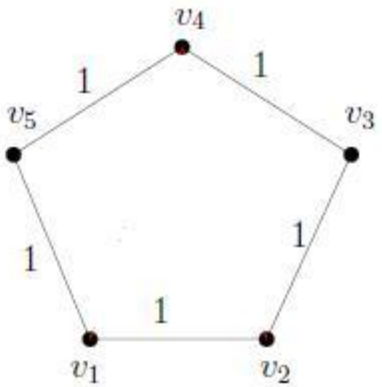
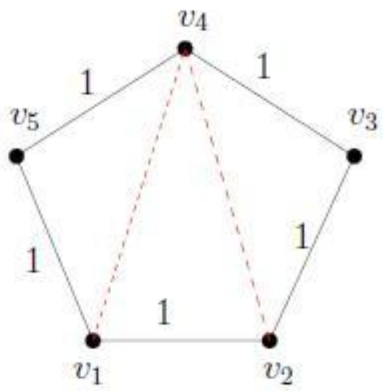
Characterizing graphs with convex and connected Cayley configuration spaces

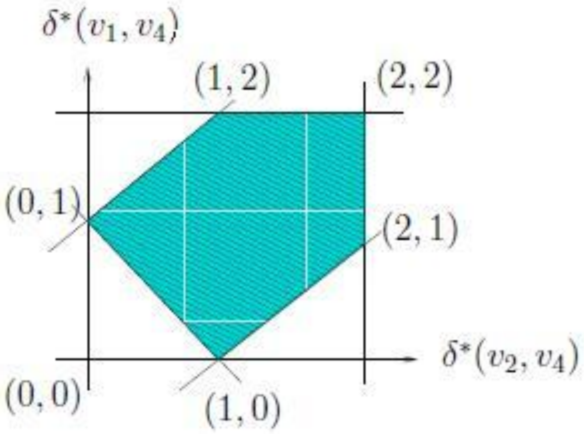
GOAL : efficient representations of realization spaces (EDCS)



- given distance assignments on the edges

- a choice of nonedges
  Cayley parameters



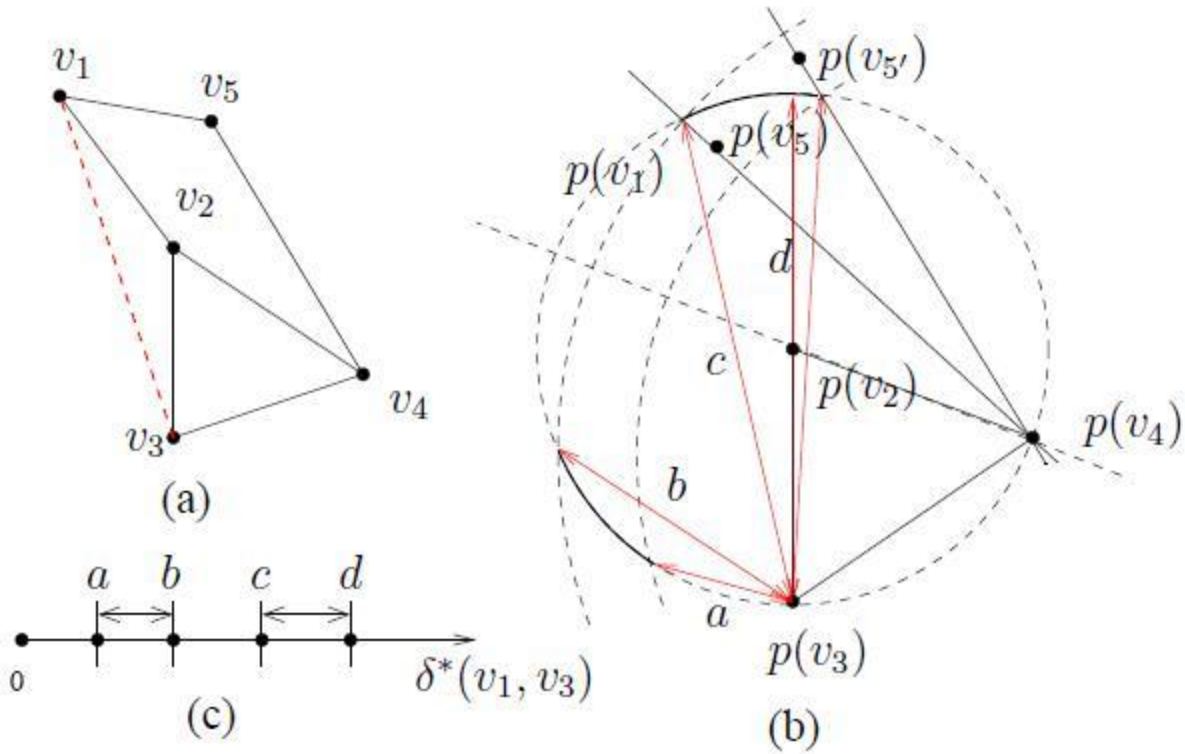


- the set of realizable distance assignments
  => Cayley configuration space.

efficiency is based on;

- **convexity** of the Cayley configuration space
- **connectedness** of the Cayley configuration space
- **sampling complexity**
- **realization complexity**
- **generic completeness**

nonconvex, disconnected example



(a) parameter is chosen to be the dashed non-edge

(b) the realization $p(v_1)$ can lie in either of the two solid arc segments

(c) disconnected 2D Cayley configuration space on the non-edge (v_1, v_3)

*Euclidean Distance Constraint System (**EDCS**) (G, δ) :*

graph G = (V,E) + an assignment of distances δ(e)

d-dimensional *realization* :

the assignment p

points in R^d => vertices in V s.t.

distance equality constraints are satisfied

$\delta(u, v) = || p(u) - p(v) ||$

F: a choice of non-edges of G

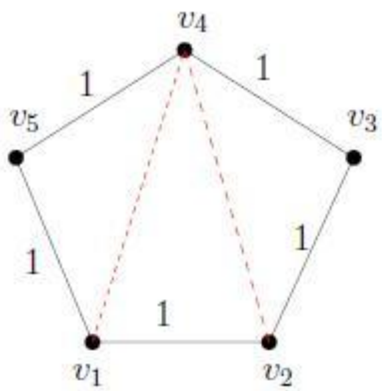
$\boldsymbol{\delta}^*$(f): distance value that the non-edge f ∈ F can take

$\boldsymbol{\delta}^*$(F): δ^*(f) tuples

augmented EDCS: (G∪F, δ(E), δ^*(F))

$\boldsymbol{\phi^d_F}$(G, $\boldsymbol{\delta}$): set of δ^*(F) s.t. *augmented EDCS* has a realization in R^d

$\boldsymbol{\phi^d_F}$(G, $\boldsymbol{\delta}$): Cayley configuration space



Using the triangle (v1,v4,v5) inequalities:

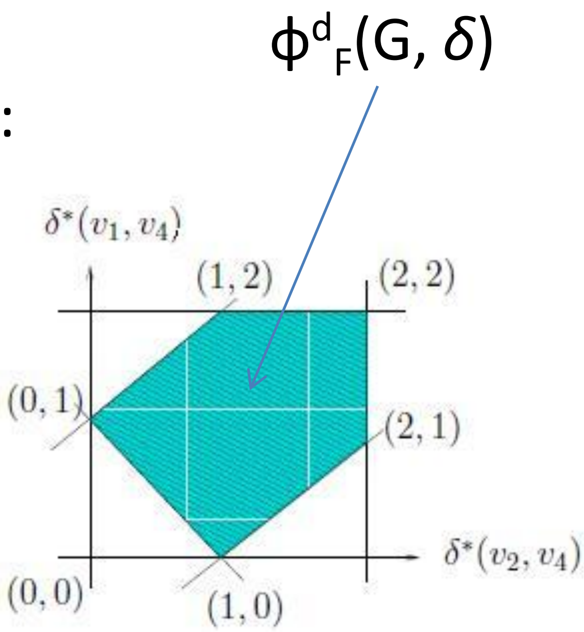
$$0 < \delta^*(f_{1\text{-}4}) < 2$$

Using the triangle (v2,v3,v4) inequalities:

$$0 < \delta^*(f_{2\text{-}4}) < 2$$

In order to let *augmented EDCS* has a realization; triangle (v1,v2,v4) inequalities should be satisfied:

$$\delta^*(f_{1\text{-}4}) + \delta^*(f_{2\text{-}4}) > 1$$
$$\delta^*(f_{1\text{-}4}) - \delta^*(f_{2\text{-}4}) < 1$$
$$\delta^*(f_{2\text{-}4}) - \delta^*(f_{1\text{-}4}) < 1$$



generic completeness:

- GUF is rigid :

each configuration => at most **finitely** many Cartesian realizations

- GUF is *not overconstrained :*

full measure, at most as many parameters as DOF of G.

=> G ∪F is wellconstrained i.e., *minimally rigid.*

sampling complexity:

- computing the set of non-edges F
- the description of the Cayley configuration space $\phi^d{}_F(G, \delta)$
  the coefficients in the *polynomial inequalities*
- the descriptive algebraic complexity
  number, terms, degree etc of the *polynomial inequalities*

---

sampling = walking through configurations in $\phi^d{}_F(G, \delta)$

$\phi^d{}_F(G, \delta)$ as a semi-algebraic set

polynomial inequalities to describe this semi-algebraic set

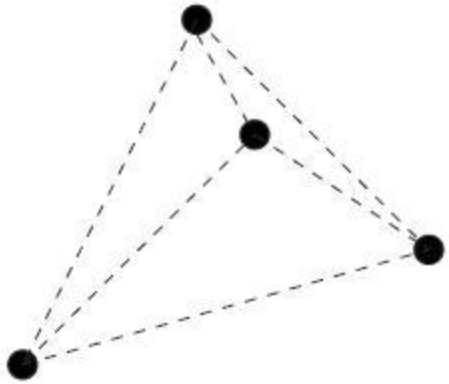
realization complexity:

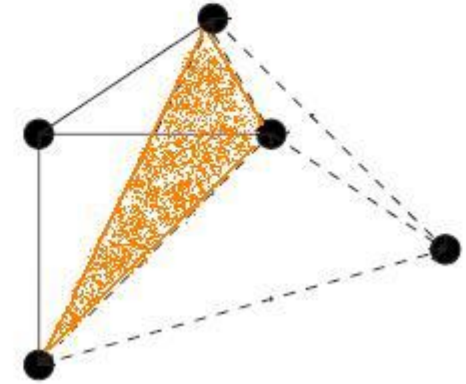
efficient map:
parametrized sample configuration => Cartesian realizations

graph-theoretic, forbidden minor characterizations for 2D and 3D EDCS that capture :

- the **class of graphs** that always admit efficient Cayley configuration spaces

- the possible choices of representation **parameters** that yield efficient Cayley configuration spaces for a given graph.

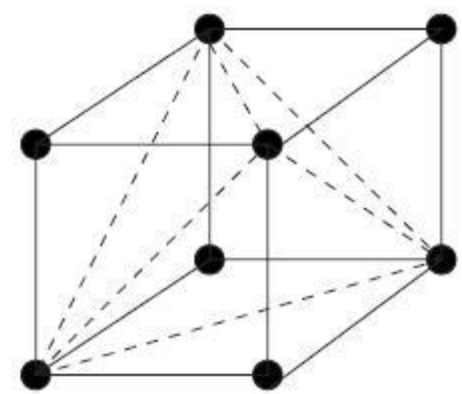


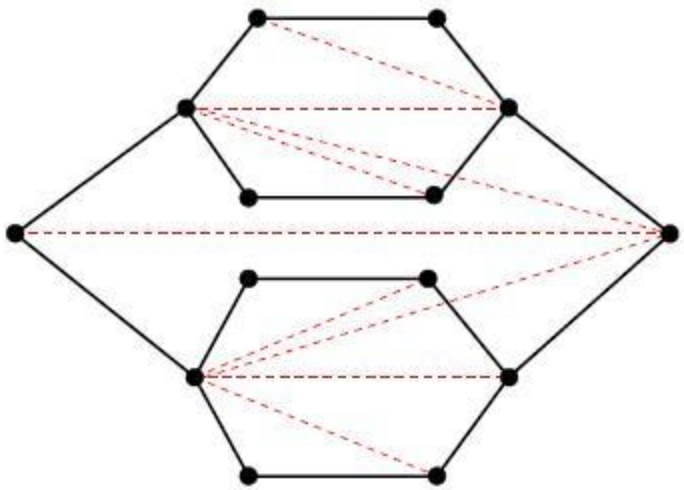


- Let G_1 and G_2 be two graphs, both containing a K_m as a sub-graph. The **m-sum** of G_1 and G_2 is the graph obtained by identifying the two K_m's.

- A graph is **m-tree** if it can be obtained through a sequence of m-sums of K_{m+1}'s.

- A graph is a **partial m-tree** if it is a sub graph of a m-tree.



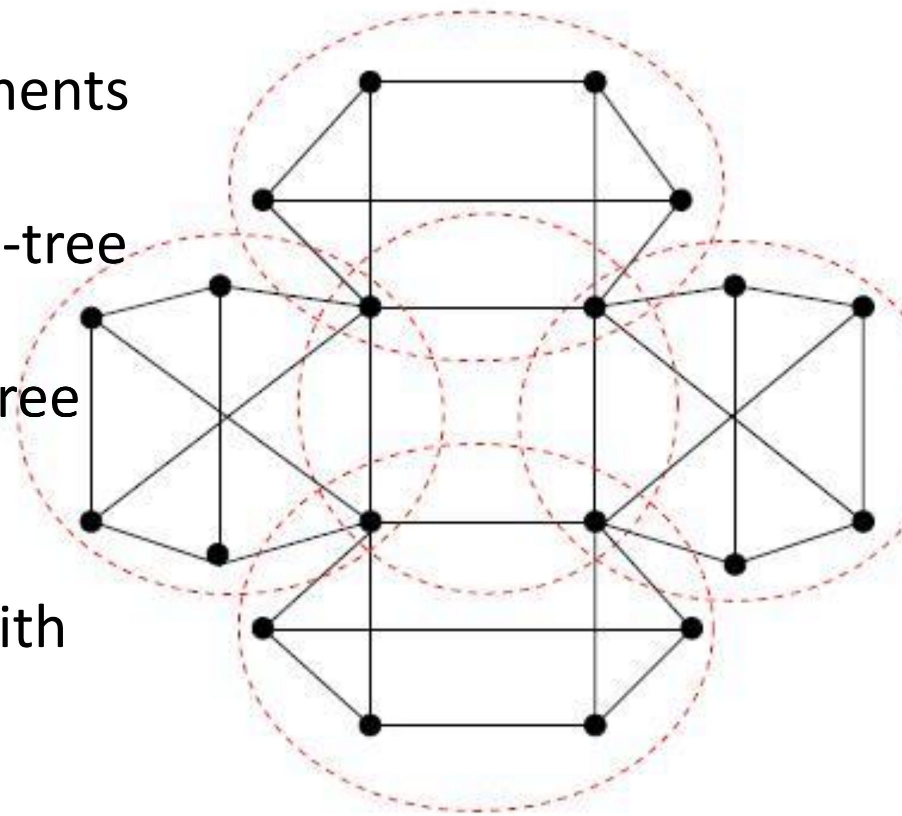


solid edges is an underconstrained partial 2-tree
with dashed edges is a 2-tree.

A 2-sum of five minimal 2-sum components

Middle 2-sum component is a partial 2-tree

but the entire graph is not a partial 2-tree



The union of the middle component with any other component is also a 2-sum component but not minimal.

Theorem 5.10

(a) *For a graph G = (V,E), the following four statements are equivalent:*

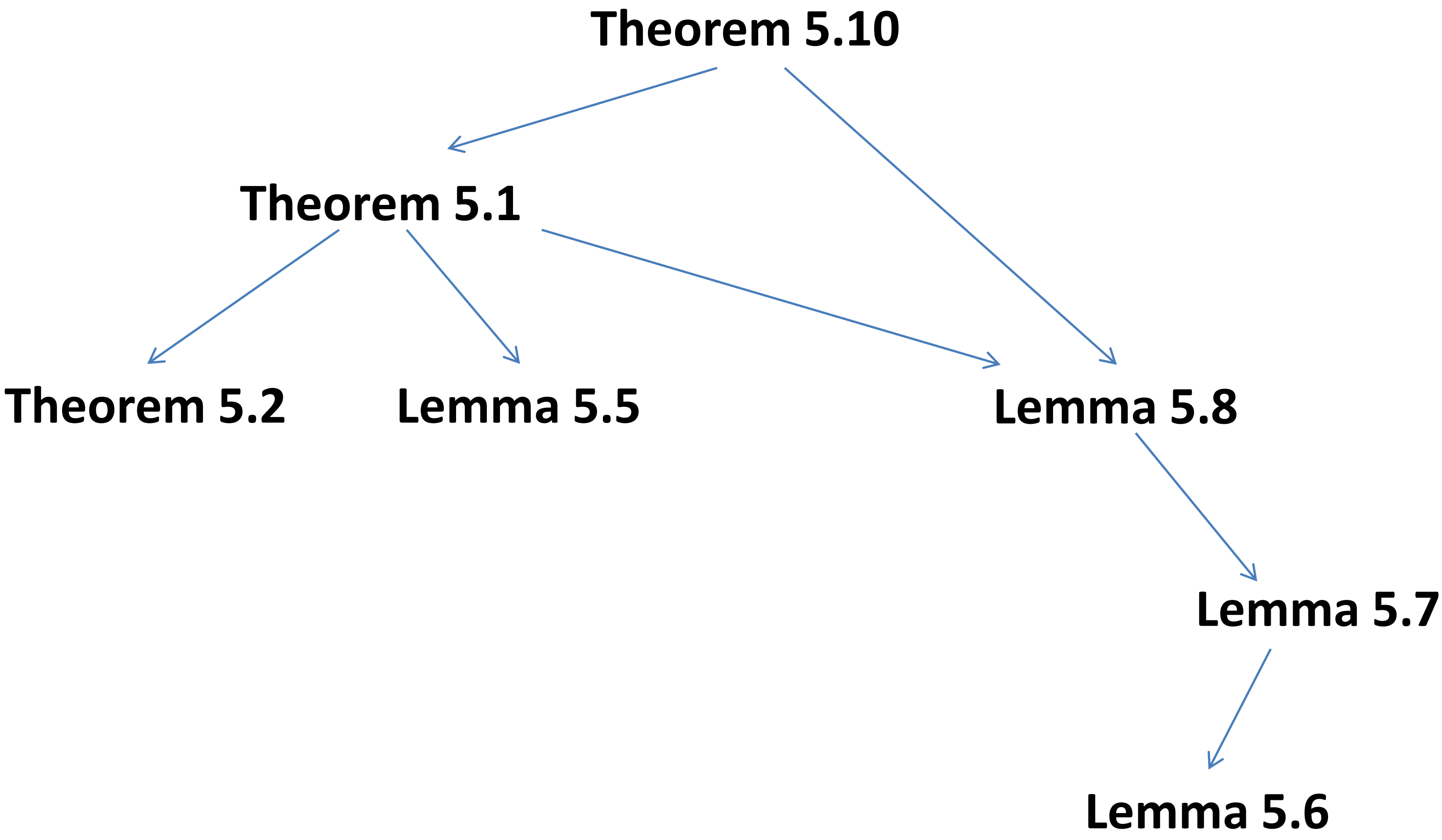
1. *∃ a non-empty set of F s.t. for all δ, $\phi^2_F(G, \delta_E)$ is* ***connected****;*
2. *∃ a non-empty set of F s.t. for all δ, $\phi^2_F(G, \delta_E)$ is* ***convex****;*
3. *∃ a non-empty set of F s.t. for all δ, $\phi^2_F(G, \delta_E)$ is a* ***linear polytope****.*
4. *G has a 2-sum component that is an underconstrained* ***partial 2-tree****.*

(b) *An underconstrained graph G always admits a generically complete linear polytope, connected or convex Cayley configuration space if and only if every underconstrained 2-sum component of G is a partial 2-tree.*

Theorem 5.11

Given a graph $G = (V,E)$ and non-empty set of non-edges F, the 2D Cayley configuration space $\phi^2_F(G, \delta_E)$ is a linear polytope, connected or convex for all δ if and only if all the minimal 2-Sum components of $G \cup F$ ***containing any subset of F*** *are partial 2-Trees.*

Furthermore, the Cayley configuration space is ***generically complete*** *if and only if all the underconstrained minimal 2-sum components of G are partial 2-trees and all the minimal 2-sum components of $G \cup F$ containing F are* ***2-trees****.*


Theorem 5.10
Theorem 5.1
Theorem 5.2
Lemma 5.5
Lemma 5.8
Lemma 5.7
Lemma 5.6

Theorem 5.1

Given a graph $G = (V,E)$ and a non-edge f, the Cayley configuration space $\phi^2_F(G, \delta_E)$ is a ***single interval*** *for all δ if and only if all the minimal 2-sum components of $G \cup f$ that contain f are* ***partial 2-trees****.*

proof of Theorem 5.1
=> direction

A : $\phi^2{}_F(G, \delta_E)$ *is a* ***single interval*** *for all* δ

B : ***all*** *the minimal 2-sum components of* $G \cup f$ *that contain* f *are* ***partial 2-trees***.

C : *G can be* ***reduced*** *to the base cases by edge contractions*

Theorem 5.2 : IFF $\overline{B}$, THEN C

Lemma 5.5 : IF C, THEN $\bar{A}$

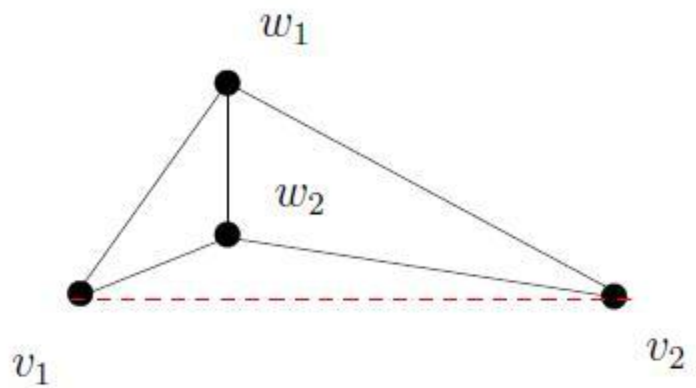
=> IF $\overline{B}$, THEN $\bar{A}$

=> IF A, THEN B

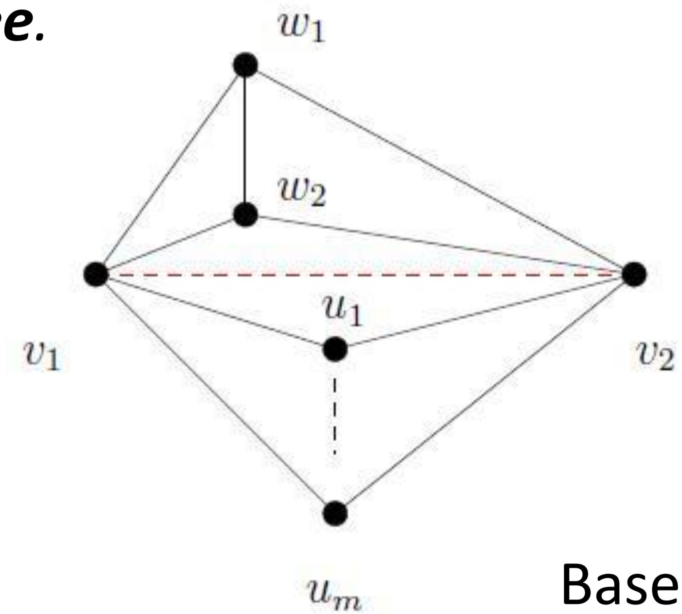
$\Rightarrow$ *direction of Theorem 5.1*

Theorem 5.2

*Given graph G = (V,E) and a non-edge f, G can be **reduced** to the base cases below by a sequence of edge contractions (no edge removals) if and only if there **exists** a minimal 2-Sum component of G∪f containing f that is **not a partial 2-tree**.*



Base Case 1



Base Case 2

Lemma 5.5

*In both base cases, there exists δ s.t. $\phi^2_F(G, \delta)$ is **not connected.***

B : ***all*** *the minimal 2-sum components of $G \cup f$ that contain f are* ***partial 2-trees****.*

C : *G can be* ***reduced*** *to the base cases by edge contractions*

=>*direction of Theorem 5.2* : IF C, THEN $\overline{\text{B}}$

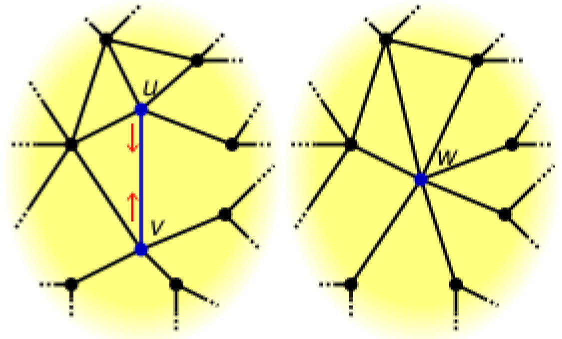
= IF B, THEN $\overline{\text{C}}$

Fact : partial 2-trees do not have K_4 minors

Observation : K_4 exists as a minor in both base cases.

minor is a graph that can be obtained by zero or more edge contractions on a subgraph of *G*.

edge contraction



B : ***all*** *the minimal 2-sum components of $G \cup f$ that contain f are* ***partial 2-trees***.

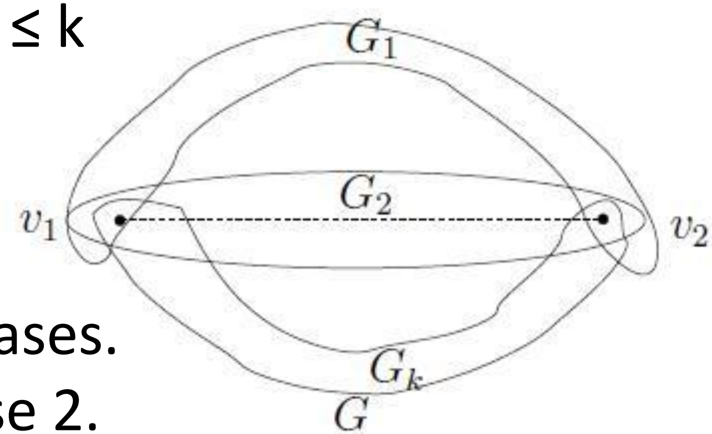
C : *G can be* ***reduced*** *to the base cases by edge contractions*

<=*direction of Theorem 5.2* : IF $\overline{\text{B}}$, THEN C

Proof by induction:

- The statement is true for the 2 base cases.
- Assume the statement is true for $|V| \leq n - 1$; prove it for $|V| = n$.

- $G_i \cup f$ is a 2-sum component of $G \cup f$, $1 \leq i \leq k$
- assume $G_1 \cup f$ is not a partial 2-tree



- **Case k > 1:**

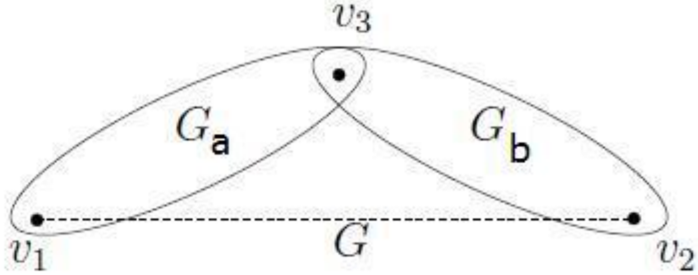
by induction $G_1 \cup f$ is reducible to any base cases.
then contract rest G_i s.t. G falls into base case 2.

Case k = 1: - If $G_1 \cup f$ is not $G \cup f$, follows induction.
- If $G_1 \cup f$ is $G \cup f$;

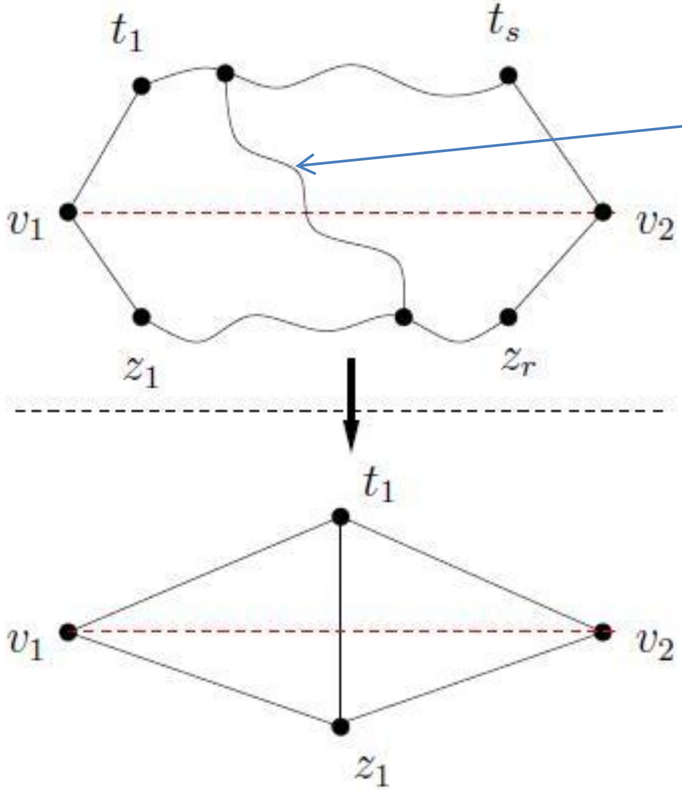
l = the maximum number of disjoint paths between v1 and v2

[Subcase l ≤ 1]:

- say $G_a \cup$ (v1, v3) is not a partial 2-tree
-follow induction



[Subcase l ≥ 2]:



this path has to exist since k=1

contract the edges
s.t. it converts the graph below
while preserving the non-edge

Lemma 5.6

If $G = (V,E)$ is the 2-sum of $G_i = (V_i, E_i)$, then for any δ, (G, δ) has a realization if and only if each (G_i, δ), (δ restricted to the edges in G_i) has a realization.

Lemma 5.7

Take a graph $G = (V,E)$ with 2-sum components $G_i = (V_i, E_i)$, and a non-edge set F that is entirely contained in an arbitrary one of the $G_i = (V_i, E_i)$, say G_1. Then for any δ,

either $\phi^2_F(G, \delta) = \phi^2_F(G_1, \delta)$ *if all the* $\phi^2_F(G_i, \delta)$*'s are non-empty*

or $\phi^2_F(G, \delta)$ *is empty.*

Lemma 5.8

(a) *If a graph G = (V,E) has a 2-sum component G′ = (V′,E′) that is an underconstrained partial 2-tree,*
then there exists a nonempty non-edge set F entirely in G′ *such that for any δ,* $\boldsymbol{\phi^2_F(G, \delta)}$ ***is a linear polytope****.*
Moreover, there is such a set F such that $\phi^2_F(G', \delta)$ *is generically complete for G′.*

(b) *If a graph G = (V,E) is an underconstrained partial 2-tree,*
then for any nonempty nonedge set F′ that preserves (V, E∪F′) as a partial 2-tree, and for all δ, $\boldsymbol{\phi^2_{F'}(G, \delta)}$ ***is a linear polytope****.*

(a)

- 2-tree is minimally rigid by Laman

find such an F that completes "partial 2-tree" to "2-tree"

- 2-tree can be written as the 2-sum of triangles

thus $\phi^2{}_F(G', \delta)$ is a *linear polytope.*
by Lemma 5.7, $\phi^2{}_F(G, \delta)$ is also *linear polytope.*

(b)

- take any subset of F′ of such an F
- $\phi^2{}_{F'}(G, \delta)$ is the projection of $\phi^2{}_F(G, \delta)$ on F′

A graph H always admits ***universally inherent*** *connected* or convex Cayley configuration spaces, if ***for every partition*** *of the edges of H into $E \cup F$, the* graph G = (V,E) always admits a connected or convex Cayley configuration space on F.

Fact: The class of 2-realizable graphs is exactly the partial 2-trees.

A n×n matrix M is a ***Euclidean square distance matrix (EDM)*** *if $\exists\, p_1, \ldots, p_n \in R^d$ for some d* such that $||p_i - p_j||^2 = M(i, j)$.

The set of all EDM's is a convex cone. (Let this cone be **EDMC**)

$$\text{EDMC} = \cup_d\ \phi^d_{V\times V}(\, G(V, \emptyset), \delta_\emptyset\,) \;=\; \cup_d\ \phi^d_{V\times V}(\,\emptyset\,) \qquad |V| = n$$

A graph is ***d-realizable*** if for every configuration of its vertices in E^N, there exist another corresponding configuration in E^d with the **same edge lengths**.

In other words; a graph is ***d-realizable*** if it satisfies the following:
For any δ,
$U_d\ (G(V, E), \delta)$ has a realization => $(G(V, E), \delta)$ has a realization in E^d

Hence ***d-realizable*** not only says:
For any δ,
$U_d\ \phi^d{}_{H/E}(G(V, E), \delta)$ is nonempty => $\phi^d{}_{H/E}(G(V, E), \delta)$ is nonempty

but also says:
For any δ, $U_d\ \phi^d{}_{H/E}(G(V, E), \delta) = \phi^d{}_{H/E}(G(V, E), \delta)$

Theorem 5.16 (updated for any d)

The following are equivalent for a graph H.

1. H is d-realizable.

2. H always admits universally inherent, connected, d-dimensional Cayley configuration spaces.

3. H always admits universally inherent, convex, d-dimensional squared Cayley configuration spaces.

Proof:

Lemma 5.14 proves (3)⇒(2)

Theorem 5.15 proves (1)⇒(3)

proof of Theorem 5.16
(3)⇒(2)

Lemma 5.14

If a graph always admits universally inherent, ***convex****, d-dimensional* ***squared*** *Cayley configuration spaces, then it also always admits universally inherent,* ***connected****, d-dimensional configuration spaces.*

Proof:

- the map $\phi^d_F(G, \delta) \rightarrow (\phi^d_F)^2(G, \delta)$ is continuous
- the inverse map is well-defined and continuous over the positive reals.
- the convexity of $(\phi^d_F)^2(G, \delta)$ implies its connectedness
- continuity of inverse map implies the connectedness of $\phi^d_F(G, \delta)$

proof of Theorem 5.16
(1)⇒(3)

Theorem 5.15

If a graph H is d-realizable, it admits universally inherent, connected (resp. convex), d-*dimensional (resp. squared) Cayley configuration spaces.*

Proof:

Since H is d-realizable then; for any δ, $\phi^d_H(\emptyset) = U_d\ \phi^d_H(\emptyset)$

$U_d\ \phi^d_H(\emptyset)$ is convex since EDMC is convex and its projection on H is convex

For any partition of H into EUF, take the section of this projection, obtained by fixing δ^* to be δ over E:

$$\phi^d_F(G(V, E), \delta) = U_d\ \phi^d_F(G(V, E), \delta)$$

$U_d\ \phi^d_H(\emptyset)$ is convex => $U_d\ \phi^d_F(G(V, E), \delta)$ is convex
since convexity is preserved by sections and projections.

Hence the Cayley configuration space $\phi^d_F(G(V, E), \delta)$ is also convex.

Theorem

proof of Theorem 5.16
(2)⇒(1) for any d

If a graph H always admits universally inherent, connected, d-dimensional Cayley configuration spaces, then it is d-realizable.

Proof:

$\phi^d_{VxV}(\emptyset)$ is rank-d stratum of EDMC

Fact: The convex hull of d-rank stratum of EDMC contains the convex hull of 1-rank stratum of EDMC.

$$\mathit{convexHull}(\phi^d_{VxV}(\emptyset)) \text{ contains } \mathit{convexHull}(\phi^1_{VxV}(\emptyset))$$

The convex hull of 1-rank stratum of EDMC is equal to the EDMC.

$$\mathit{convexHull}(\phi^1_{VxV}(\emptyset)) = \cup_d \phi^d_{VxV}(\emptyset) = \text{EDMC}$$

This is because for EDM D, we have a realization $X = (x_1, x_2, x_3, \ldots, x_n)$, $x_i \in R^n$ and we know $D_{ij} = ||x_i - x_j||^2$.

Project the realization onto 1-dimensional space, we can get n realizations $X^1, X^2, \ldots, X^n$ where X^i is the projection of X to i_{th} axis.

Theorem

proof of Theorem 5.16
(2)⇒(1) for any d

If a graph H always admits universally inherent, connected, d-dimensional Cayley configuration spaces, then it is d-realizable.

Proof continuous:

Let D^i be the corresponding EDM for X^i.

It is easy to see that $D_{ij} = ||x_i - x_j||^2 = \sum_k D_{ij}^k$. Hence $D = \sum_k D^k$.

Fact: The convex hull of d-rank stratum of EDMC is contained in the EDMC.

$$\mathit{convexHull}(\, \phi^d_{VxV}(\emptyset)\,) \text{ is contained in } \cup_d \ \phi^d_{VxV}(\emptyset) = \text{EDMC}$$

Thus the convex hull of d-rank stratum of EDMC is equal to the EDMC.

$$\mathit{convexHull}(\, \phi^d_{VxV}(\emptyset)\,) = \cup_d \ \phi^d_{VxV}(\emptyset) = \text{EDMC}$$

We can project those two cones on the graph G, and we have the projection of the convex hull of d-rank stratum of EDMC on G is equal to the projection of the EDMC on G.

$$\mathit{projection}_{on\ G}(\ \mathit{convexHull}(\, \phi^d_{VxV}(\emptyset)\,)\ ) = \cup_d \ \phi^d_{H}(\emptyset)$$

Theorem

If a graph H always admits universally inherent, connected, d-dimensional Cayley configuration spaces, then it is d-realizable.

Proof continuous:

Since the convex hull of the projection is equal to the projection of the convex hull, we have the convex hull of the projection of d-rank stratum of EDMC on G is equal to the projection of EDMC on G.

$$\mathit{convexHull}(\,\phi^d_H(\,\emptyset\,)\,) = \cup_d\ \phi^d_H(\,\emptyset\,)$$

Since we know the projection of d-rank stratum of EDMC on G is convex, the convex hull of the projection of d-rank stratum of EDMC on G is equal to the projection of d-rank stratum of EDMC on G.

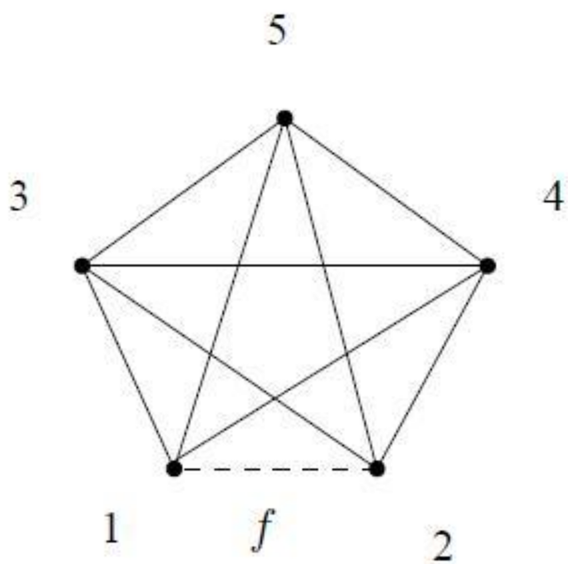
since $\phi^d_H(\,\emptyset\,)$ is convex, $\quad \mathit{convexHull}(\,\phi^d_H(\,\emptyset\,)\,) = \phi^d_H(\,\emptyset\,)$

Hence the projection of d-rank stratum of EDMC on G is equal to the projection of EDMC on G. Thus G is d-realizable.

$\phi^d_H(\,\emptyset\,) = \cup_d\ \phi^d_H(\,\emptyset\,) \quad$ => G is d-realizable.

Conjecture 6.2

Given a graph G that is not generically over-constrained and a non-edge f, the Cayley configuration space $\phi^2_F(G, \delta)$ *is a single interval for all generic δ (i.e., for δ that admit a 2D generic realization of* (G, δ) *), if and only if all the minimal 2-sum components containing f are 2-realizable and (partial 2-trees).*



minimal 2-sum component containing f is not 2-realizable

generically globally rigid

$\phi^2_f(G, \delta)$ is single point

Conjecture 6.3

Let H be a 3-realizable graph on vertex set V . Take any partition of the edge set of H into $E \cup F$, and consider the graph G = (V,E) and any EDCS (G, δ). Then there is a ***$O(|V|^q)$*** *time algorithm to write down the* ***description of the Cayley configuration space*** ϕ^3_F*(G, δ) as a semi-algebraic set of low degree (say, no more than 4).*

Conjecture 6.4

Given graph G that is a partial 3-tree and non-edge f

- *if $G \cup f$ has no K_5 or $K_{2,2,2}$ minor, then G has a connected 3D Cayley configuration space on f;*

- *if $G \cup f$ has a K_5 or $K_{2,2,2}$ minor then G has a connected 3D Cayley configuration space on f if and only if the 2 vertices of f must be identified in order to get a K_5 or $K_{2,2,2}$ minor in G.*