Vildan Atalay
UFID: 4984-7393
Geometric Complexity

Realizability of Graphs

A relization of a graph G is a function which assigns to each vertex i of G a point p_i in some Euclidean space. When we draw a realization, we generally also draw the edges between vertices as straight lines. Two vertices may be assigned to the same point in a realization and edges may intersect and even overlap.

Definition 1: A graph G is d-realizable if, given any realization $p_1, \dots, p_n$ of the graph in some finite dimensional Euclidean space, there exists a realization $q_1, \dots, q_n$ in E^d with the same edge lenghts: $| p_i – p_j | = | q_i – q_j |$ for all $\{i, j\} \in E(G)$.

Examples:

1. A path is 1-realizable, because we can arrange the vertices in order on a line with the appropriate distance between any two points.
2. A tree is also 1-realizable.
3. A triangle is not 1-realizable.

In this paper, we look at the question of which graphs are d-realizable for $d \leq 3$:

Theorem 1: *A graph G is 1-realizable if and only if it does not have K_3 as a minor (i.e., G is a forest).*

Theorem 2: *A graph G is 2-realizable if and only if it does not have K_4 as a minor.*

Theorem 3 (Main Theorem): *A graph G is 3-realizable if and only if it does not have either K_5 or the 1-skeleton of the octahedron as a minor.*

Theorem 1: A graph G is 1-realizable if and only if it is a forest (a disjoint collection of trees).

Proof: Any forest with any specified edge lengths can be realized in one dimension. If a graph is not a forest, then it contains a cycle as a subgraph. This cycle can be realized in the Euclidean plane (three edges have length 1, others 0). There is no realization in the line with the same edge lengths. So, a graph cpntaining a cycle is not 1-realizable. □

In general, if a graph G is d-realizable, then any subgraph of G is also d-realizable.

If we require all edges to have positive length in this proof, it would not change the d-realizability of graphs. Let G be a graph, v = |V(G)| and e = |E(G)|. Consider the function f: $R^{dv} \rightarrow R^e$ which takes a realization of G in R^d and returns the length of each edge of G. The image of f applied to a closed ball of radius M is a compact set in R^e. Thus, the set of edge lengths which cannot be realized in R^d inside a closed ball of radius M is an open set in R^e. If a graph G has realization $p = (p_1, \dots, p_n)$ in R^N with some zero length edges that is not realizable in R^d with the same edge lengths, then a sufficiently small perturbation of $p = (p_1, \dots, p_n)$ to a configuration with no zero length edges in R^N will still not be realizable with the same edge lengths in R^d.

Theorem 2: A graph G is 2-realizable if and only if it does not have K_4 as a minor.

Definition 2: A minor of a graph G is any graph obtained from G by a sequence of edge deletions and edge contractions (identify the two vertices belonging to an edge and then remove any loops or multiple edges).

Theorem 4: If a graph G is d-realizable and H is a minor of G, then H is d-realizable.

Definition 3: A graph property is called minor monotone if it is closed under the operation of taking minors.

Theorem 5 (The Graph Minor Theorem): Every minor monotone graph property has a finite list of forbidden minors; i.e. there exists a finite list of graphs $G_1, \dots, G_n$ such that a graph G satisfies the graph property if and only if G does not have any G_i as a minor.

Definition 4: A graph is series parallel if it is a subgraph of a graph that is constructed from a K_2 by repeatedly attaching subdivided edges to two adjacent vertices.

Theorem 6 (Wagner 1937): A graph G is series parallel if and only if G does not contain K_4 as a minor; i.e. K_4 is the only forbidden minor for series parallel graphs.

Theorem 2: *A graph G is 2-realizable if and only if it does not have K_4 as a minor.*

Proof: Suppose G does not have K_4 as a minor. G is series parallel. We assume that G is maximally series parallel (we cannot add more edges). A maximally series parallel graph can be constructed from K_2 by attaching subdivided edges with exactly one subdivion between two adjacent vertices.

The graph K_2 is 2-realizable. If we attach a subdivided edge to adjacent vertices with edge lengths satisfying the triangle inequality to a graph that is realized in R^2, the resulting graph can also be realized in R^2. By induction, all maximally series parallel graphs are 2-realizable.

For the opposite direction, suppose that a graph G is 2-realizable. K_4 is not 2-realizable (K_4 can be realized in R^3 as the skeleton of a tetrahedron). Thus, G cannot contain K_4 as a minor. □

Theorem 3 (Main Theorem): A graph G is 3-realizable if and only if it does not have either K_5 or the 1-skeleton of the octahedron as a minor.

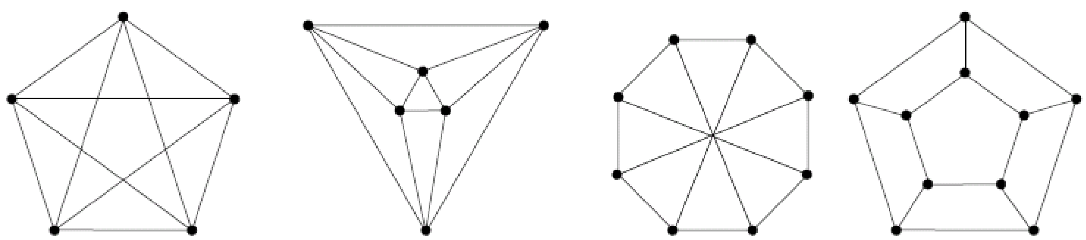
Definition 5: Let G_1 and G_2 be two graphs, both containin a K_k as a subgraph. The k-sum of G_1 and G_2, denoted $G_1 \oplus_k G_2$, is the graph obtained by identifying the two K_k's.

Definition 6: A graph is a k-tree if it can be obtained through a sequence of k-sums of K_{k+1}'s. A graph is a partial k-tree if it is a subgraph of a k-tree.

If G_1 and G_2 are both d-realizable and both contain K_d as a subgraph, $G_1 \oplus_d G_2$ is also d-realizable.

Forests are equivalent to partial 1-trees, 1-realizable graphs are partial 1-trees. Series parallel graphs are partial 2-trees, 2-realizable graphs are partial 2-trees. Thus, all partial d-trees are d-realizable.

Theorem 7 (Arnborg, Proskurowski, and Corneil 1990): The forbidden minors for partial 3-trees are K_5, the 1-skeleton of the octahedron ($K_{2,2,2}$), V_8, and C_5 x C_2.



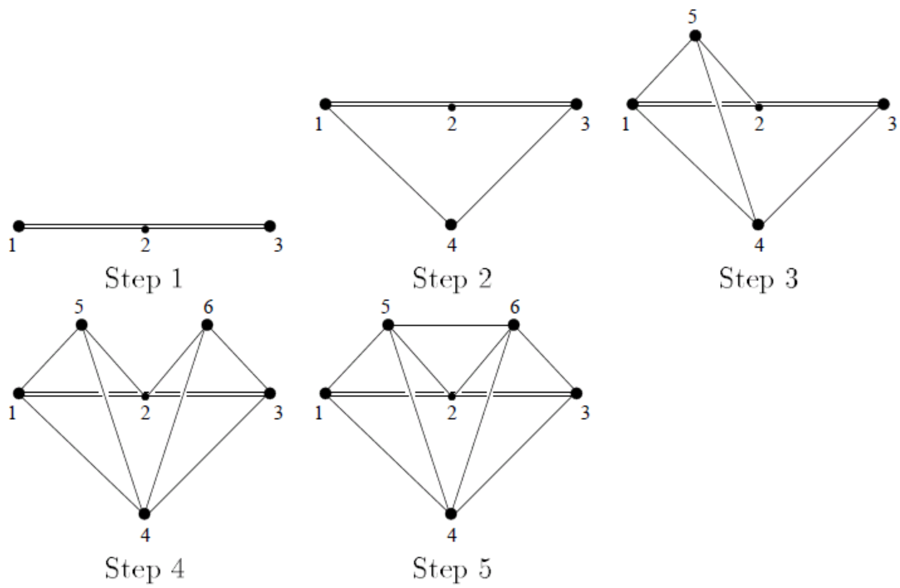
K_5 Octahedron $= K_{2,2,2}$ V_8 $C_5 \times C_2$

If any of these graphs is not 3-realizable, then it is a forbidden minor for 3-realizability. We know that K_5 is not 3-realizable.

Theorem 8: The 1-skeleton of the octahedron ($K_{2,2,2}$) is not 3-realizable.

Proof: We construct a realization of the octahedron in E^4 which cannot be realized in E^3.

Step 1. Start with a degenerate triangle with edge lengths 1, 1 and 2.
Step 2. Attach vertex 4 to this degenerate triangle at vertices 1 and 3 with edge lengths $\sqrt{2}$ and $\sqrt{2}$.
Step 3. Attach vertex 5 to vertices 1, 2 and 4. Place this vertex in the third dimension. Make all of the new edges have length 1.
Step 4. Attach vetrex 6 to vertices 2, 3 and 4. In three dimensions, place this vertex either above or below the plane. Make all of the new edges have length 1. There are only two possible realizations in E^3.
Step 5. Add a final edge between vertices 5 and 6. In E^3, this edge can have two possible lengths $\sqrt{2}$ and 2. In E^4, this edge can be any length.



Thus, the octahedron is not 3-realizable. □

Theorem 9: *The graph V8 is 3-realizable.*

Definition 7: A tensegrity, denoted G(p) is a configuration $p = (p_1, \dots, p_n)$ and a graph G, where each edge of the graph is labeled as a cable, strut, or bar, and where each vertex is labeled as being pinned or unpinned. Cables are allowed to decrease in length and shown as dotted lines (or stay the same length). Struts are allowed to increase in length and shown as double lines (or stay the same length). Bars are forced to remain the same length and shown as single lines. Pinned vertices are forced to remain where they are.

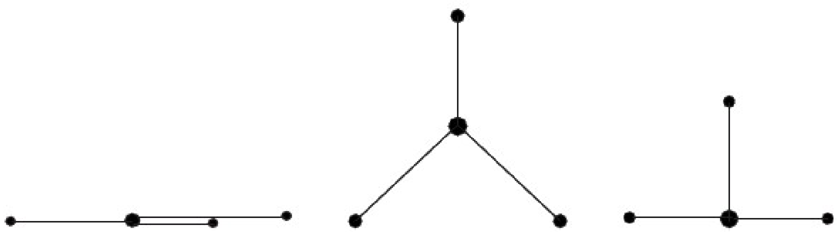
Definition 8: Fix a graph G. Let p and q be two configurations of G. If G(q) satisfies the cable, strut, and bar conditions of G(p), then we say that G(p) dominates G(q), which we denote by $G(p) \geq G(q)$. More precisely, $G(p) \geq G(q)$ if for every pinned vertex i , $p_i = q_i$, and for every edge {i, j },

- $| p_i - p_j | \geq | q_i - q_j |$ if {i, j } is a cable,
- $| p_i - p_j | = | q_i - q_j |$ if {i, j } is a bar,
- $| p_i - p_j | \leq | q_i - q_j |$ if {i, j } is a strut.

Definition 9: An equilibrium stress for G(p) is an assignment of real numbers $\omega_{ij} = \omega_{ji}$ to each edge $\{i, j\} \in E(G)$ such that for each unpinned vertex i of G,

$$\Sigma_{j:\ \{i,j\} \in E(G)}\ \omega_{ij} (p_i - p_j) = 0$$

We denote a stress by a vector ω, where the components of the vector range over the edges of G. We often refer to an equilibrium stress as simply a stress. The existence of an equilibrium stress is what we use to argue that certain vertices span low enough dimension.



Collinear Coplanar Coplanar

Definition 10: A tensegrity G(p) is unyielding if any other configuration q with G(p) $\geq$ G(q) has the same edge lengths as in p.

Theorem 10: If G(p) is an unyielding tensegrity with exactly one strut or cable, then G(p) has an equilibrium stress that is non-zero on at least one edge.

Proof: Suppose the only equilibrium stress on G(p) is zero on all edges. R(p) is a surjective linear transformation. So, there exists an open neighborhood U about p $\in R^{Nv}$ and an open neighborhood V about f (p) such that f maps U onto V. So, there exists a realization q such that p and q have the same bar lengths but q has a longer strut length or a shorter cable length. Thus, G(p) is not unyielding. □

Lemma 1: Let G(p) be a tensegrity with exactly one strut or cable and suppose that G is connected and remains connected if the strut or cable is removed. Then there exists a configuration q with G(p) $\geq$ G(q) and G(q) unyielding.

Proof: Let v = |V(G)| and f : $R^{v.(v-1)} \rightarrow R^1$. Every configuration of G spans at most v − 1 dimensions; thus, every configuration of G exists in R^{v-1}. Consider the set of configurations that have the same bar lengths as in G(p) and that have at least one of the vertices pinned. This set is compact: since G with the strut or cable removed is connected, every vertex has a maximum distance it can be from the pinned vertex. Thus, f attains a maximum on this set. This maximum occurs at an unyielding configuration G(q), since there is no configuration with the same edge lengths and a longer strut length.

Lemma 2: If a realization p of V_8 has any of the following situations, then it can be folded into E^3:

1. Four vertices are collinear;
2. Three vertices, not consecutive on the outer cycle, are collinear;
3. Each vertex in the set {3, 6, 7, 8} (or an isomorphic set) lies in the same plane or line as its neighbor.

Proof: If vertices 1, 2, 4, 5, and 7 are in E^3, then each remaining vertex can be rotated about the plane spanned by its neighbors into E^3.

If four vertices or three non-consecutive vertices are collinear, then there are 5 such vertices that span a 3-dimensional subspace. Thus, situations 2 and 2 can be folded into E^3.

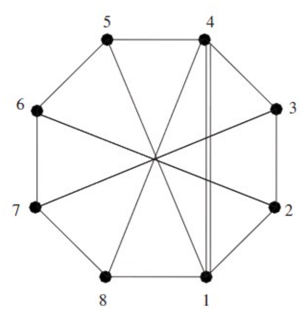
Start with four vertices assumed to be in E^3 and, one by one, show that the remaining vertices must also be in E^3. If vertex i lies in a plane with its neighbors and vertex i and two of its neighbors are in E^3, then either the third vertex is in E^3 or vertex i is collinear with its other two neighbors (meaning the third vertex is in E^3).

Vertices 2, 3, 6, and 7 span at most a three-dimensional space. We assume that p_2 , p_3, p_6, $p_7 \in E^3$. Vertex 6 lies in the same plane as its neighbors, 2, 5, and 7. 2, 6 and 7 are already in E^3. If 2, 6, and 7 do not span all of the plane, then they lie on a line and the entire realization can fold into E^3 by situation 2. Otherwise, the entire plane is in E^3, so vertex 5 is also in E^3. Vertex 7 lying in a plane with its neighbors forces vertex 8 to be in E^3. Vertex 3 lying in a plane with its neighbors forces vertex 4 to be in E^3. Vertex 8 lying in a plane with its neighbors forces vertex 1 to be in E^3. □

Theorem 9: V_8 is 3-realizable.

Proof: Let p be a realization of V_8. Let G(p): V_8 plus a strut.

This tensegrity has an unyielding realization q such that G(q) has a non-zero stress. We should show a non-zero stress on the tensegrity implies that one of the situations of the previous lemma. We will call a vertex zero vertex if all of the incident edges have a stress of zero. We will call a vertex non-zero vertex if at least one of the incident edges has a non-zero stress.



Case 1: *Vertex 5, 6, 7, or 8 is a zero vertex.*

Suppose vertex 5 is a zero vertex. Vertex 6 is adjacent to 5, so the edge {5, 6} has a stress of zero. Vertex 6 either has exactly two incident edges with non-zero stress (2, 6 and 7 are collinear, V8 can fold into E3) or all edges incident to vertex 6 have zero stress, making 6 a zero vertex. In this case the non-zero stress lies entirely on vertices 1, 2, 3, and 4. To have a non-zero stress all four of these vertices have to be collinear.

Case 2: *Vertex 2 or 3 is a zero vertex.*

Either 2 and 3 are both zero vertices or situation 2 occurs. The non-zero stress lies entirely on $\{1, 4, 5, 6, 7, 8\}$. If 2 is a zero vertex, then 6 is a zero vertex or 5, 6, and 7 are collinear. If 3 is a zero vertex, then 7 is a zero vertex or 6, 7, and 8 are collinear. Thus, either case 1 occurs, or vertices 5–8 are all collinear.

Case 3: *None of vertices 2, 3, 5, 6, 7, or 8 is a zero vertex.*

Then each of these vertices must lie in the same plane as their neighbors, so we have situation 2. □

Theorem 11: *The graph $C_5 \times C_2$ is 3-realizable.*

One of the following must occur;

- $C_5 \times C_2$ can be folded into E^3,
- Pin the stressed vertices and add a strut between two other vertices, resulting in another stressed configuration that can be folded into E^3, or
- There is a sequence of realizations of $C_5 \times C_2$ that can be realized in E^3 converging to the unyielding configuration (unyielding configuration has a realization in E^3).

Lemma 3: Let G(p) be a tensegrity, with vertex i having degree 2. Let the neighbors of vertex i be vertices j and k. Let H be the graph obtained from G by removing vertex i and its incident edges and adding the edge {j, k} (if the edge does not already exist). Let the configuration q be the same as p but not including vertex i . If G(p) has an equilibrium stress, then H(q) has an equilibrium stress.

Proof: Let ω be the equilibrium stress on G(p). Let α be the equilibrium stress on H(q). For the edge {j, k}, the stress is:

$$\alpha_{jk} = \omega_{ij} \frac{\|\mathbf{p}_i - \mathbf{p}_j\|}{\|\mathbf{p}_j - \mathbf{p}_k\|} + \omega_{jk}.$$

Since ω is an equilibrium stress, this is equivalent to:

$$\alpha_{jk} = \omega_{ik} \frac{\|\mathbf{p}_i - \mathbf{p}_k\|}{\|\mathbf{p}_j - \mathbf{p}_k\|} + \omega_{jk}.$$

α satisfies the equilibrium condition at vertices j and k. □

Lemma 4: If there are realizations of $C_5 \times C_2$ that cannot be realized in E^3, then there exists such a realization where the lengths of the edges along a cycle cannot be added and subtracted to give zero.

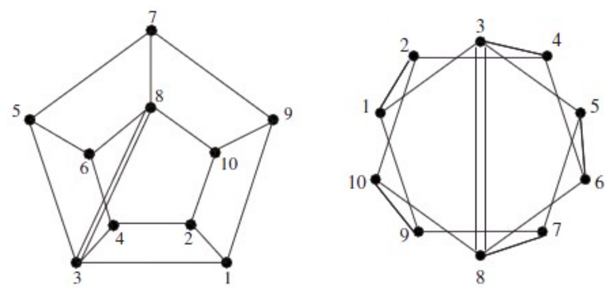
Proof: Let $v = |V(C_5 \times C_2)|$. Let $e = |E(C_5 \times C_2)|$ and $f : R^{3v} \to R^e$.

The image of a closed ball of radius M under this f is a compact set, since f is a continuous function. Thus, the set of edge lengths with no realization in E^3 inside a closed ball of radius M is an open set. Thus, given a list of edge lengths that cannot be realized in E^3 where the lengths of the edges along one of the cycles can be added and subtracted to give zero, there must be a nearby list of edge lengths without this property that can also not be realized in E^3. □

Lemma 5: Let G(p) be a tensegrity with a stress that is non-zero on all edges. If G has n vertices, then the dimension of the span of the n vertices is at most $n-2$. Additionally, if G is not the complete graph on n vertices, then the dimension of the span of the n vertices is at most $n-3$.

Proof: The only tensegrity in E^{n-2} with n vertices and a stress that is non-zero on all edges has $G = K_n$. □

Lemma 6: Consider the tensegrity G(p) shown in figure. If there is a stress that is non-zero on every edge of this tensegrity, then the configuration p lies in E^3.



Proof: Every vertex that has degree 3 must be either coplanar or collinear with its neighbors. If a degree 3 vertex is collinear with two of its neighbors, then it must be collinear with the third neighbor. We should show that a vertex is in E^3. Use "if vertex *i* lies in a plane or a line with its neighbors and vertex *i* and two of its neighbors are known to be in E^3 and to span the plane or line, then the third vertex is also in E^3".

The four vertices 1, 2, 9, and 10 span at most three dimensions. Vertex 1 lies in a plane (or a line) with its neighbors, 2, 3, and 9, so vertex 3 is in E^3. Vertex 2 lies in a plane (or line) with its neighbors, so vertex 4 is in E^3. Vertex 9 lies in a plane (or line) with its neighbors, so vertex 7 is in E^3. Vertex 10 lies in a plane (or line) with its neighbors, so vertex 8 is in E^3. Vertex 7 lies in a plane (or line) with its neighbors, so vertex 5 is in E^3. Vertex 5 lies in a plane (or line) with its neighbors, so vertex 6 is in E^3. Therefore, if there is a non-zero stress on every edge of this tensegrity, all vertices are in E^3. □

To simplify the later proofs, we use the notation <a> to mean the affine span of the vertex a and the neighbors of a. We use the notation <a> ⇒ b to mean that;

- b is a neighbor of a,
- <a> is a line or a plane (that is, a is coplanar or collinear with its neighbors),
- a and its neighbors other than b are known to be in E^3,
- and because of this b is also in E^3.

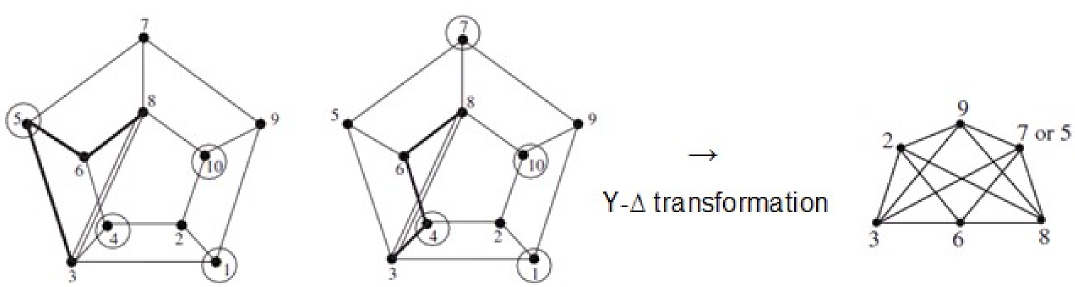
Definition 11: A Y–Δ transformation is an operation applied to a graph that removes a vertex i with degree 3, and adds edges between all pairs of vertices adjacent to vertex i .

Lemma 7: Let G be a graph and let $p = (p_1, \ldots, p_n)$ be a realization of the graph. Let H and $q = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ be the graph and realization obtained by performing a Y – Δ transformation on a vertex i of G. If there is a realization of H in E^3 with the same edge lengths as in q, then there is a realization of G in E^3 with the same edge lengths as in p.

Proof: We can construct the realization of G by realizing H, and then folding the vertex i into E^3 along the plane formed by the vertices adjacent to i. □

Lemma 8. Let G(p) be an unyielding configuration of the tensegrity $C_5 \times C_2$ plus a strut. Suppose that there is a cycle of length 4 which includes the strut that is collinear. Then there is a realization of $C_5 \times C_2$ in E^3 with the given edge lengths.

Proof. Assume 3 edges in bold are collinear. Vertices 3, 6, and 8 are collinear. By the previous lemma, if the resulting configuration can be realized in E^3, the original configuration can also be realized in E^3. 3, 6, and 8 are collinear, so 3, 6, 8, 2, and 9 are all in E^3. We can rotate the remaining vertex, vertex 7 or 5, about the plane spanned by vertices 3, 6, 8, and 9 and into E^3. □



Theorem 11: The graph $C_5 \times C_2$ is 3-realizable.

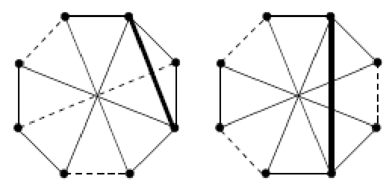
Proof: For the proof we should consider the subgraph of $C_5 \times C_2$ that does not contain removed vertices and edges. It suffices to show that the resulting graph is in the appropriate dimension. We can show that $C_5 \times C_2$ is 3-realizable by considering all cases of removed edges and vertices. □

We showed that the graphs V_8 and C_5 x C_2 are 3-realizable. So, there can be other graphs which are not 3-realizable but do not have K_5 or the octahedron as a minor.

We can eliminate this possibility by showing that any graph containing V_8 or C_5 x C_2 as a minor either contains K_5 or the octahedron as a minor or is 3-realizable.

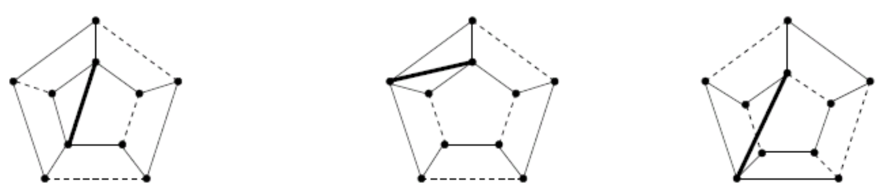
Lemma 9: If any edge is added between non-adjacent vertives of V_8, the resulting graph has K_5 as a minor.

Proof: There are two ways to add an edge to V_8. When we contract the dotted edges, the resulting graph is K_5. □



Lemma 10: If any edge is added between non-adjacent vertices of C_5 x C_2, the resulting graph has either the octahedron or K_5 as a minor.

Proof: There are three ways to add an edge to C_5 x C_2. Contracting the dotted edges produces the octahedron or the K_5. □



Contracts to Octahedron   Contracts to Octahedron   Contracts to K_5

Definition 12: A graph H is a subdivision of a graph G is H can be obtained from G by replacing every edge {i, j} of G with a path from vertex i to vertex j.

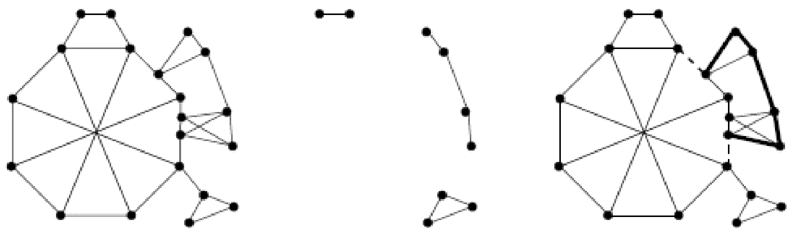
Lemma 11: Let H be a graph whose vertices are of maximum degree 3. If a graph G has H as a minor, then G contains a subdivision of H as a subgraph.

Theorem 3 (Main Theorem): *The forbidden minors for 3-realizability are K_5 and the octahedron.*

Proof: We should show that if a graph G does not have K_5 or the octahedron as a minor, then it is 3-realizable. G is connected.

If G does not contain V_8 or C_5 x C_2 as a minor, then it is a partial 3-tree and 3-realizable.

Suppose G contains V_8 as a minor. G must contain a subdivion of V_8. We will crate G as a subgraph of the 2-sum of 3-realizable graphs.



1 2 3

Remove the subdivion of V_8 from G and consider the components of the resulting graph. Each component has to be connected in G to exactly one of the subdivided edges of V_8.

If a component connects to two subdivided edges, then there should be a path in G from the subdivided version of {i, j} to the subdivided version of edge {k, l}. The subdivided edges can be contracted in G so that the path goes from i to k, which contradicts lemma 1.

Let $V_{\{i,j\}}$ be the union of all vertices from the components associated with the subdivided edge {i, j} and the vertices from the subdivided edge.

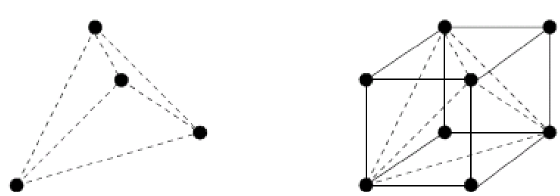
Add the edges to G that correspond to the contraction of the subdivided edges. Call this new graph H. We should create H as a 2-sum of smaller graphs. Let $H_{\{i,j\}}$ be the induced subgraph of H on the vertices $V_{\{i,j\}}$. Then, H is a 2-sum of V_8 and all the $H_{\{i,j\}}$ by attaching along the edges {i, j}.

The graphs $H_{\{i,j\}}$ are minors of G, and cannot contain K_5 or the octahedron as a minor. So $H_{\{i,j\}}$ are 3-realizable. Thus, H is 3-realizable. G is a subgraph of H. G is 3-realizable. □

Every 3-realizable graph is a subgraph of a graph that can be obtained by a sequence of 3-sums and 2-sums involving K_4, V_8 and $C_5 \times C_2$.

Example 1: The 1-skeleton of the cube is a partial 3-tree, and therefore 3-realizable.

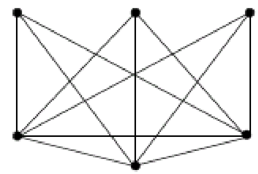
Take the 3-sum of 1-skeleton of the tetrahedron with four other K_4's.



K_4 K_4 3-summed with four K_4's

Example 2: The graph $K_{3,3}$ is a partial 3-tree, and therefore 3-realizable.

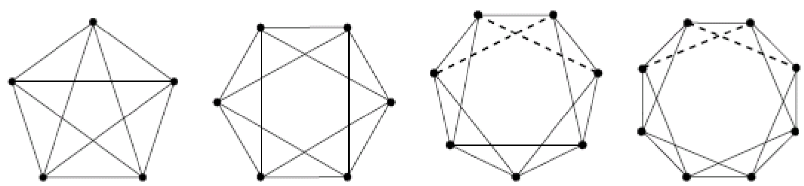
Take the 3-sum of a triangle with three K_4's.



3-tree

Example 3: The Cauchy graph on $n \geq 5$ vertices Ch_n is 4-realizable.

Ch_n is the graph obtained from a cyclic graph by placing an edge between every other vertex. Ch_n is a minor of Ch_{n+2}. Ch_5 is K_5, Ch_6 is octahedron and neither of them are 3-realizable.



Ch_5 Ch_6 Ch_7 Ch_8

Discussion and Open Problems

Theorem 12: Any graph G with e edges is d-realizable if e < (d+1)(d+2) / 2. Furthermore, G is still d-realizable if c, and G is not the complete graph K_{d+1}.

Conjecture 1. If a graph G has e edges and e < (d+1)(d+2) / 2, then G is a partial d-tree. Furthermore, if G has e = (d+1)(d+2) / 2, and G is not the complete graph K_{d+1}, then G is still a d-tree.