Geometric Complexity CIS6930	September 9,2011

Lecture 1 through 6

Lecturer: Dr. Meera Sitharam

Scribe: Jialong Cheng, Menghan Wang

Contents

1 Schoenberg's Theorem (1935)

Problem 1 *Given an $n \times n$ matrix $\Delta = (\delta_{ij})_{n\times n}$,*

1. *Does there exists a* Euclidean realization, *i.e., a set of points $p_1, p_2, \dots, p_n \in \mathbb{R}^d$ for some d, s.t. $\forall i, j$, $||p_i - p_j||^2 = \delta_{ij}$?*
2. *find such a realization;*
3. *when d is given, does there exist a Euclidean realization? If so, find one.*

It is not hard to show the following condition is necessary for the existence of a Euclidean realization: δ_{ij} should form a metric, i.e., satisfy the triangle inequality: $\forall i, j, k$, $\delta_{ij} + \delta_{jk} \geq \delta_{ik}$. A necessary and sufficient condition was due to Schoenberg ([1, 2, 3]):

Theorem 1 *A Euclidean realization exists if and only if matrix Δ is negative semidefinite.*

Proof: A Euclidean realization exists means we can find the coordinates of $p_1, p_2, \ldots, p_n \in \mathbb{R}^d$ for some d. Without loss of generality, we can assume $d = n$, since the affine span of n points cannot be of dimension higher than n. Let $P = (p_1, p_2, \ldots, p_n)$, then the Gram matrix would be $\Gamma = P^T P$. The Gram matrix is an $n \times n$ matrix and P exists if and only if Γ is positive semidefinite. Next we will show the relationship between Δ and Γ. Suppose the centroid of the points is the origin, then we have $\sum_i g_{ij} = \sum_i p_i^T p_j = 0$ and $\sum_j g_{ij} = \sum_i p_i^T p_j = 0$. Since $\delta_{ij} = ||p_i - p_j||^2$, we have:

$$\begin{aligned}\delta_{ij} &= ||p_i||^2 + ||p_j||^2 - 2p_i^T p_j \\ &= p_i^T p_i + p_j^T p_j - 2p_i^T p_j \\ &= g_{ii} + g_{jj} - 2g_{ij}\end{aligned}$$

Hence

$$\begin{aligned}\sum_i \delta_{ij} &= \sum_i g_{ii} + \sum_i g_{jj} - 2\sum_i g_{ij} \\ &= \sum_i g_{ii} + n \cdot g_{jj},\end{aligned}$$

$$\begin{aligned}\sum_j \delta_{ij} &= \sum_j g_{ii} + \sum_j g_{jj} - 2\sum_j g_{ij} \\ &= \sum_j g_{jj} + n \cdot g_{ii},\end{aligned}$$

and

$$\begin{aligned}\sum_{i,j} \delta_{ij} &= \sum_j \sum_i g_{ii} + n \cdot g_{jj} \\ &= 2n \sum_i g_{ii}.\end{aligned}$$

Thus

$$g_{ij} = 1/2[1/n \sum_i \delta_{ij} + 1/n \sum_j \delta_{ij} - \delta_{ij} - 1/n \sum_{i,j} \delta_{ij}]$$

Hence $\Gamma = -1/2[I - 1/nJ^T]\Delta[I - 1/nJ]$, where I is the identity matrix and J is the all 1 matrix. Thus P exists if and only if Γ is positive semidefinite, if and only if Δ is negative semidefinite. (In some papers, Δ is defined to have entries that are negative squared distances. Then we should say Δ is positive semidefinite.) ■

In fact, Schoenberg's theorem with additional condition that rank is at most d is exactly Euclidean realizability in d-dimensions[3].

Theorem 2 *A Euclidean realization in d-space exists if and only if matrix Δ is negative semidefinite and rank$(\Delta) \leq d + 1$.*

Proof: From the proof of Theorem 1, we know a Euclidean realization in d-space exists if and only if we can find the coordinates of $p_1, p_2, \ldots, p_n \in \mathbb{R}^d$, if and only if rank$(P) = d$, if and only if Γ is positive semidefinite with rank at most d, if and only if Δ is negative semidefinite with rank at most $d+1$. ■

With that, we have the following definition of *Euclidean distance matrix* (EDM).

Definition 1 *An $n \times n$ symmetric matrix Δ is a* Euclidean distance matrix (EDM)*, if D is negative semidefinite.*

Thus we have Algorithm 1 to compute a Euclidean realization from a Euclidean distance matrix Δ.

Algorithm 1 Using SVD

Use $\Gamma = -1/2[I - 1/nJ^T]\Delta[I - 1/nJ]$ to find Γ
Apply Singular Value Decomposition (SVD) on Γ to get $\Gamma = USV$, where

$$V = U^T = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1d} & 0 & \dots & 0 \\ p_{21} & p_{22} & \dots & p_{2d} & 0 & \dots & 0 \\ & \dots & & \dots & & \dots & \\ p_{n1} & p_{22} & \dots & p_{nd} & 0 & \dots & 0 \end{bmatrix}$$

and $S = \text{Diag}\{\lambda_1, \lambda_2, \dots, \lambda_d, 0, \dots, 0\}$ is a diagonal matrix with eigenvalues of Δ on the diagonal. Then U is the matrix of the coordinates of $p_1, p_2, \dots, p_n$, where each column corresponds to the coordinate of a point.

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2 Cayley-Menger Condition: alternate equivalent condition for Euclidean realizability

In Section 1, we find the realization of Euclidean distance matrix Δ from its Gram matrix $\Gamma = P^T P$. Γ is associated with Δ: $\Gamma = 1/2[I - 1/nJ^T]\Delta[I - 1/nJ]$. From this we derived Schoenberg's theorem, that is,

$$\begin{gathered} \Delta \text{ has Euclidean realization [in dimension } d] \\ \iff \Gamma \text{ is symmetric PSD [of rank } d] \\ \iff \Delta \text{ is symmetric Negative Semi-Definite [of rank } d+1]. \end{gathered}$$

Here, we introduce a different method, Algorithm 2, to find the Euclidean realization of Γ.

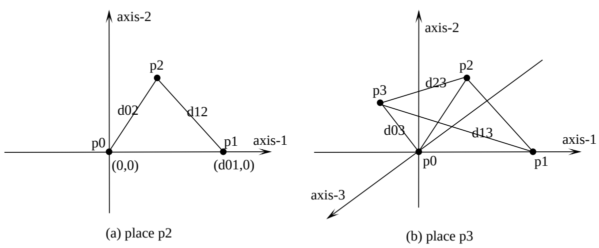

Figure 1: A figure demonstrating the algorithm to realize a complete inner product matrix

Algorithm 2 Solving quadratics

If we are given a Euclidean distance matrix Δ or inner product matrix Γ, then when the dimension d is fixed, we can just solve a system of polynomial equations one for each inner product entry of Γ where the variables are the coordinates of each point $p_1, \ldots, p_n$ in the desired dimension d. When d is not fixed, we simply solve for one point at a time. Each time we add a point p_k, we solve for its coordinates using its distances to *all* previously placed points, p_0 to p_{k-1}. In this way, p_k can be represented by at most $k-1$ coordinate values. Thus we keep the dimension and the complexity of the algorithm as low as possible.

Note: there is a unique placement of p_k modulo rotations and reflections if one exists. See Figure 1 for an illustration.

Example 2 *Let point* p_k *have coordinates* $\{p_k^1, p_k^2, ...\}$

First we show an example of placing three vertices. They can be placed on a 2D plane.

1. *Put* p_0 *at the origin:* $p_0 = \{0, 0, ...\}$
2. *Put* p_1 *on the first axis:* $p_1 = \{\delta_{01}, 0, 0, ...\}$
3. *To place* p_2*:*

$$\begin{cases} (p_0^1 - p_2^1)^2 + (p_0^2 - p_2^2)^2 = \delta_{02}^2 \\ (p_1^1 - p_2^1)^2 + (p_1^2 - p_2^2)^2 = \delta_{12}^2 \end{cases}$$

$$\Longrightarrow \begin{cases} p_2^1 = \frac{\delta_{01}^2 + \delta_{02}^2 - \delta_{12}^2}{2\delta_{01}} \\ p_2^2 = \frac{\sqrt{(\delta_{20}+\delta_{21}+\delta_{01})(\delta_{20}+\delta_{21}-\delta_{01})(\delta_{20}-\delta_{21}+\delta_{01})(-\delta_{20}+\delta_{21}+\delta_{01})}}{2\delta_{01}} \end{cases}$$

Now we can formalize this question.

Question 2 *What geometric conditions are necessary and sufficient for a Euclidean realization?*

In 2D, the condition is seen intuitively to be relatively simple. We only require the Gram matrix $\Gamma = P^T P$ to be a metric space. This is equivalent to requiring the *discriminant* of the distance quadratic system to be positive, so that real solutions exist. It is also equivalent to letting $\triangle p_0 p_1 p_2$ have positive volume (area), or *determinant of the volume matrix* to be greater than or equal to 0.

Definition 3 *Given an* $n \times n$ *matrix* $\Delta = (\delta_{ij})_{n\times n}$*, the volume matrix* $\hat{\Delta}$ *is the* $(n+1)\times(n+1)$ *matrix obtained from* Δ *by bordering* Δ *with a top row* $(0, 1, ..., 1)$ *and a left column* $(0, 1, ..., 1)^T$*. Now* $\det \hat{\Delta}$ *is called the* Cayley-Menger determinant *[4].*

$$V_{n-1}^2 = \frac{(-1)^n}{2^{n-1}(n-1!)^2} det(\hat{M}),$$

where V_{n-1} *equals the volume of the* $n-1$ *dimension simplex formed by the* n *vertices.*

Example 4 *We are going to place another point p_3 to Example 2.The Cayley-Menger determinant of Δ is:*

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \delta_{01}^2 & \delta_{02}^2 & \delta_{03}^2 \\ 1 & \delta_{10}^2 & 0 & \delta_{12}^2 & \delta_{13}^2 \\ 1 & \delta_{20}^2 & \delta_{21}^2 & 0 & \delta_{23}^2 \\ 1 & \delta_{31}^2 & \delta_{32}^2 & \delta_{33}^2 & 0 \end{vmatrix}$$

This gives the volume of the tetrahedron formed by the four points.

In addition, every $k \times k$ submatrix of Δ gives the volume matrix of a $k-1$ dimension simplex contained in the tetrahedron. The metric space condition is the Cayley-Menger condition for $k = 3$, i.e., we want the area of all triangles to be nonnegative. This discussion leads to the following theorem.

Theorem 3 *(Cayley-Menger condition) A complete matrix $\Delta = (\delta_{ij})_{n\times n}$ ($n \times n$) has a Euclidean realization if and only if for all $k \times k (k \le n)$ submatrices S of Δ, $\det(\hat{\delta_S}) \ge 0$.*

Here $\delta_S = \delta_{i\in S, j\in S}, S \subseteq 1, ..., n, |S| = k$.

For Δ to have Euclidean realization in dimension d, we need the additional condition that when $|S| = k \ge d+2$, $\det(\hat{\delta_S}) = 0$.

Note: the Cayley-Menger condition is equivalent to Theorem 2.

3 Partial matrices and linkages

Definition 5 *A* partial distance matrix *is a matrix with some entries specified that has an extension to a Euclidean distance matrix.*

Definition 6 *A* linkage *is a pair $L = (G, \delta)$ consisting of a graph $G = (V, E)$ and a function $\delta : E(G) \to \mathbb{R}_{\ge 0}$ assigning nonnegative lengths to the edges. Often the edges of G are referred as* bars*.*

Now let us consider Algorithms 1 and 2 for linkages (G, δ).

Algorithm 3 modified Algorithm 1

The Gram matrix Γ is incomplete and hence SVD cannot directly work. A semidefinite programming approach can be used to complete Γ and find the realization at the same time. This is called the *Euclidean Distance Matrix Completion Problem(EDMCP)* and has been well studied in the literature ([16]). Positive semidefiniteness of Γ is a convex condition, hence the feasible region for (the entries of) Γ is a convex region, making it easier to search.

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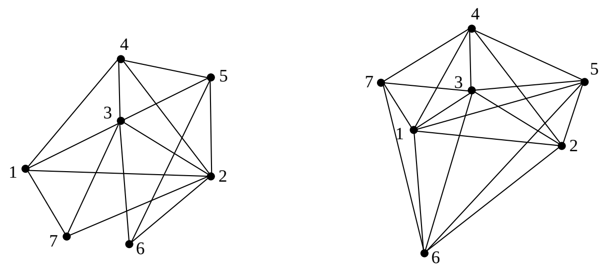
4 Triangularizable multivariate quadratic systems

In this section, we provide a general algebraic viewpoint for Algorithm 2. As mentioned earlier, realizing a Euclidean distance matrix Δ is the same as solving a system of quadratics. Some of these quadratic systems can be triangularized, which is an analogy of "triangulating" a linear system.

Algorithm 4 modified Algorithm 2

Algorithm 2 may not work because while placing each point, there is more than 1 placing (non-equivalent modulo Euclidean motions), resulting in a combinatorial explosion of paths, most of which may not find a successful realization. In fact, the problem of existence of Euclidean realization for linkage (G, δ) is NP-hard (a reduction from 3-SAT can be found in [17]).
But for a special class of graphs called "trilateralizations" (in 2D, see Figure 2) and the corresponding "k-lateralizations" (in $(k-1)$-space) the above algorithm still works. For these graphs, the algorithm gives a unique solution at each step.



(a) trilateralization graph

(b) 4-lateralization graph

Figure 2: k-lateralizations

Example 7 *Suppose a quadratic system has n equations. After triangularization, we have:*

- *The first equation is a quadratic equation of* x_1, x_2, ..., x_n.
- *The second equation is a quadratic equation of* x_2, x_3, ..., x_n.

  ...

- *The* $(n-1)^{th}$ *equation is a quadratic equation of* x_{n-1}, x_n.
- *The* n^{th} *equation is a quadratic equation of* x_n.

Suppose all coefficients are in $\mathbb{Q}$. *To solve this system, we*

1. *Compute* x_n *from the* n^{th} *equation. This takes one square root, and* x_n *is one extension field of* $\mathbb{Q}$ *with one square root.*
2. *Substitute* x_n *in the* $(n-1)^{th}$ *equation, and solve* x_{n-1}. x_{n-1} *is the extension field of* $\mathbb{Q}$ *with two square roots.*
3. *Substitute* x_n *and* x_{n-1} *in the* $(n-2)^{th}$ *equation ... ...*

Proceeding in this way, all variables lie in an extension field of $\mathbb{Q}$ *obtained by nested square roots, i.e., nested quadratic extensions, which is also called* tower field of the rationals. *In complexity terms, triangularizable quadratic systems (while still NP-hard as pointed out before due to two possible solutions for each equation) are less complex than general quadratic systems for which the only know algorithms have double exponential complexity.*

Definition 8 *A triangularizable quadratic system is* oriented *if the solution of each equation is fixed by the addition of further (consistently dependent) quadratics.*

Note: oriented triangularizable quadratic system can be solved by a linear time algorithm.

Putting together Algorithm 1 and Algorithm 2, i.e., Theorem 2 and Theorem 3, we have the following observation:

Observation 4 *Solving an oriented triangularizable quadratic system has the same complexity as factorizing the negative semidefinite matrix Δ.*

For partial matrices, or linkages (G, δ), the corresponding quadratic systems may not be triangularizable, except in rare cases. Such linkages are called *ruler and compass constructible.*

Note: the k-lateralization graphs are not only ruler and compass constructible, i.e., their corresponding quadratic systems are not just triangularizable, but also oriented and hence they can be solved in linear time.

5 Geometric embedding problems

The problem mentioned in Section 1 is a geometric embedding problem. More specifically, it is a structure preserving optimal geometric embedding problem. I.e., we want to find an embedding as a geometric structure in a minimal dimensional space while preserving distances as close as possible. The problem has many applications, such as molecular distance constraint system, sensor network localization, and structural problem in computer aided design, etc.

Note: we can use many different metrics, such as the shortest-distance metric, the tree metric and the cut metric etc. as entries of Δ. All these metrics represent the same idea that we are trying to convert a combinatorial structure by embedding it as a geometric structure in an optimized dimension that preserves the metric.

Several questions arise here:

Problem 3 *Given a complete matrix Δ,*

(a) find EDM $\tilde{\Delta}$ s.t. $||\Delta - \tilde{\Delta}||$ is minimized. This problem can be solved with convex programming, since the feasible region of the EDM matrix $\tilde{\Delta}$ is convex. More specifically, we can use semidefinite programming to solve this constraint optimization problem.

(b) find EDM $\tilde{\Delta}$ with rank$(\tilde{\Delta}) \leq d$ s.t. $||\Delta - \tilde{\Delta}||$ is minimized. The feasible region here is no longer convex, but we can use PCA to find $\tilde{\Delta}$. (Δ is the set of vectors we want to approximate, $\tilde{\Delta}$ is the set of vectors obtained by PCA, and d is the number of principal components.)

(c) find EDM $\tilde{\Delta}$ with minimum rank s.t. $||\Delta - \tilde{\Delta}|| \leq \epsilon$.

Here, the norm is not specified. We can use operator norm, Frobenius norm, etc.

Question 4 *If Δ is a Euclidean distance matrix , how many realizations are there modulo translations and rotations in a d-dimensional space?*

The answer to Question 4 is as follows: Let rank$(\Delta) = r$, then

- if $r > d + 1$, then there is no realization;
- if $r = d + 1$, then there is one realization;
- if $r < d + 1$, then there are infinitely many realizations.

Homework 1 Show that the dimension of the space of the possible realizations is equal to $r - d - 1$.

We already discussed algorithms for realizing special classes of partial matrices, i.e., linkages, the question is: can we formalize algorithms for realizing (as efficiently as possible) a *general* partial matrix ([18, 19]. Another related question that has been studied is to give characterizations of classes of partial matrices that have efficient realizations ([20, 21]).

We know that graphs, via linkages, correspond to the "pattern" of nonzero entries of partial matrices. Hence in the following, we use linkages and their corresponding partial matrices interchangeably.

Problem 5 *Given a linkage $L = (G, \delta)$,*

(a) Does there exist an EDM $\tilde{\Delta}$ s.t.$\tilde{\Delta}|_G = L$?

(b) find EDM $\tilde{\Delta}$ s.t. $||\tilde{\Delta} - L||_G$ is minimized.

(c) find EDM $\tilde{\Delta}$ with rank$(\tilde{\Delta}) \leq d$ s.t. $||L - \tilde{\Delta}||_G$ is minimized.

(d) find EMM $\tilde{\Delta}$ with minimum rank s.t. $||L - \tilde{\Delta}||_G \leq \epsilon$.

In some of the previous work, researchers have transformed Problem 5 to Problem 3 by completing L with shortest path metric or other metric. If we know L is realizable, we cah ask the following question.

Question 6 *Which partial matrices are uniquely completable (or completable in at most finitely many ways) with or without specifying Euclidean dimension? For linkages, the question is : for which linkages does there exist a unique (finitely many) realization(s) in dimension d (for some distance assignment)?*

Question 6 is highly related to rigidity theory.

- a *rigid* linkage has finitely many realizations in a fixed dimension;
- a *globally rigid* linkage has a unique realization in a fixed dimension;
- a *universally rigid* linkage has a unique realization in all dimensions.

6 Introduction to Genericity

Question 7 *Given a matrix $\Delta = (\delta_{ij})_{n \times n}$, we are trying to find whether it has a certain property. Do we need to know the distance values? Or we only need to know the graph G?*

Question 7 is related to "generic" property. There are many different ways to define genericity. The following is an attempt.

Definition 9 *(attempt) A property is generic if it holds for all δ_{ij} assignments to G, or it does not hold for any δ_{ij} assignment to G.*

This is a rather strong genericity condition and very few properties can satisfy it.

Example 10 *Convex configuration space (a property that we will discuss in Section 7) satisfies strong genericity condition.*

Let us try another definition of genericty.

Definition 11 *(another attempt) A property is generic if it holds for almost all δ_{ij} assignments to G, or it does no hold for almost all δ_{ij} assignments to G.*

This is the definition more commonly used. Figure 3 and Figure 4 demonstrate this definition of genericity.

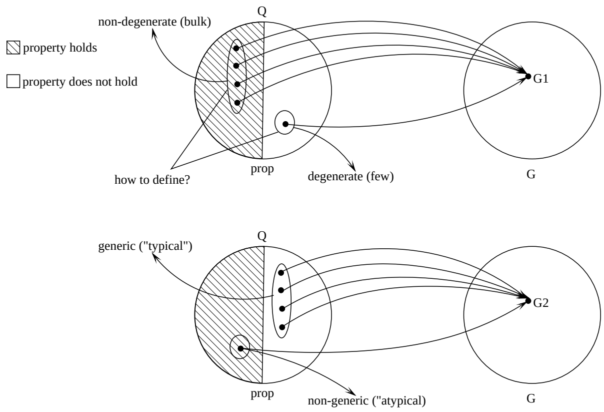


Figure 3: Generic Property

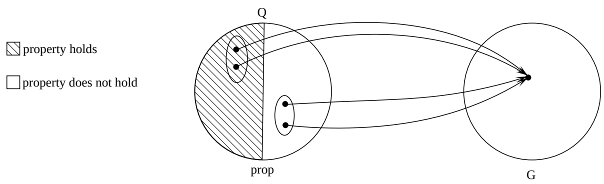


Figure 4: Non-generic Property

However, we still have some issues to address.

Question 8 *How should we define "almost all linkages"? Which are the "non-generic(degenerate) linkages"?*

Question 9 *How should we define non-generic (degenerate) linkages with respect to some property?*

Remark: we not only have to define generic properties (of linkages) but also generic linkages *with respect to* a given property.

Remark: if a property of linkages is nongeneric, it means there are some graphs for which some large number of generic linkages (corresponding to that graph) have the property, and some equally large number do not. We can still characterize and study those graphs, for which *all* corresponding linkages are generic with respect to that property.

The exact definition of genericity varies under different situations. We must take into consideration the property under discussion. We usually choose a definition of genericity that is more natural to the given property and makes it more convenient for theorem proving.

One example of a definition of genericity is the following:

Definition 12 *Let P be a set of points in $\mathbb{R}^d$. Let L be a partial matrix, i.e., a linkage (G, δ). As mentioned before, G the pattern or support of entries of L. Then P is a* realization *of L if $\Delta|_G = L$, where Δ is the Euclidean distance matrix corresponding to P.*

Example of a "property" of realization (P, L)

- (P, L) is such that "P is the unique realization satisfying $\Delta|_G = L$".
- (P, L) is such that "P is one of the finitely many realizations satisfying $\Delta|_G = L$".

Thus we can give a definition of generic linkages.

Definition 13 *P is* generic *w.r.t. a property, if prop(P) $\Leftrightarrow$ prop(P'), for all P' in the neighborhood of P. A property is generic if for all generic P, prop(P) or $\neg$prop(P). A linkage L is generic for a property if it has at least 1 generic realization for that property.*

7 Related characterization problems

The problems mentioned in Section 5 have a set of corresponding characterization problems in rigidity theory. Those characterization problems lie on the interface of convex analysis, algebraic geometry and graph theory.

Problem 10 *Consider the following property of linkages $L = (G, \delta)$, "$L : \exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$".*

(a) Is this property generic?

(b) If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, s.t. $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.

(c) If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, s.t. $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$.

Problem 10 is completely open.

Remark: Problem 10 cannot use the notion of generic linkages that we have in Section 6, since the current genericity of linkages is defined when there is a realization. To make Problem 10 well-defined, we need to develop a different notion of generic linkage.

Problem 11 *Consider the following property of linkages $L = (G, \delta)$, " $L : \exists$ finitely many EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$.".*

(a) Is this property generic?

(b) If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.

(c) If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$.

Problem 11 is completely open.

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(b) If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, $\exists$ a unique EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ a unique EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.

(c) If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, s.t. $\exists$ a unique EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G$.

The property in Problem 12 is *universal rigidity*. It has been proved that the property is not generic by Görtler, Thurston, et al ([11]).

Problem 13 *Consider the following property of linkages $L = (G, \delta)$, " $L : \exists$ finitely many EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$.".*

(a) Is this property generic?

(b) If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.

(c) If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$.

The property in Problem 13 is *rigidity*. Rigidity is a generic property and Laman's Theorem([5]) gives a characterization in $2D$. But there is no known combinatorial characterization for $d \geq 3$. If we do not fix the rank here, the problem becomes Problem 11 and is completely open.

Problem 14 *Consider the following property of linkages $L = (G, \delta)$, "L: $\exists$ a unique EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$.".*

(a) Is this property generic?

(b) If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, $\exists$ a unique EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ a unique EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.

(c) *If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, $\exists$ a unique EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$.*

The property in Problem 14 is *global rigidity*. It has been proved that the property is generic by Görtler, Thurston, et al ([6]). But there is no known combinatorial characterization for $d \geq 3$.

Problem 15 *Consider the following property of linkages $L = (G, \delta)$, " L : $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G \Rightarrow \exists$ EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$".*

(a) *Is this property generic?*

(b) *If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, s.t.$\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G \Rightarrow \exists$ EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G \Rightarrow \exists$ EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$, since the property now is generic.*

(c) *If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, s.t. $\exists$ EDM $\tilde{\Delta}$ s.t. $L = \tilde{\Delta}|_G \Rightarrow \exists$ EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$.*

The property in Problem 15 is *d-realizability.*

Definition 14 *A graph G is d-*realizable *if for all δ_{ij} assigned to edges of G, G has realization implies there exists a realization in d-dimension. I.e., if there is a way of realizing a set of bar lengths in any dimension, there is a realization of the same set of bar lengths in d-dimension. (To not be confused with Euclidean realizability in d dimensions, we should all G d-flattenable.)*

For d-realizability, a commonly explored characterization is the *forbidden minor* characterization.

Definition 15 *A class C of graphs G has a* finite forbidden minor characterization, *if there exists a fixed, finite set M of minors, such that $G \in C$ if and only if G doesn't contain any $K \in M$ as a minor.*

Example 16 *Kuratowski's planarity theorem: A graph G is planar if and only if it contains no $K_{3,3}$ or K_5 minor [13].*

Existence of finite forbidden minor characterization guarantees a polynomial time graph algorithm ([14]). We can find characterization of 1-realizable and 2-realizable graphs in [22, 23] and characterizing of 3-realizable graphs is still open in [15]. But characterizing of d-realizable graphs for $d > 3$ is still open.

Problem 16 *Consider the following property of linkages $L = (G, \delta)$, "L : $\{ \tilde{\Delta}$ is an EDM $: L = \tilde{\Delta}|_G \}$ is convex".*

(a) *Is this property generic?*

(b) *If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, $\{ \tilde{\Delta}$ is an EDM $: L = \tilde{\Delta}|_G \}$ is convex. This is equivalent to $\forall$ generic $L = (G, \delta)$, $\{ \tilde{\Delta}$ is an EDM $: L = \tilde{\Delta}|_G \}$ is convex, since the property now is generic.*

(c) *If not, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, $\{ \tilde{\Delta}$ is an EDM $: L = \tilde{\Delta}|_G \}$ is convex.*

We do not know whether the property in Problem 16 is generic. Again, there are still only partial results for $d = 3$ ([12]).

8 More on characterization Problems

Problem 17 *For each of the following property, we want to answer:*

1. *Is this property generic?*
2. *If so, then characterize graph G for which $\exists$ generic $L = (G, \delta)$, s.t. the property is satisfied.*
3. *If the property is not generic, then characterize graph G for which $\forall$ generic $L = (G, \delta)$, the property is satisfied.*

1. Rigidity [partial EDM has finitely many realizations in dimension d]
   - Generic in any dimension d (with an intuitive way to define genericity). Hence rigidity is a property of graphs.
     ——————————Ruijin
   - $d = 2$, Laman's theorem (1970) characterizes rigidity of graphs [5]
     ——————————Ruijin
   - $d = 3$, OPEN.
     Partial results: natural extension of Laman's theorem fails (Cheng & Sitharam &Streinu [8])
     ——————————Nam
2. Global rigidity [partial EDM has unique realization in dimension d]
   - Generic in any d [6]
     ——————————Jialong
   - $d = 2$: Genericity and characterization of graphs [7]
     ——————————Menghan
3. Universal rigidity [partial EDM has unique realization]
   - Not generic ([9] [10] [11])
     ——————————Liqian
   - Polynomial algorithms to find the unique realization if there is one
     ——————————-Liqian
4. Space of d-dimension realizations has a convex parametrization (draw from a specified clan)
   - Generic, 2d characterization ([12])
     ——————————Aysegul
   - Higher dimension: convexity of d-dimension configuration space equivalent to d-realizability (Gao, Cheng, Sitharam [12])
     ——————————Aysegul
5. d-realizability characterization
   - 2-realizability
     A graph is 2-realizable $\iff \nexists$ a K_4 graph (series parallel graph, 2-width trees, partial 2-trees)
     ——————————Vildan
   - 3-realizability: 4 forbidden minors ([15])
     ——————————Vildan

9 Introduction to Laman's theorem

Problem 18 *Characterize G for which $\exists$ generic $L = (G, \delta)$, $\exists$ finitely many EDM $\tilde{\Delta}$ of rank d s.t. $L = \tilde{\Delta}|_G$ This is called* linkage rigidity*. Such a characterization is possible only if the property is generic. Therefore, we also have to define an appropriate notion of genericity.*

Laman's theorem [5] solves this problem for $d = 2$.

Theorem 5 *A graph $G = (V, E)$ is rigid if and only if there exists $E' \subseteq E$ s.t.*
(1) $\forall S \subseteq V$, $2|V_S| - |E'_S| \geq 3$
(2) $2|V| - |E'| = 3$

The only if direction has been demonstrated by Maxwell in 1864 [27].

First we have to define genericity, s.t. we can show rigidity of G is generically independent of the actual entries in Δ. In other words, we want to show that rigidity is a "generic" property.

There exists several different Proofs for Laman's theorem by: 1. Laman ([5]; 2. Lovasz & Yemini ([24]); 3.Tay ([25]); 4. Theran & Streinu ([26]).

We will show Laman's original proof.

Sketch of Proof:

- Step 1: Prove that G has (1) & (2) if and only if G has a Henneberg construction (a Henneberg construction is a graph theoretical construction, not realization)

  The if direction is simple.

- Step 2: Prove that G has a Henneberg construction if and only if it has a *generic* rigid realization.

  We will first show that G has a Henneberg construction if and only if G has a generic *infinitesimally rigid* (will be defined when we introduce the proof of Laman's Theorem) realization. Then we will show that infinitesimal rigidity implies rigidity and generic rigidity implies infinitesimal rigidity.

- Step 3: (show rigidity is generic) Prove that G has one generic rigid realization if and only if all generic realizations of G are rigid.

  Here, *rigid* means this realization is the only realization locally(in a neighborhood of that same realiztion). It is called *framework rigidity* (or local rigidity).

- Step 4: Prove that there are only finitely many realizations for any generic Δ corresponding to G. In other words, prove that generic framework rigidity implies generic linkage rigidity.

  Equivalently, $\exists$ finitely many realizations of G / G is generically rigid $\Longleftrightarrow$ all generic realizations of G are infinitesimally rigid.

■

Remark: infinitesimal rigidity always implies rigidity. On the other hand, rigidity only implies infinitesimal rigidity for generic frameworks with respect to infinitesimal rigidity.

Henneberg-constructible graphs $\mathcal{H}$

Henneberg-constructible graph is a inductive (incremental, iterative) definition of a class of graphs.

Definition 17 *A Henneberg constructible graph G is defined inductively.*

- *Base case: a single point $\in \mathcal{H}_1$; a line segment $\in \mathcal{H}_2$*
- *Induction: Given $G_k \in \mathcal{H}_k$, we can construct $G_{k+1} \in \mathcal{H}_{k+1}$ with one step of Henneberg construction from G_k.*

  Type I construction: for any two vertices u, w in G_k, add a vertex v with two edges (u, v), (w, v). See Figure 5.

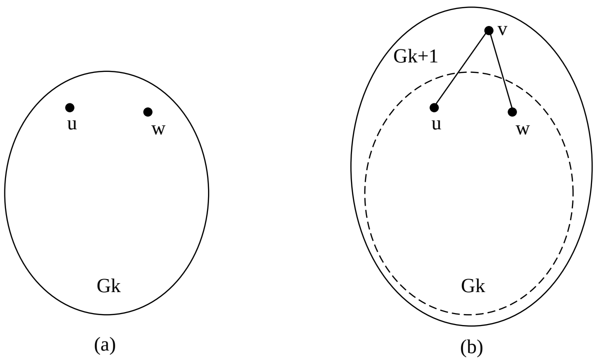

Figure 5: Henneberg-I construction

  Type II construction: for any three vertices u, w, t in G_k with at least one edge e among them, add a vertex v with three edges (u, v), (w, v), (t, v), and remove e. See Figure 6.

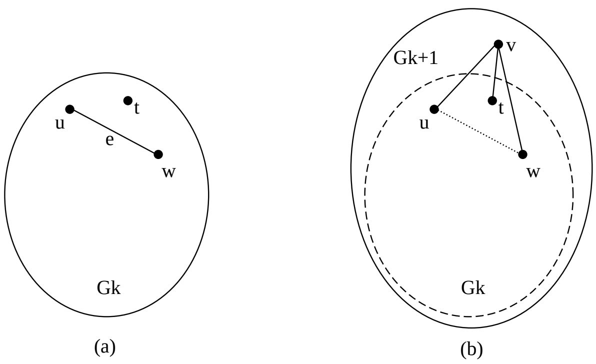

Figure 6: Henneberg-II construction

Homework 2 Come up with an algorithm to determine whether a given graph G is Henneberg-constructible.

For further reading, please refer to `http://www.convexoptimization.com/dattorro/cone_of_euclidean_distance_matrices.html`.

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