Geometric Complexity CIS6930	Sep 19 2011
Lecture 7-12	
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Contents

1 Definitions

Definition 1. *Graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$, $V=\{1,2,\dots,n\}, K=\{(i,j)|i,j\in V\}$

Definition 2. *Embedding* $p:V\to\mathbb{R}^m$, $P=(p_1^1,p_2^1,\dots,p_1^n,p_2^n,\dots,p_m^n)$, *$P$ is a vector of dimension mn.* $P\in\mathbb{R}^{mn}$.

Intuitively, embedding is assignment of position to each vertices.

Definition 3. *Framework $G(p)$ is a graph $G(V,E)$ and a embedding $p:V\to\mathbb{R}^m$.*

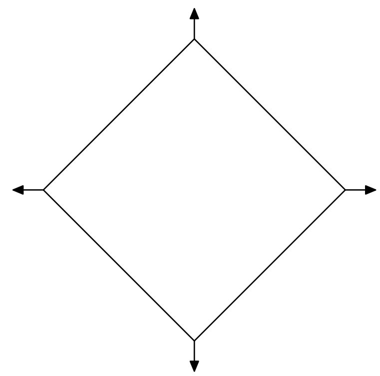

Figure 1: example of infinitesimal motion

Definition 4. *Distance map* $\rho(p)_G : \mathbb{R}^{mn} \to \mathbb{R}^{|E|}$. $\rho(p)_{ij} = ||\mathbf{p}_i - \mathbf{p}_j||^2$

Definition 5. *Jacobian Matrix* $\mathbb{J}(f(x))$ *of a differentiable map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is a m-by-n matrix s.t. the elements are:* $\mathbb{J}(f(x))_{i,j} = \frac{\partial f_i}{\partial x_j}$.

Definition 6. *Rigidity matrix* $R(p)_G = \frac{1}{2}\mathbb{J}(\rho(p)_G)$.

The rigidity matrix has this shape:

$$\left(\begin{array}{cccccccccccc} & & & & ith\text{ column} & & & & jth\text{ column} & & & \\ & & & & & & \vdots & & & & & \\ (ij)th\text{ row} & 0 & \dots & 0 & p_i - p_j & 0 & \dots & 0 & p_j - p_i & 0 & \dots & 0 \\ & & & & & & \vdots & & & & & \end{array}\right)$$

Every row corresponding to one edge in the graph.

The (ij)th row are all 0 except the ith column and jth column.

Definition 7. *Infinitesimal motion* $u : V \to \mathbb{R}^m$ *s.t.* $(\mathbf{u}(i) - \mathbf{u}(j)) \cdot (\mathbf{p}_i - \mathbf{p}_i) = 0$.

Intuitively, infinitesimal motion is assignment of velocity to each vertices s.t. the velocity difference along the direction of each edge is 0.

Figure 1 shows a example of infinitesimal motion.

u is also a vector of $\mathbb{R}^{mn}$. If we write the above formula in matrix form, we can see:

$$R(p) \cdot u = 0$$

so:

$$u \in ker(R(p))$$

Definition 8. *$U(G(p))$ infinitesimal motion space of Framework $G(p)$. By the definition of infinitesimal motion, $U(G(p)) = ker(R(p)_G)$.*

Definition 9. *Framework $G(p)$ is* infinitesimal rigid *if $U(G(p))$ is the infinitesimal motion space of Euclidean transformation.*

2 Outline of proof of laman's theorem - combinatorial characterization of generic rigidity in 2D

Our ultimate goal it to prove the following Laman's Theorem.

Theorem 10. *A graph G is Laman iff G is rigid.*

Figure 2 show the road map of the proof of the theorem. The number on the arrows indicates the Theorem/Proposition related to it.

In **section 3.1**, theorem 17 will show that infinitesimal rigidity is a generic property of graph, which means if there exist one generic framework which is infinitesimal rigid, all generic framework is infinitesimal rigid.

In **section 3.2**, proposition 18 and 20 will show that infinitesimal rigidity and rigidity of framework are related. And theorem 21 will show that generic infinitesimal rigidity is equivalent to generic rigidity for a graph G.

In **section 4** we will prove a graph G is Laman iff G has a generic infinitesimal rigid framework $G(p)$. To do this, we first show a generic infinitesimal rigid framework $G(p)$ must satisfy Laman's condition in theorem 25, then we show a minimal rigid graph must have a Henneberg construction in proposition 31. At last we will show a Henneberg constructable graph must have a generic infinitesimal rigid framework in theorem 33.

3 Generic Rigidity

Definition 11. *A framework $G(p)$ is generic w.r.t property prop iff: $\exists$ neighbourhood $\delta(p) s.t. \forall q \in \delta(p)$, p satisfy prop $\Leftrightarrow$ q satisfy prop.*

Definition 12. *A property is generic w.r.t graph if for all graph G, either all the generic framework $G(p)$ satisfy the property or all do not satisfy.*

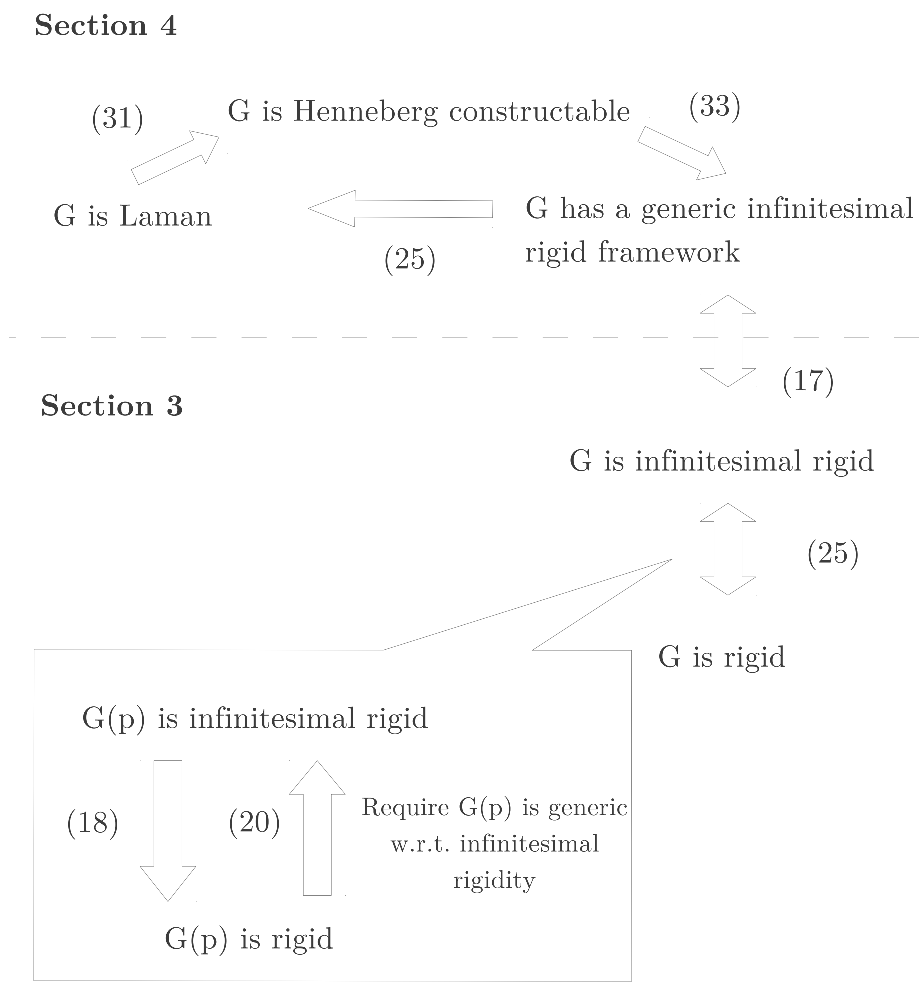

Figure 2: Roadmap of Proof

In this section, we will prove that infinitesimal rigidity is a generic property.

We can also prove that a graph G is infinitesimal rigid $\Leftrightarrow$ G is rigid.

3.1 Genericity w.r.t infinitesimal rigid

Observation 13. *The dimension of the infinitesimal motion space determines infinitesimal rigidity.*

It follows because the Euclidean motion space is always contained in the infinitesimal motion space, and both are linear spaces. So, when the latter has the same dimension as the former, the spaces must be the same.

Proposition 14. *Framework $G(p)$ is infinitesimal rigid in $\mathbb{R}^m$ iff $dim(U(G(p))) = \binom{m+1}{2}$.*

Proof. Since Euclidean transformation in $\mathbb{R}^m$ has dimension $\binom{m+1}{2}$, it follows directly from definition 9 and observation 13. □

To easier understand infinitesimal rigidity, we need another definition of generic w.r.t infinitesimal rigidity:

Definition 15. *A framework $G(p)$ is generic iff $rank(R(p)_g) = \max_q rank(R(q)_g), \forall q : V \to \mathbb{R}^m$.*

Proposition 16. *For a graph G, $\forall p$, $\forall neighbourhood \delta(p)$, there exist $q \in \delta(p)$ s.t. $R(q)_g$ has maximum rank.*

Proof. Suppose embedding p s.t. $rank(R(p)_G)$ is not maximum, then there must exist a set of edge E' s.t. the corresponding rows in $R(p)_G$ is dependent while for some other embedding q, those rows are independent.

For each $E' \subseteq E$, if rows in $R(q)_{E'}$ are independent for some embeddings, we define the set of embedding $\mathcal{X}_{E'} = \{p : \text{rows in } R(q)_{E'} \text{ are dependent}\}$. Notice that $\mathcal{X}_{E'}$ is a curve in $\mathbb{R}^{|V|m}$ (it's the embeddings which make the all minor of $R(q)_{E'}$ zero). Define $\mathcal{X} = \cup_{E' \in E} \mathcal{X}_{E'}$. From the previous discussion, we know that for all p s.t. $rank(R(p)_G)$ is not maximum, $p \in \mathcal{X}$. So the set of embeddings with not maximum rank rigidity matrix is a subset of the union of finitely many curves. So in any neighbourhood of any point, there must be a embedding of maximum rank. □

It can be proved that Definition 15 is equivalent to the definition in section 1 using the Proposition 16.

Exercise 1. *Prove definition 11 is equivalent to definition 15 w.r.t infinitesimal rigid.*

Theorem 17. *For a graph G, if exist one generic framework $G(p)$ which is infinitesimal rigid, then all the generic framework is infinitesimal rigid.*

Proof Idea: The dimension of kernel space of rigidity matrix is related to it's rank. We can first look at the rank of the rigidity matrix and then show that rigidity matrix of all generic framework actually has the same rank.

Proof. From definition 15 we know that the generic framework's rigidity matrix has the maximum rank. So all the generic frameworks' rigidity matrix has the same rank.

From Definition 8:

$$\therefore dim(U(G(p)) = dim(ker(R(p)_G)) = mn - rank(R(p)_G) \tag{1}$$

From the equation (1), all generic framework has the same dimension of infinitesimal motion.

If $\exists G(p)$, $G(p)$ is infinitesimal rigid, which means $dim(U(G(p))) = \binom{m+1}{2}$, then $\forall q$, $G(q)$ is generic w.r.t infinitesimal rigidity, $dim(U(G(q)) = \binom{m+1}{2}$. So all the generic framework is infinitesimal rigid. □

Theorem 17 actually show that infinitesimal rigid is a generic property of Graph.

3.2 Infinitesimal rigid and rigid

Proposition 18. *If a framework G(p) is infinitesimal rigid, then G(p) is rigid.*

Proof. We can prove it by contradiction.

If $G(p)$ is not rigid, by definition, $\forall$ neighbourhood $\delta(p)$, $\exists q \in \delta(p)$, $\rho(q) = \rho(p)$, q is not a Euclidean transformation of p. let $q = p + h$. $\|h\| < \delta$.

We have:

$$\begin{aligned} \rho(p)_{ij} - \rho(q)_{ij} &= \|p_i - p_j\|^2 - \|q_i - q_j\|^2 \\ &= \|h_i - h_j\|^2 - 2(p_i - p_j)(h_i - h_j) = 0 \end{aligned}$$

so:

$$\frac{\|h_i - h_j\|^2}{\|h\|} = 2(p_i - p_j)\frac{(h_i - h_j)}{\|h\|} \tag{2}$$

We can choose a series of δ^i s.t. $\delta^{i+1} < \delta^i$, $\delta^i \to 0$. So the corresponding h^i,s.t. $\|h^i\| \to 0, \frac{(h_i - h_j)}{\|h\|} \to (h_i^* - h_j^*)$,

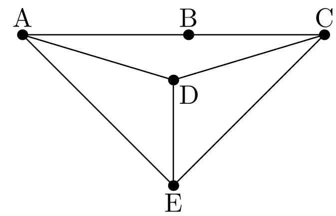

Figure 3: counter example

$$
\begin{aligned}
2(p_i - p_j)(h_i^* - h_j^*) &= \lim_{n\to\infty} 2(p_i - p_j)\frac{(h_i^n - h_j^n)}{\|h^n\|} \\
&= \lim_{n\to\infty} \frac{\|h_i^n - h_j^n\|^2}{\|h^n\|} \\
&= 0
\end{aligned}
$$

since q^i is not Euclidean transformation of p, h^* is not the infinitesimal motion of Euclidean transformation. So $G(p)$ is not infinitesimal rigid.

So proposition 18 is true. □

Remark Unlike Proposition 20, Proposition 18 doesn't require the framework to be generic w.r.t. infinitesimal rigidity.

Remark A natural variation of Proposition 18 is not true.

Proposition 19. *If a framework $G(p)$ is generic w.r.t rigidity and it's rigid, it's not necessarily infinitesimal rigid.*

Proof. This can be proved by giving a simple counter example. In the above framework, A,B and C are collinear. This Framework is generic rigid but is not infinitesimal rigid. □

But if we add some restriction on the framework, this direction can be true.

Proposition 20. *If a framework G(p) is generic w.r.t infinitesimal rigidity and G(p) is rigid, then G(p) is infinitesimal rigid.*

Proof. This can be proved using implicit function theorem.

Since framework $G(p)$ is generic w.r.t infinitesimal rigidity, which means $R(p)_G$ has maximum rank. So no $m+1$ points lie on a hyperplane of dimension m-1. So we can rule out the Euclidean transformation by transforming p into q s.t. the first $(m+1)$ points q_i s.t $q_1 = (0,\dots,0)$,$q_2 = (x'_1, 0, \dots, 0)$,..., and $\|q_i - q_j\| = \|p_i - p_j\|$. There are $\binom{m}{2}$ non-zero value in all the coordinates of $q_i, i = 1,\dots,m+1$. In a neighbourhood $\delta(p)$, we can do the same thing for all $p^* \in \delta(p)$ to get a q^*. The set of q^* is still a smooth manifold of dimension $mn - \binom{m+1}{2}$, $q^* \in \mathbb{R}^{mn-\binom{m+1}{2}}$. So we can re-define the distance map as $\rho^* : \mathbb{R}^{mn-\binom{m+1}{2}} \to \mathbb{R}^{|E|}$, and re-define the rigidity matrix $R^*(q)_G$ as the Jacobian matrix of the new ρ^*. So the $R^*(q)_G$ has only $mn - \binom{m+1}{2}$ columns. $R*(q)_G$ has the same rank as $R(p)_G$. Under this definition, $G(p_0)$ is rigid means in a neighbourhood $\delta(q_0)$, the zero set of function $\rho^*(q)$ is a isolated point $\{p_0\}$,

If p is not infinitesimal rigid, then $rank(R(p)_G) < mn - \binom{m+1}{2}$, so $rank(R^*(q)_G) < mn - \binom{m+1}{2}$. Lets say $rank(R^*(q)_G) = k$, $k < mn - \binom{m+1}{2}$. So we can find a edge set $E' \subseteq E$,$|E'| = k$, E' corresponding to those k independent rows. In $R^*(q)_{E'}$ we can find k independent columns. Since $R^*(q)_{E'}$ is part of the Jacobian of $\rho^*(q)$, each column corresponding to one component in vector q. Let say $y = (y_1, y_2, \dots, y_k)$, y_i corresponding to the kth independent columns; $x = (x_1, \dots, x_{mn-\binom{m+1}{2}-k})$, x_i corresponding to the rest columns. Since rows in $R^*(q)_{E'}$ are independent, the k-by-k submatrix corresponding to y is invertible. By implicit function theorem, in a neighbourhood of q, exists a continuous and differentiable function g s.t. $y = g(x)$ and $\rho^*(x,y)_{E'} = \rho^*(q)_{E'}$. So $G'(p)$ is not rigid, where $G' = G'(V, E')$. Since $R^*(q)_G$ has maximum rank, which means the rows in $R^*(q)_{E'}$ span all the row space of $R^*(q)_G$, so $G(q)$ is not rigid, it's contradicting to our assumption.

So p is infinitesimal rigid. □

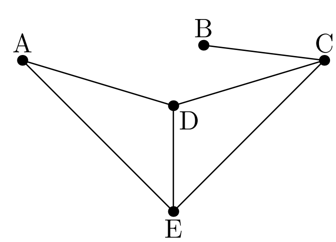

Figure 4: independent edges of figure 3

Remark The genericity condition in proposition 20 is important because without this condition, the non-rigidity of sub-graph $G'(V, E')$ will not imply the non-rigidity of the whole graph G. For example, in figure 3, the independent edge set E' is the lower 2 triangles and one of the upper edge as shown in figure 4. But $G(p)$ is not generic w.r.t.

infinitesimal rigidity so when $G'(p)$ deforming to another framework $G'(q)$, the length of the rest edge will not be preserved.

Infinitesimal rigidity and rigidity are closely related. Actually we can use infinitesimal rigidity to prove that rigidity is a generic property of graph. Further more, we can even show that graph rigidity is equivalent to graph infinitesimal rigidity.

Theorem 21. *If one generic framework $G(p)$ w.r.t. rigidity is rigid, all generic framework $G(q)$ w.r.t. rigidity is rigid.*

Proof Idea: From proposition 16 we know in any neighbourhood there must exist a generic framework w.r.t. infinitesimal rigidity. And from theorem 17 and theorem 20 we know rigidity and infinitesimal rigidity are equivalent for generic framework w.r.t infinitesimal rigidity. We can use this as a tool to prove the theorem.

Proof. Suppose $G(p)$ is a generic framework w.r.t rigidity and $G(p)$ is rigid. By definition, $\exists\delta(p)$ s.t. $\forall q \in \delta(p)$, $G(q)$ is also rigid. By proposition 16, there must exist an embedding $q' \in \delta(p)$ s.t. $G(q')$ is generic w.r.t infinitesimal rigidity. $G(q')$ is rigid, by proposition 20, $G(q')$ is also infinitesimal rigid. By theorem 17, all generic $G(q)$ w.r.t infinitesimal rigidity is infinitesimal rigid.

$\forall G(p^*)$ which is generic w.r.t rigidity, there is a neighbourhood $\delta_{p*}(p^*)$ s.t. $\forall q \in \delta_{p^*}(p^*)$, $G(q)$ and $G(p^*)$ has same rigidity. And we can always find a embedding $q^* \in \delta_{p^*}(p^*)$ s.t. $G(q^*)$ is generic w.r.t. infinitesimal rigidity. And $G(q^*)$ is infinitesimal rigid, therefore it's rigid. So $G(p*)$ is rigid. □

Theorem 21 actually shows that rigidity is a generic property of graph.

Theorem 22. *A graph G is rigid iff G is infinitesimal rigid.*

Proof Idea: From proposition 16 we know in any neighbourhood there must be exist generic framework w.r.t. infinitesimal rigidity. And from theorem 17 we know generic framework w.r.t. infinitesimal rigidity is related to infinitesimal rigidity of graph. So we can use this framework as a bridge between rigidity and infinitesimal rigidity.

Proof. This can be proved in a very similar way of the proof of Theorem 21.

If a graph G is rigid, means $\exists p$, $G(p)$ is generic w.r.t. rigid and is rigid. And $\exists\delta(p)$, $\forall q \in \delta(p)$, $G(q)$ is rigid. By proposition 16, $\exists q' \in \delta(p)$, $G(p)$ is generic w.r.t. infinitesimal rigidity and $G(p)$ is rigid. By proposition 20 $G(p)$ is also infinitesimal rigid. So by theorem 17, G is infinitesimal rigid.

If a graph G is infinitesimal rigid. Assume G is not rigid, we can find a embedding p s.t. $G(p)$ is not rigid and $\forall q \in \delta(p)$ is not rigid. By proposition 16, $\exists q' \in \delta(p)$, $G(p)$ is generic w.r.t. infinitesimal rigidity and $G(p)$ is infinitesimal rigid. By proposition 18, $G(p)$ is also rigid, which is contradicting to our assumption. So G is rigid. □

4 Laman's Theorem

Definition 23. *A graph $G(V, E)$ is minimal rigid if:*

1. $|E| = 2|V| - 3$
2. $\forall G'(V', E) \in G$, $|E'| \leq 2|V'| - 3$.

Definition 24. *A graph $G(V, E)$ is Laman if $\exists G'(V, E') \in G$ s.t. G' is minimal rigid.*

4.1 Infinitesimal rigidity and laman graph

Theorem 25. *If a framework $G(p)$ is infinitesimal rigid in $\mathbb{R}^2$, G is Laman.*

Proof Idea: Form proposition 14 we know that the rigidity matrix of all infinitesimal rigid framework $G(p)$'s has same rank, which means there is a set of rows in the rigidity matrix which are independent. And each row in rigidity matrix corresponding to one edge. So if we only look at those independent edges we will find a minimal rigid graph.

Proof. $G(p)$ is infinitesimal rigid means that the rigidity matrix $R(p)_G$ has rank $2n - 3$, where $n = |V|$. So we can make a new rigidity matrix $R(p)_{G'}$ of $2n - 3$ independent rows by removing row from $R(p)_G$. Since every row in the rigidity matrix corresponding to an edge in the graph. Removing rows from $R(p)_G$ is equivalent to removing edges from G. So the matrix $R(p)_{G'}$ corresponding to a sub-graph $G' \subseteq G$ with $2n - 3$ edges.

For all $G''(V'', E'') \subseteq G'$, rows in $R(p)_{G''}$ are the corresponding rows of $R(p)_{G'}$. So all the rows in $R(p)_{G''}$ are independent. So:

$$rank(R(p)_{G''}) = |E''|$$

From equation (1):

$$rank(R(p)_{G''}) = 2n - dim(U(G''(p)) \leq 2n - dim(D(G''(p)) = 2|V''| - 3$$

So $|E''| \leq 2|V''| - 3$. □

4.2 Laman graph and Henneberg construction

Definition 26. *Henneberg construction is a graph construction algorithm:*

1. *It start at a single edge with 2 vertices.*
2. *We can construction a new graph from a existing graph using on of the following 2 steps:*
   - *type 1. Add a vertex* $\mathbf{a}$ *and connect it to 2 different vertices of the existing graph.*
   - *type 2. Add a vertex* $\mathbf{a}$ *and connect it to 3 different vertices* $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ *of the existing graph. Then remove a edge* $(\mathbf{b}_1, \mathbf{b}_2)$.

Lemma 27. *If graph G is minimal rigid, no vertex in G is of degree 1.*

Proof. Assume G is minimal rigid and vertex $\mathbf{a} \in G$ is degree 1, consider the rest part of the graph $G'(V', E')$ where $V' = V/\mathbf{a}$. It's obvious that $|V'| = |V| - 1$, $|E'| = |E| - 1$. Since G is minimal rigid, $|E| = 2|V| - 3$.

so:

$$|E'| = |E| - 1 = 2|V| - 3 - 1 = 2|V'| - 2 > 2|V'| - 3$$

which contradicting our assumption.

□

Lemma 28. *If graph G is minimal rigid, there is at least 1 vertex in G whose degree is less then 4.*

Proof. Assume G is minimal rigid and all vertices degree is greater or equal to 4, then:

$$|E| = \frac{\sum_{i \in V} degree(i)}{2} \geq 2|V|$$

which contradicting our assumption. □

Lemma 29. *Let $G(V, E)$ a minimal rigid graph. $K(V_k, E_k) \subset G$,$L(V_L, E_L) \subset G$, $|E_k| = 2|V_K| - 3$,$|E_L| = 2|V_L| - 3$, $|V_k \cap V_L| \geq 2$. Then the graph $G'(V', E') = K \cup L$ satisfy: $|E'| = 2|V'| - 3$.*

Proof.

$$|E'| \geq |E_L| + |E_K| - E_{L\cap K} \geq 2|V_L| - 3 + 2|V_K| - 3 - (2|V_{L\cap K}| - 3) = 2|V_{K\cup L}| - 3 = 2|V'| - 3$$

By assumption $|E'| \leq 2|V'| - 3$. So $|E'| = 2|V'| - 3$ □

Lemma 30. *If graph $G(V, E)$ is minimal rigid and $\mathbf{a} \in V$ is of degree 3, its 3 neighbours are $\mathbf{b}_i, i \in \{1, 2, 3\}$, then there is at least one pair of $(\mathbf{b}_i, \mathbf{b}_j), i, j \in \{1, 2, 3\}$, there are no sub-graph $L(V_L, E_L) \subseteq G$ s.t. $\mathbf{b}_i \in V_L, \mathbf{b}_j \in V_L, |E_L| = 2|V_L| - 3$*

Proof. If all 3 pairs has such a Graph. Let's say they are $L_i(V_i, E_i)$, $\mathbf{b}_1, \mathbf{b}_2 \in L_1$, $\mathbf{b}_2, \mathbf{b}_3 \in L_2$, $\mathbf{b}_3, \mathbf{b}_1 \in L_3$, and $|E_i| = 2|V_i| - 3$.

If $\forall i, j \in \{1, 2, 3\}$, $|V_i \cap V_j| = 1$, then:

$$|E_{1\cup 2\cup 3}| = \sum_i |E_i| = 2\sum_i (|V_i|) - 9 = 2(|V_{1\cup 2\cup 3}| + 3) - 9 = 2(|V_{1\cup 2\cup 3}|) - 3$$

If two of L_i, let say L_1 and L_2, $|V_1 \cap V_2| \geq 2$. By lemma 29, $|E_{1\cup 2}| = 2|V_{1\cup 2}| - 3$. Then obviously $|V_{1\cup 2} \cap V_3| \geq 2$. By lemma 29:

$$|E_{1\cup 2\cup 3}| = 2|V_{1\cup 2\cup 3}| - 3$$

In both case, $|E_{1\cup 2\cup 3}| = 2|V_{1\cup 2\cup 3}| - 3$. Let's look at the graph $G'(V', E') = L_1 \cup L_2 \cup L_3 \cup \mathbf{a}$.

$$|V'| = |V_{1\cup 2\cup 3}| + 1$$
$$|E'| = |E_{1\cup 2\cup 3}| + 3$$

So:
$$|E'| = 2|V_{1\cup 2\cup 3}| - 3 + 3 = 2|V'| - 2 > 2|V'| - 3$$

which contradicting our assumption. □

Proposition 31. *A minimal rigid graph $G(V, E)$ has a Henneberg construction.*

Proof. We can prove it by induction.

1. If $|V| = 2$, the graph is a single edge. It's the start point of Henneberg construction.

2. Suppose all minimal rigid graph with k vertices have a Henneberg construction. Let's consider a minimal rigid graph $G(V, E)$ with k+1 vertices.

   By Lemma 27 and 28, there is at least one vertex of degree 2 or 3.

   If there is a vertex $\mathbf{a}$ of degree 2, consider the sub-graph without $\mathbf{a}$: $G'(V', E')$, where $V' = V/\mathbf{a}$, $|V'| = |V| - 1 = k$, $|E'| = |E| - 2$. Since G is minimal rigid, so:

   $$|E'| = |E| - 2 = 2|V| - 3 - 2 = 2|V'| - 3$$

   $\forall G''(V'', E'') \subset G'$, also $G'' \subset G$, by assumption:

   $$|E''| \leq 2|V''| - 3$$

   So G' is also minimal rigid.

   By induction, G' has a Henneberg construction, and G can be extended from G' by type 1 step. So G also has a Henneberg construction.

   If there is a vertex $\mathbf{a}$ of degree 3. Let's say the three edge attached to $\mathbf{a}$ are $(\mathbf{a}, \mathbf{b}_1)$,$(\mathbf{a}, \mathbf{b}_2)$,$(\mathbf{a}, \mathbf{b}_3)$. By lemma 30, there is a pair, let's say $(\mathbf{b}_1, \mathbf{b}_2)$, there are no graph $L \subseteq G$ s.t. $\mathbf{b}_1, \mathbf{b}_2 \in L$ and $|E_L| = 2|V_L| - 3$. Observe that edge $(\mathbf{b}_1, \mathbf{b}_2) \notin E$, otherwise the graph $L(\{\mathbf{b}_1, \mathbf{b}_2\}, \{(\mathbf{b}_1, \mathbf{b}_2)\})$ satisfy $|E_L| = 2|V_L| - 3$. So we can get a new graph $G'(V', E')$ by removing vertex $\mathbf{a}$ from G and add edge $(\mathbf{b}_1, \mathbf{b}_2)$. So:

   $$|E'| = |E| - 3 + 1 = 2|V| - 3 - 2 = 2|V'| - 3$$

   $\forall G''(V'', E'') \subseteq G'$, if $(\mathbf{b}_1, \mathbf{b}_2) \notin E''$, then $G'' \subseteq G$, by assumption $|E''| \leq 2|V''| - 3$. If $(\mathbf{b}_1, \mathbf{b}_2) \in E''$, the graph $G'''(V'', E''/(\mathbf{b}_1, \mathbf{b}_2)) \in G$. By assumption and lemma 30, $|E''/(\mathbf{b}_1, \mathbf{b}_2)| \leq 2|V''| - 2$. So $|E''| \leq 2|V''| - 3$.

   So G' is also minimal rigid.

   By induction, G' has a Henneberg construction, and G can be extended from G' by type 2 step. So G also has a Henneberg construction.

Combine 1 and 2, all minimal rigid graph $G(V, E)$ has a Henneberg construction.

□

Proposition 32. *If graph $G(V, E)$ has a Henneberg construction, it's minimal rigid.*

Proof. It's very easy to proved by induction. □

4.3 Henneberg construction and Infinitesimal rigidity

Theorem 33. *If a graph G has a Henneberg construction, then G is infinitesimal rigid in $\mathbb{R}^2$.*

Proof Idea: By theorem 17, G is infinitesimal rigid in $\mathbb{R}^2$ means exist one embedding $p \in \mathbb{R}^{2n}$ s.t framework $G(p)$ is generic w.r.t infinitesimal rigid and is infinitesimal rigid. So to prove theorem 33, we just need to find such a embedding for each graph with Henneberg construction.

Lemma 34. *If a inhomogeneous quadratic function $f(x, y) = ax^2+bxy+cy^2+dx+ey+f$ vanish at 3 non-collinear points $p_i \in \mathbb{R}^2$ and the midpoint of the pair $\frac{p_i+p_j}{2}$, then the function vanish at any point.*

Proof. Since p_i are non-collinear, so we can present any point $p \in \mathbb{R}^2$ as $p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$, $\sum_i \alpha_i = 1$. Using this barycentric coordinate we can translate function f into a homogeneous function of α_i. Then plug $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$ into the function we can easily see all the coefficient are zero. $\square$

Lemma 35. *If 3 points $p_i \in \mathbb{R}^2$ and 3 vector $u_i \in \mathbb{R}^2$ s.t. $(p_i - p_j) \cdot (u_i - u_j) = 0, \forall i, j$, then:*

$$f(p) = \begin{vmatrix} p - p_1 & (p - p_1) \cdot u_1 \\ p - p_2 & (p - p_2) \cdot u_2 \\ p - p_3 & (p - p_3) \cdot u_3 \end{vmatrix} \text{ vanish at any value of } p$$

Proof. Observe that $f(p)$ is a quadratic inhomogeneous function of p.

$$f(p_1) = \begin{vmatrix} \mathbf{0} & 0 \\ p_1 - p_2 & (p_1 - p_2) \cdot u_2 \\ p_1 - p_3 & (p_1 - p_3) \cdot u_3 \end{vmatrix} = 0$$

Also, $f(p_2) = 0$, $f(p_3) = 0$.

$$f(\frac{p_1 + p_2}{2}) = \begin{vmatrix} \frac{p_1 - p_2}{2} & (p_1 - p_2) \cdot u_1 \\ \frac{p_2 - p_1}{2} & (p_2 - p_1) \cdot u_2 \\ \vdots & \vdots \end{vmatrix}$$

The sum of first and second row of above determinant is 0, so

$$f(\frac{p_1+p_2}{2}) = 0$$

Also, $f(\frac{p_2+p_3}{2}) = 0$, $f(\frac{p_1+p_3}{2}) = 0$.

By lemma 4.3 $f(p) = 0$ for all value of p.

$\square$

Proof of Theorem 33:

We can prove it by induction.

1. If $|V| = 3$, the graph is a triangle. It's infinitesimal rigid.

2. Suppose all graph with k vertices which have a Henneberg construction is infinitesimal rigid. Let's consider graph $G(V, E)$ with k+1 vertices and has Henneberg construction.

   If G is extended from a Henneberg constructable graph $G'(V', E')$ by type 1 step, which means there is a vertex $\mathbf{a} \in V$, $\mathbf{a} \notin V'$, $degree(a) = 2$. By induction, G' is infinitesimal rigid, there is a embedding $p \in \mathbb{R}^{2k}$ s.t. $dim(ker(R(p)_{G'})) = 3$. I can make a new embedding $p^* \in \mathbb{R}^{2(k+1)}$, $p^*_i = p_i, i \neq a$. Let's say $\mathbf{a}$'s neighbour in G are $\mathbf{b}_1, \mathbf{b}_2$. Adding $\mathbf{a}$ into G' is equivalent to adding 2 equations:

   $$(p_a - p_1) \cdot (u_a - u_1) = 0$$

   $$(p_a - p_2) \cdot (u_a - u_2) = 0$$

   If we choose p^*_a not lying on the line of p_1 and p_2, this system has only one solution.

   So $dim(ker(R(p^*)_G)) = dim(ker(R(p)_{G'})) = 3$. $G(p^*)$ is infinitesimal rigid.

   If G is extended from a graph $G'(V', E')$ by type 2 step, which means there is a vertex $\mathbf{a} \in V$, $\mathbf{a} \notin V'$, $degree(\mathbf{a}) = 3$. Let's say the 3 neighbour of a are $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, and the edge $(\mathbf{b}_1, \mathbf{b}_2) \in E'$, $(\mathbf{b}_1, \mathbf{b}_2) \notin E$, the graph $G'''(V'', E''')$ is made by removing $(\mathbf{b}_1, \mathbf{b}_2)$ from G'.

   By induction, G' is infinitesimal rigid, there is a embedding $p \in \mathbb{R}^{2k}$ s.t. $dim(ker(R(p)_{G'})) = 3$. If we choose a new embedding $p^* \in \mathbb{R}^{2(k+1)}$, $p^*_i = p_i, i \neq a$, the rigidity matrix $R(p^*)_{G'''}$ is $R(p)_{G'}$ removing one row. Since G' is minimal rigid, which means all the rows in $R(p)_{G'}$ are independent, so all the rows in $R(p^*)_{G'''}$ are also independent. So $dim(ker(R(p^*)_{G'''})) = 4$. Let's say the basis vector of the kernel space of $R(p^*)_{G'''})$ is $\lambda^1, \lambda^2, \lambda^3, \lambda^4$, where $\lambda^i, i = 1, 2, 3$ are the basis of Euclidean motion. $u_i = \sum_k \alpha_k \lambda^k_i$.

Framework $G(p^*)$ is $G'''(p^*)$ adding 3 edges $(\mathbf{a}, \mathbf{b}_i)$, which is equivalent to adding 3 equations into the system:

$$(p_a - p_i) \cdot (u_a - u_i) = 0$$

We can write these equations in matrix form:

$$\begin{pmatrix} p_a - p_1 \\ p_a - p_2 \\ p_a - p_3 \end{pmatrix} u_a = \begin{pmatrix} (p_a - p_1) \cdot u_1 \\ (p_a - p_2) \cdot u_2 \\ (p_a - p_3) \cdot u_3 \end{pmatrix}$$

This system has solution iff:

$$\begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot u_1 \\ p_a - p_2 & (p_a - p_2) \cdot u_2 \\ p_a - p_3 & (p_a - p_3) \cdot u_3 \end{vmatrix} = 0$$

Since $u_i = \sum_k \alpha_k \lambda_i^k$, so:

$$\begin{aligned} \begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot u_1 \\ p_a - p_2 & (p_a - p_2) \cdot u_2 \\ p_a - p_3 & (p_a - p_3) \cdot u_3 \end{vmatrix} &= \begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot \sum_k \alpha_k \lambda_1^k \\ p_a - p_2 & (p_a - p_2) \cdot \sum_k \alpha_k \lambda_2^k \\ p_a - p_3 & (p_a - p_3) \cdot \sum_k \alpha_k \lambda_3^k \end{vmatrix} \\ &= \sum_k \alpha_k \begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot \lambda_1^k \\ p_a - p_2 & (p_a - p_2) \cdot \lambda_2^k \\ p_a - p_3 & (p_a - p_3) \cdot \lambda_3^k \end{vmatrix} \end{aligned}$$

Since $\lambda_i, i = 1, 2, 3$ are basis of Euclidean motion and lemma 35:

$$\begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot \lambda_1^k \\ p_a - p_2 & (p_a - p_2) \cdot \lambda_2^k \\ p_a - p_3 & (p_a - p_3) \cdot \lambda_3^k \end{vmatrix} = 0, k = 1, 2, 3$$

So:

$$\begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot u_1 \\ p_a - p_2 & (p_a - p_2) \cdot u_2 \\ p_a - p_3 & (p_a - p_3) \cdot u_3 \end{vmatrix} = \alpha_4 \begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot \lambda_1^4 \\ p_a - p_2 & (p_a - p_2) \cdot \lambda_2^4 \\ p_a - p_3 & (p_a - p_3) \cdot \lambda_3^4 \end{vmatrix}$$

The solution of equation $\begin{vmatrix} p_a - p_1 & (p_a - p_1) \cdot \lambda_1^4 \\ p_a - p_2 & (p_a - p_2) \cdot \lambda_2^4 \\ p_a - p_3 & (p_a - p_3) \cdot \lambda_3^4 \end{vmatrix} = 0$ is a curve in $\mathbb{R}^2$. If we choose p_a no on this curve, then $\alpha_4 = 0$, equation 2 has only one solution.

So $dim(ker(R(p^*)_G)) = 3$, $G(p^*)$ is infinitesimal rigid. G is infinitesimal rigid.

□