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Lecture 22-26

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1 Introduction

In this section, let's begin with some basic definitions related to universal rigidity. Then we will show that (1) universal rigidity is stronger than global rigidity, and (2) it is not a generic property for a graph. $V(G)$ and $E(G)$ are respectively the vertex set and the edge set of a simple edge-weighted graph G, where each edge (i, j) has a positive weight d_{ij}.

Definition 1 *An d-*configuration *p is a finite set of points p^1, ... ,p^n in $\mathbb{R}^d$ whose affine hull is $\mathbb{R}^d$. A* framework*, denoted by $G(p)$, in $\mathbb{R}^d$ is a simple graph $G = (V, E)$ on the vertices $1, ..., n$ together with an d- configuration p, where each vertex i of G is located at point p^i.*

For a given graph G the distance map $\rho(p)_G : \mathbb{R}^{mn} \to \mathbb{R}^{|E|}$ is the function assigning to each edge of G its squared edge length in the framework, and the component of $\rho(p)_G$ in the direction of an edge $\{i, j\}$ is $\rho(p)_{ij} = |p_i - p_j|$.

Definition 2 *Two frameworks $G(p)$ in $\mathbb{R}^d$ and $G(q)$ in $\mathbb{R}^s$ are said to be* equivalent *if $\rho(p)_G = \rho(q)_G$. Two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^d$ are said to be* congruent *if they are related by an element of group of Eucl(d) of rigid motions of $\mathbb{R}^d$, which can be represented as $p = q$.*

Definition 3 *A d-configuration is* proper *if it does not lie in any affine subspace of $\mathbb{R}^d$ of dimension less than d. It is* generic *if its first d coordinates (i.e., the coordinates not constrained to be 0) do not satisfy any algebraic equation with rational coefficients.*

Definition 4 *A framework $G(p)$ in $\mathbb{R}^d$ is said to be* (locally) rigid *if for some $\varepsilon > 0$, there does not exist a framework $G(q)$ in $\mathbb{R}^d$ which meets $\rho(p)_G = \rho(q)_G$ such that $\left|p^i - q^i\right| < \varepsilon$ for all $i = 1, ..., n$. A framework $G(p)$ in $\mathbb{R}^d$ is said to be* globally rigid *if there does not exist a framework $G(q)$ in the same space $\mathbb{R}^d$ which is equivalent to $G(p)$. A framework $G(p)$ in $\mathbb{R}^d$ is said to be* universally rigid *if there does not exist a framework $G(q)$ in $\mathbb{R}^s$, which is equivalent to $G(p)$, for any s, $1 \le s \le n - 1$.*

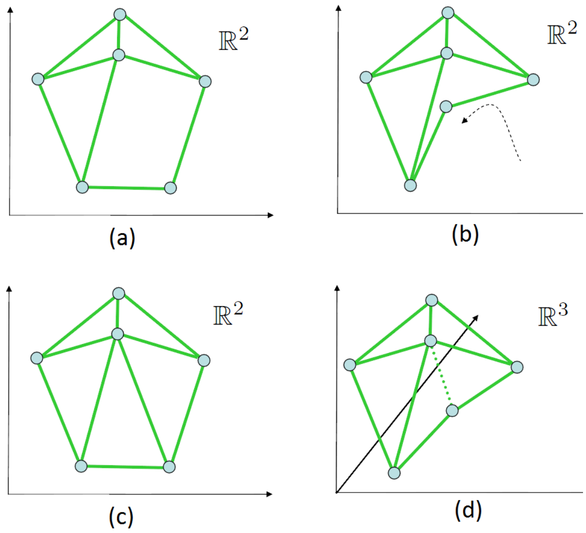

Figure 1: As shown in (a), there is a generic (locally) rigid graph in a plane, which rules out continuous flexes in $\mathbb{R}^2$ that preserve edge lengths. But there is another equivalent graph in the plane, as given in (b). By adding an additional edge, we can achieve a generic globally rigid graph, as shown in (c), where the lengths fully determine the embedding in the smaller space $\mathbb{R}^2$. But in a high dimensional space, such as $\mathbb{R}^3$, the embedding is not unique, as illustrated in (d). Thus, global rigidity is stronger than local rigidity. And universal rigidity, in turn, stronger than local rigidity.

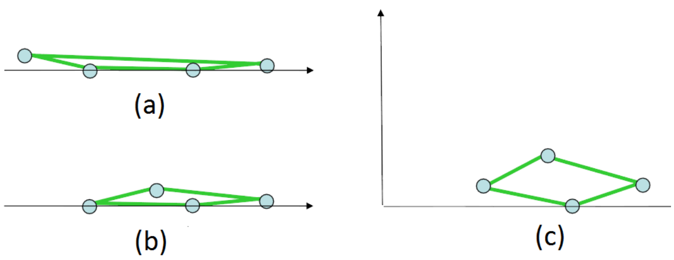

Figure 2: In fig. (a), 4-cycle in the line is universally rigid. But in fig. (b), it is not universally rigid, and it can deform to another equivalent but not congruent configuration in $\mathbb{R}^2$ while preserving the edge length, as given in (c).

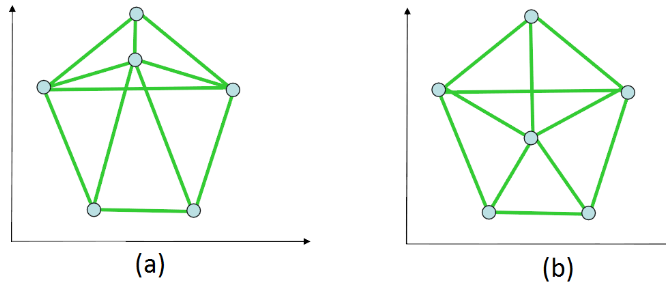

Figure 3: The same graph, but different type of rigidity. (a) is universally rigid, (b) is globally rigid, but not universally rigid.

It immediately follows that global rigidity implies local rigidity but the converse is not true. On the other hand, universal rigidity, in turn, implies global rigidity but the converse is not true either.

Unlike local and global rigidity, universal rigidity is not a generic property of a graph. Fig. 2 and Fig. 3 give two examples for the fact that some frameworks are universally rigid and others are not for the same graph[3]. On the other hand, some graphs, such as a simplex, and any d- lateration graph are generically universally rigid. A d-lateration graph has the property that one can order its vertices such that the first $d+2$ vertices are part of a simplex in G and each following vertex is connected to $d+1$ previous vertices. However, it should be emphasized that universal rigidity is still a generic property of a framework. As in fig. 3(a), the neighborhood of the universally rigid framework is still universally rigid. In fig. 4, graphs increase types of rigidity when more edges are added.

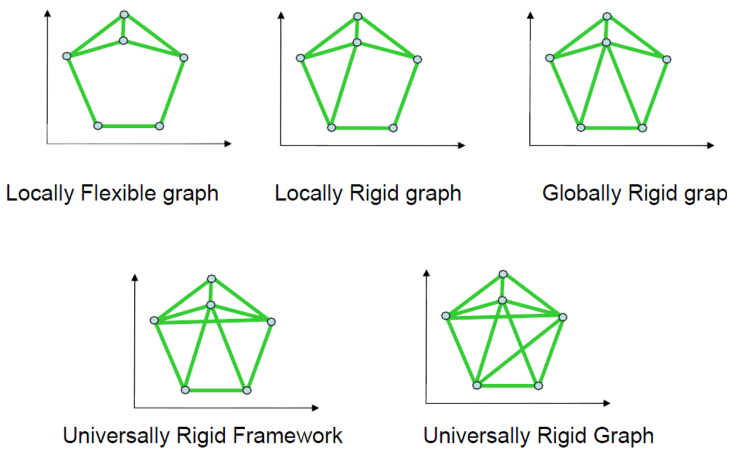

Figure 4: Planar graph embeddings with increasing types of rigidity [3]. From left to right, generically locally flexible graph, generically locally rigid graph, generically globally rigid graph, generic universally rigid framework, generically universally rigid graph.

2 Necessary conditions for universal rigidity

A framework $G(p)$ in $\mathbb{R}^d$ is said to be dimensionally rigid if there does not exist a framework $G(q)$ in $\mathbb{R}^s$, which is equivalent to $G(p)$, for any $s \geq d+1$. Alfakih [4] proved that a given framework $G(p)$, not necessarily generic, is universally rigid if it is both globally rigid and dimensionally rigid. Alfakih also presented in [4] the following sufficient condition for dimensional rigidity of frameworks.

Lemma 1 *Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^d$ for some $d \leq n-2$, and let Z be the Gale matrix corresponding to $G(p)$. Further, let $\bar{r} = n-1-d$ and let z_i^T denote the ith row of Z. If there exists an $\bar{r} \times \bar{r}$ symmetric positive definite matrix Ψ such that $z_i^T \Psi z_j = 0$, $\forall (i,j) \notin E$, then $G(p)$ is dimensionally rigid.*

We are more interested in the case when the framework is generic. Theorem 2 to theorem 4 are a list of main theorems [1] for the necessary conditions for universal rigidity. Let $G(p)$ be a given generic framework in $\mathbb{R}^d$ with n vertices for some $d \leq n-1$.

Theorem 2 *If $G(p)$ is dimensionally rigid, then $G(p)$ is universally rigid.*

Theorem 3 *If there exists an $\bar{r} \times \bar{r}$ symmetric positive definite matrix Ψ such that $z_i^T \Psi z_j = 0$, $\forall (i,j) \notin E$, then $G(p)$ is universally rigid.*

Theorem 4 *If there exists a stress matrix S for $G(p)$ such that S is positive semidefinite (PSD) with rank $\bar{r} = n-1-d$, then $G(p)$ is universally rigid.*

The section 4 is dedicated to a proof of theorem 2. theorem 3 is an immediate corollary of lemma 1 and theorem 2 [1]. For theorem 4, we will shows that it is equivalent to theorem 2 in the next section.

Since universal rigidity implies global rigidity, it is interesting to note that whereas S in the sufficient and necessary condition for generic global rigidity is required to be non-singular [5, 6], S in the sufficient [1] and necessary condition [3] for generic universal rigidity is required to satisfy the stronger notion of positive definiteness.

The framework $G(P)$ is in *general* position in $\mathbb{R}^d$ if no $(d+1)$ points of $p_1, ..., p_n$ are affinely dependent. For example, points are in general position in $\mathbb{R}^2$ if no 3 of them are collinear. Note that whether or not n rational points are in general position can be checked in time polynomial in n for any fixed dimension d, while the generic position condition is uncheckable. Recently, Alfakih [9] proved that theorem 4 is still true if the assumption of a framework in generic position is replaced by the weaker assumption of a framework in general position. Let's list this theorem.

Theorem 5 *Let $G(p)$ be a framework of n vertices in* general *position in $\mathbb{R}^d$, $d <= n-1$. Then $G(p)$ is universally rigid if there exits a stress matrix S such that S is PSD and rank of S is $\bar{r}$.*

3 Gale matrix and stress matrix

Since the necessary conditions for universal rigidity are given by the Gale matrix (theorem 3)and the stress matrix (theorem 4), we will show the definition of them at the first part of this section. Then we will state and prove lemma 6 to discover relationship between Gale matrix and stress matrix which finally servers to the proof of the equivalence of theorem 3 and "if" part of theorem 4.

Let e denote the vector of all 1's in $\mathbb{R}^n$, and $P = [p^1, p^2, ..., p^n]$ be the position matrix. Assumes the centroid of the points $p^1, ..., p^n$ coincides with the origin, we have $P^T e = 0$. P can also be considered as n vectors, it has full row rank if it spans a d-dimensional space. Under affine transformations, e^T provides a basis for translation while P provides a basis for rotation. We can define extended position matrix of $G(p)$ as a $(d+1) \times n$ matrix

$$A = \begin{bmatrix} P \\ e^T \end{bmatrix}. \tag{1}$$

Let $\overline{r} = n - 1 - d$. The *Gale matrix* Λ corresponding to $G(p)$ is defined as the $n \times \overline{r}$ matrix whose columns form a basis for the null space of A. We take advantage of the fact that Λ is not unique, we use a special sparse Gale matrix Z for convenience. Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \tag{2}$$

where Λ_1 is $(d+1) \times \overline{r}$ and Λ_2 is $\overline{r} \times \overline{r}$. Since Λ has full column rank, we can assume without loss of generality that Λ_2 is non-singular. Then Z is defined as

$$Z := \Lambda\Lambda_2^{-1} = \begin{bmatrix} \Lambda_1\Lambda_2^{-1} \\ I_{\overline{r}} \end{bmatrix}. \tag{3}$$

An equilibrium stress for $G(p)$ is a vector $\omega = \omega_{ij}$ in $\mathbb{R}^{|E|}$ such that $\sum_{j:(i,j)\in E} \omega_{ij}(p^i - p^j) = 0$ for all $i = 1, ..., n$. Given an equilibrium stress, let $S = (s_{ij})$ be the $n \times n$ symmetric matrix defined by:

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if}(i,j) \in E, \\ 0 & \text{if}(i,j) \notin E, \\ \sum\limits_{k:(i,k)\in E} \omega_{ik} & \text{if} i = j. \end{cases} \tag{4}$$

S is called the *stress matrix* associated with ω. By definition, stress matrix can represent the "force" of each edge while keeps summation of all the forces of each vertex equals zero. The following lemma [1] shows that the Gale matrix Z corresponding to $G(p)$ and the stress matrix S associated with an equilibrium stress ω of $G(p)$ are closely related.

Lemma 6 *Let S be the stress matrix and Z be the Gale matrix. Then we can find some $\overline{r} \times \overline{r}$ symmetric matrices Ψ to meet $S = Z\Psi Z^T$. Furthermore, let z_i^T be the ith row of Z. If Ψ' is any $\overline{r} \times \overline{r}$ symmetric matrix such that $z_i^T \Psi' z_j = 0$ for all $(i,j) \notin E$, then $Z^T \Psi' Z$ is a stress matrix associated with an equilibrium stress ω for $G(p)$.*

Proof: Let S be the stress matrix associated with an equilibrium stress ω for $G(p)$. Then
$(e^T S)_{ij} = \sum_k e_{ik}^T s_{kj} = \sum_k s_{kj} = 0$, and
$(PS)_{,j} = \sum_k p^k s_{kj} = \sum_{k:(i,k)\in E} p^k s_{kj} + p^j s_{jj} = \sum_{k:(i,k)\in E} p^k(-\omega_{kj}) + p^j \sum_{k:(i,k)\in E} \omega_{ik} = 0.$
Hence $AS = 0$ and the columns of S belong to the kernel of A which is spaned by Z. Thus $S = ZL$ for some $\overline{r} \times \overline{r}$ matrix L. But since S is symmetric and Z has full column rank, it follows that the columns of L^T also belong to the kernel of A. Therefore $S = Z\Psi Z^T$ for some $\overline{r} \times \overline{r}$ symmetric matrix Ψ.

On the other hand, let Ψ' be any $\overline{r} \times \overline{r}$ symmetric matrix such that $z_i^T \Psi' z_j = 0$ for all $(i,j) \notin E$ and let $S' = (s'_{ij}) = Z\Psi' Z^T$. Then $s'_{ij} = 0$ for all $(i,j) \notin E$ and $e^T S' = 0$ and $P^T S' = 0$. Therefore, for $i \neq j$, $\omega_{ij} = -s'_{ij}$ is an equilibrium stress for $G(p)$ and the result follows. ■

With lemma 6, the "if" part of theorem 4 is equivalent to theorem 3.

Also, the following result [1, 5] follows immediately from the Lemma 6, which reveal the maximal rank of stress matrix,

Corollary 7 *Let S be the stress matrix, then $rank(S) \leq \overline{r} = n - 1 - d$.*

4 Proof of theorem 2

As stated in the at the very beginning in sec. 2, it has been proved that a given framework $G(p)$ is universally rigid if it is both globally rigid and dimensionally rigid [4]. So in order to prove theorem

2, we only need to prove if a framework is not globally rigid then it is dimensionally flexible. Without lose of generality, we will assume the framework is rigid. Otherwise, we will add some edges to make the new framework rigid, but not redundantly rigid. According to Henrickson's theorem [11], the new framework is still globally flexible. We will prove that if a framework is rigid but not globally rigid, then there exists another equivalent but not congruent framework in higher dimensional space. To uniquely represent a framework, we use the projected Gram matrix.

Let B be the Gram matrix, i.e., $B = PP^T$, and let V be an $n \times (n-1)$ matrix such that $V^T e = 0$, $V^T V = I_{n-1}$. V^T is the projection matrix which reduce one dimension by ruling out the translation. The $(n-1) \times (n-1)$ *projected Gram matrix* X is defined by $X := V^T BV = V^T P^T PV$. Let E_{ij} be the $n \times n$ matrix with ones in the (i, j)th and (j, i)th entries and zeros elsewhere. It forms a basis for the Gram matrix. For each missing edge, define $M^{ij} := -1/2V^T E^{ij} V$ in the projected space. For each $(i, j) \notin E$, define the matrix

$$M(y) = \sum_{(i,j)\notin E} y_{ij} M^{ij}. \tag{5}$$

Let X_1 be the projected Gram matrix corresponding to p_1, the set of all frameworks $G(q)$ that are equivalent to $G(p_1)$ is given by

$$\{G(X(y)) : y \in R^{\overline{m}} \text{ and } X(y) = X_1 + M(y) \geq 0\}. \tag{6}$$

where $\overline{m}$ is number of edges of G. The following three lemmas are served to prove that a framework is not $d + 1$- connected if a it is not globally rigid, that is, there exits a nonzero y in $\mathbb{R}^{\overline{m}}$ such that $X(y) = X_1 + M(y) \geq 0$ and $rank(X) = d$.

Let $\kappa_V(X)$ be the linear map defined on the set of symmetric matrices of order $n - 1$ by:

$$\kappa_V(X) := \text{diag}(VXV^T)ee^T + ee^T\text{diag}(VXV^T) - 2VXV^T. \tag{7}$$

It is has been proved that the set $M_{ij} : (i, j) \notin E$ forms a basis for the kernel of $H \circ \kappa_V(\cdot)$, where H is the adjacency matrix of graph G and $H \circ \kappa_V(X)$ denotes the Hadamard product of matrices H and $\kappa_V(X)$ [7]. Alfakih [8] also proved the following technical lemma which establishes a relationship between the Gale matrix Z and the projected Gram matrix X.

Lemma 8 *Let $Q = [WU]$ be the orthogonal matrix whose columns are the eigenvectors of X, where the columns of U form an orthonormal basis for the null space of X. Then*
(1) $VU = ZA$ for some non-singular matrix A, i.e., VU is a Gale matrix.
(2) $VW = PA'$ for some non-singular matrix A'.

The next two lemmas are crucial for our proof of Theorem 2.

Lemma 9 *Let U, W be the matrices as defined in Lemma 8. Then the following statements are equivalent:*
(1) There exists a nonzero $\overline{r} \times \overline{r}$ symmetric matrix Φ such that $(p_i - p_j)^T \Phi(p_i - p_j) = 0$, $\forall (i, j) \in E$.
(2) There exists a nonzero $y = (y_{ij}) \in \mathbb{R}^{|E|}$ such that $M(y)U = 0$, where the $M(y)$ is defined in eqn. 5.

Proof: It is straightforward to find that $\left(p^i - p^j\right)^T \Phi \left(p^i - p^j\right) = \left(P^T \Phi P\right)_{ii} + \left(P^T \Phi P\right)_{jj} - 2\left(P^T \Phi P\right)_{ij}$. On the other hand, from lemma 8, we have $VW\Phi'W^TV^T = P^T(A\Phi'A^T)P = P^T\Phi P$, where $\Phi = A\Phi'A^T$. Thus, the equation $\kappa_V(W\Phi'W^T)_{ij} = \left(P^T\Phi P\right)_{ii} + \left(P^T\Phi P\right)_{jj} - 2\left(P^T\Phi P\right)_{ij}$ can be achieved. The Statement 1 holds if and only if $H \circ \kappa_V(W\Phi'W^T) = 0$ for some nonzero symmetric matrix Φ'. According to the remark before lemma 8, the set $M_{ij} : (i, j) \notin E$ forms a basis for the kernel of $H \circ \kappa_V(\cdot)$. So the later statement is true if and only there exits a nonzero y such that $W\Phi'W^T = M(y)$. As W and U are orthogonal, the last statement holds if and only if the second condition in the lemma is true. ■

Lemma 10 *Let $G(p)$ be a generic framework in $\mathbb{R}^d$ and let each vertex of G have degree at least d. Then there does not exist an $\overline{r} \times \overline{r}$ symmetric nonzero matrix Ψ such that $z_i^T \Psi' z_j = 0$ for all $(i,j) \in E$.*

Connelly gave a proof for this lemma [5], it is also a very important lemma for the necessary condition of global rigidity. Now let prove the theorem 2:

Proof: As stated at the beginning of this section, we will prove that if a framework is rigid but not globally rigid, then it is dimensionally flexible. Suppose $G(p_1)$ rigid but not globally rigid in $\mathbb{R}^d$, there must exist another framework $G(q)$ in $\mathbb{R}^d$, which is equivalent to $G(p_1)$. Therefore, there exits a nonzero y in $\mathbb{R}^{|E|}$ such that $X(y) = X_1 + M(y) \geq 0$ and $rank(X) = d$. In matrix form,

$$Q^T(X_1 + M(y))Q = \begin{bmatrix} \Lambda + W^T M(y) W & W^T M(y) U \\ U^T M(y) W & U^T M(y) U \end{bmatrix} \geq 0, \tag{8}$$

where Λ is the $d \times d$ diagonal matrix consisting of the positive eigenvalues of X_1. The semidefinite property of X requires that $U^T M(y) U \geq 0$, and the kernel of $U^T M(y) U \subset$ the kernel of $W^T M(y) U$. Since $rank(X) = d$, both matrices $U^T M(y) U$ and $M^T M(y) U$ should be zero, which is equivalent to $M(y)U$ is zero. By lemma 9, there exists a nonzero $\overline{r} \times \overline{r}$ symmetric matrix Φ such that $(p_i - p_j)^T \Phi (p_i - p_j) = 0, \forall (i,j) \in E$. By Lemma 10, other $G(p_1)$ is not generic, or it is not $d+1$-connected.

Since we only consider the generic framework here, the only possible case is that $G(p_1)$ is not $d+1$-connected. Intuitively, any d vertices in a generic framework form a mirror in an $\mathbb{R}^{d-1}$ hyperplane whose removal separates the graph into at least two unconnected pieces. Henrickson [11] has proved that if a rigid graph is not vertex $(d+1)$-connected, there is a framework $G(q)$ that is partial reflection of $G(p_1)$. In fact, one can find a continuous path between $G(p_1)$ and $G(q)$, that is, partially rotate one unconnected piece with constant edge lengths. Since $G(p_1)$ is rigid, the continuous partial rotation must occurs in a higher dimensional space, thus there exists an equivalent but non- congruent framework $G(r)$ in a higher dimensional space. ■

In fact, we can represent the continuous path that connected $G(p_1)$ and $G(q)$ by the projected Gram matrix. Here is another proof of theorem 2.

Proof: Let $G(p_1)$ and $G(q)$ be incongruent frameworks in $\mathbb{R}^d$. X_1 and $X(y)$ are their corresponding projected Gram matrices. We have $X(y) = X_1 + M(y)$ where $y \in \mathbb{R}^{\overline{m}}$. Each nonzero element of y corresponds to a missing edge which is not preserved in partially reflection. Since both X_1 and $X(y)$ are semidefinite, $X(ty) = X_1 + M(ty)$ is also semidefinite, for $0 < t < 1$. Each specified projected Gram matrix $G(X(ty))$ can uniquely determine a framework that is equivalent but not congruent to $G(p_1)$. Moreover $G(X(ty))$ is the continuous deformation of $G(p_1)$ with constant edge lengths. Since the framework is locally rigid, the continuous deformation can not occurs in $\mathbb{R}^d$. Thus, if a framework is rigid but not globally rigid, we always can find an incongruent framework in a higher dimensional space, which means the framework is not dimensionally rigid. ■

Finally the theorem 1.18 in [6] states that if a graph G is not generically globally rigid in $\mathbb{R}^d$, then any generic framework $G(p_1)$ can be connected to some incongruent framework $G(q)$ by a path of frameworks of G in $\mathbb{R}^{d+1}$ with constant edge lengths. This is stronger than our proof for theorem 2.

5 Sufficient conditions for universal rigidity

Theorem 4 gives a necessary condition of universal rigidity. The converse of theorem 4 is also true and it is proved by Gortler [3].

Theorem 11 *Let $G(p)$ be a framework of n vertices in generic position in $\mathbb{R}^d$, $d <= n-1$. If $G(p)$ is universally rigid, then there exists a stress matrix S such that S is positive semidefinite (PSD) and the rank of S is $\overline{r} = n - d - 1$.*

However, it remains an open question whether or not the converse of theorem 5 holds if the assumption of a framework is in general position. Alfakih gave a confirmative proof [2] for the case that the graph is a $d+1$-lateration graph. Such frameworks were shown to be universally rigid in [10].

Theorem 12 *Let $G(p)$ be a framework of n vertices in general position in $\mathbb{R}^d$, $d \leq n-1$, where G is a $(d+1)$-lateration graph. Then $G(p)$ admits a positive semidefinite (PSD) stress matrix with rank $\bar{r} = n-d-1$.*

Sketch of Proof: An $n \times n$ symmetric matrix S' that satisfies, $AS' = 0$, is called a pre-stress matrix. Our constructive proof of theorem 12 first generates a PSD pre-stress matrix $S^n = ZZ^T$ with rank $\bar{r}$. Then uses this pre-stress matrix as a basis to generate a PSD stress matrix with the same rank. Since a stress matrix S is a pre-stress matrix which also satisfies $s_{ij} = 0, \forall (i,j) \notin E$, we need to zero out the entries which should be zero but are not, i.e., the entries $s^n_{ij} \neq 0, i < j$ and $(i,j) \notin E$. We do this in reverse order by column, first, we zero out the entries $s^n_{in} \neq 0$, for $i < n$ and $(i,n) \notin E$, then do the same for columns $(n-1), (n-2), ..., (d+3)$. This "purification" process will keep the pre-stress matrix PDS and maintain rank $\bar{r}$.

During the kth purification step, we construct a vector $t^k \in \mathbb{R}^n$ with the elements,

$$t^k_i = -s^k_{ik}, \; \forall (i,k) \notin E, \;\; t^k_k = 1, \;\; \text{and } t^k_i = 0, \; \forall i > k, \tag{9}$$

and solve the system of linear equation for the remaining entries in t^k,

$$\sum_{(i,k)\in E} t^k_i a_i = - \sum_{(i,k)\notin E} t^k_i a_i. \tag{10}$$

Then let $S^{k-1} = S^k + t^k (t^k)^T$, we have

- $AS^{k-1} = 0$.
- $S^{k-1} \geq 0$, $rank(S^{k-1}) = \bar{r}$.
- $s^{k-1}_{ij} = 0$ for all $j \geq k$, and $(i,j) \notin E$.

After step $k = (d+3)$, will have a matrix S^{d+2} that satisfies,

$$AS^{d+2} = 0 \;\; \text{and} \;\; s^{d+2}_{ij} = 0, \;\; \forall (i,j) \notin E. \tag{11}$$

This is the stress matrix. ∎

Here is an example to construct a stress matrix for a 3-lateration framework in $\mathbb{R}^2$. The position matrix is given by

$$P = \begin{pmatrix} -1 & 1 & 0 & 2 & 1 & -1 & 2 \\ 1 & 1 & 0.5 & 0 & -1 & 1 & 0 \end{pmatrix} \in R^{2\times 7}. \tag{12}$$

The graph is shown in fig. 5. We can find a special sparse Gale matrix Z corresponding to the framework $G(p)$,

$$Z = \begin{pmatrix} 1.5 & 5 & 0 & -1.25 \\ -0.5 & 0 & -1 & 0 \\ -2 & -8 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -0.75 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

and compute the pre-stress matrix $S^7 = ZZ^T$,

$$S^7 = \begin{pmatrix} 28.8125 & -0.75 & -44.25 & 11.5 & 5 & 0.9375 & -1.25 \\ -0.75 & 1.25 & 1 & -2.5 & 2 & -1 & 0 \\ -44.25 & 1 & 69 & -18 & -8 & -0.75 & 1 \\ 11.5 & -2.5 & -18 & 9 & -2 & 2 & 0 \\ 5 & 2 & -8 & -2 & 5 & -2 & 0 \\ 0.9375 & -1 & -0.75 & 2 & -2 & 1.5625 & -0.75 \\ -1.25 & 0 & 1 & 0 & 0 & -0.75 & 1 \end{pmatrix}. \tag{14}$$

The last column and row of S^7 meets the purification condition, we have $S^7 = S^6$. Next we can

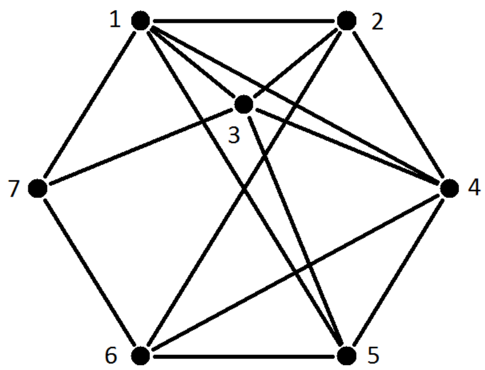

Figure 5: A 3-lateration framework in dimension 2 on n=7 nodes.

calculate the column vector $t^6 = (-0.9375, -0.0625, 0.75, 0.875, -1.625, 1, 0)^T$, $S^5 = S^6 + t^6(t^6)^T$. Then can find $t^5 = (11.3047, -2.1016, -16.4063, 6.2031, 1, 0, 0)^T$, and get the desired stress matrix $S^4 = S^5 + t^5(t^5)^T$,

$$S^4 = \begin{pmatrix} 157.4874 & -24.4489 & -230.4207 & 80.8041 & 17.8281 & 0 & -1.25 \\ -24.4489 & 5.6705 & 35.4319 & -15.5909 & 0 & -1.0625 & 0 \\ -230.4207 & 35.4319 & 338.7275 & -119.1138 & -25.625 & 0 & 1 \\ 80.8041 & -15.5909 & -119.1138 & 48.2444 & 2.7813 & 2.875 & 0 \\ 17.8281 & 0 & -25.625 & 2.7813 & 8.6406 & -3.625 & 0 \\ 0 & -1.0625 & 0 & 2.875 & -3.625 & 2.5625 & -0.75 \\ -1.25 & 0 & 1 & 0 & 0 & -0.75 & 1 \end{pmatrix}. \tag{15}$$

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