Smallest enclosing circles and more

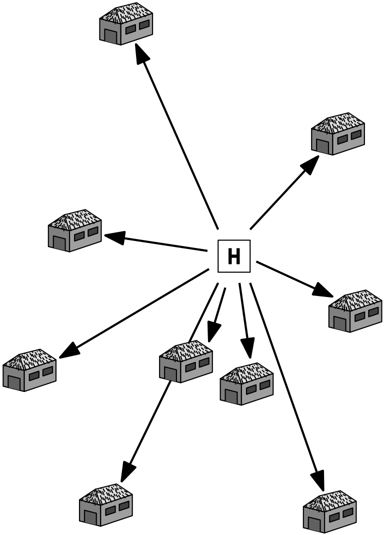
Computational Geometry

Lecture 4: Smallest enclosing circles and more

Facility location

Given a set of houses and farms in an isolated area. Can we place a helicopter ambulance post so that each house and farm can be reached within 15 minutes?

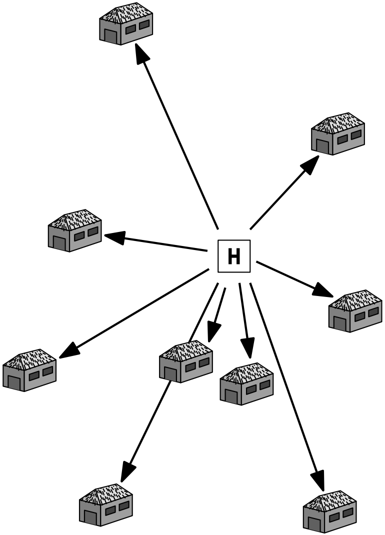
Where should we place an antenna so that a number of locations have maximum reception?



Facility location in geometric terms

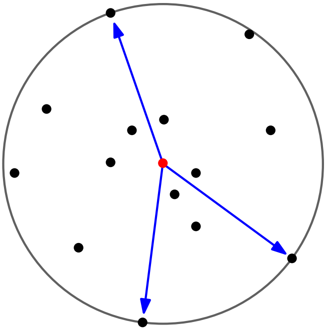
Given a set of points in the plane. Is there any point that is within a certain distance of these points?

Where do we place a point that minimizes the maximum distance to a set of points?



Facility location in geometric terms

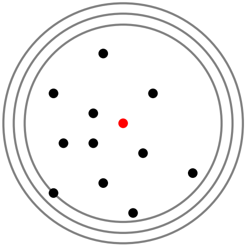
Given a set of points in the plane, compute the smallest enclosing circle

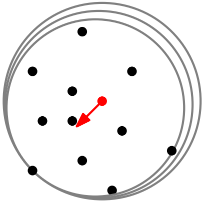


Smallest enclosing circle

Observation: It must pass through some points, or else it cannot be smallest

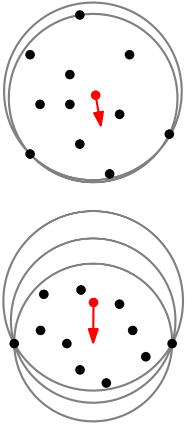
- Take any circle that encloses the points, and reduce its radius until it contains a point p
- Move center towards p while reducing the radius further, until the circle contains another point q





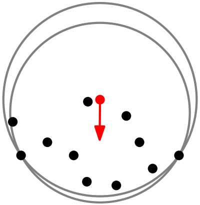
Smallest enclosing circle

- Move center on the bisector of p and q towards their midpoint, until:
  (i) the circle contains a third point, or
  (ii) the center reaches the midpoint of p and q



Smallest enclosing circle

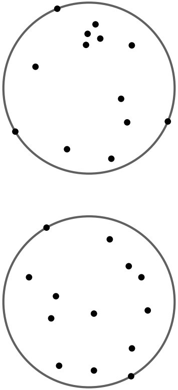
Question: Does the "algorithm" of the previous slide work?



Smallest enclosing circle

Observe: A smallest enclosing circle has (at least) three points on its boundary, or only two in which case they are diametrally opposite

Question: What is the extra property when there are three points on the boundary?



Randomized incremental construction

Construction by randomized incremental construction

incremental construction: Add points one by one and maintain the solution so far

randomized: Use a random order to add the points

Putting in random order

The points may be given in any order, the algorithm will just reorder them

- Let j be a random integer in $[1,n]$
- Swap p_j and p_n
- Recursively shuffle $p_1,\ldots,p_{n-1}$

Putting in random order takes $O(n)$ time

Expected running time

Every one of the $n!$ orders is equally likely

The expected time taken by the algorithm is the *average* time over all orders

$$\frac{1}{n!} \cdot \sum_{\Pi \text{ permutation}} \text{time if the random order is } \Pi$$

Adding a point

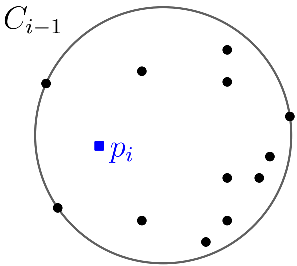
Let $p_1, \ldots, p_n$ be the points in random order

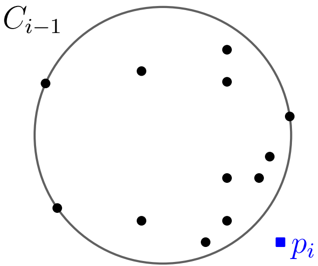
Let C_i be the smallest enclosing circle for $p_1, \ldots, p_i$

Suppose we know C_{i-1} and we want to add p_i

- If p_i is inside C_{i-1}, then $C_i = C_{i-1}$
- If p_i is outside C_{i-1}, then C_i will have p_i on its boundary

Adding a point

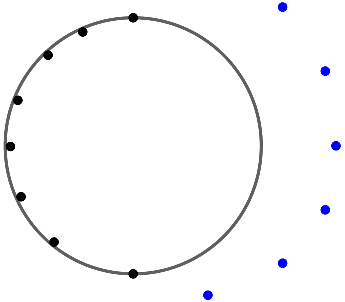




Adding a point

Question: Suppose we remembered not only C_{i-1}, but also the two or three points defining it. It looks like if p_i is outside C_{i-1}, the new circle C_i is defined by p_i and some points that defined C_{i-1}. Why is this false?

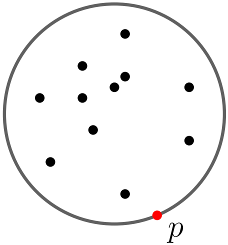
Adding a point



Adding a point

How do we find the smallest enclosing circle of $p_1 \ldots, p_{i-1}$ with p_i on the boundary?

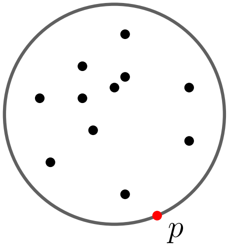
We study the *new(!)* geometric problem of computing the smallest enclosing circle with a given point p on its boundary



Smallest enclosing circle with point

Given a set P of points and one special point p, determine the smallest enclosing circle of P that must have p on the boundary

Question: How do we solve it?



Randomized incremental construction

Construction by randomized incremental construction

incremental construction: Add points one by one and maintain the solution so far

randomized: Use a random order to add the points

Adding a point

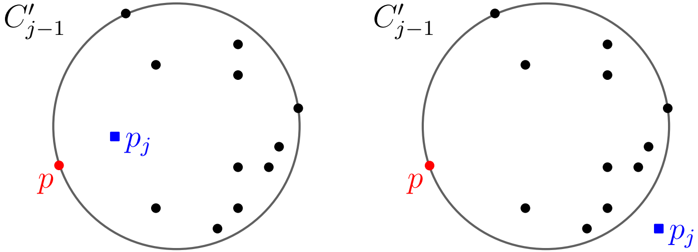
Let $p_1, \ldots, p_{i-1}$ be the points in random order

Let C'_j be the smallest enclosing circle for $p_1, \ldots, p_j$ $(j \leq i-1)$
and with p on the boundary

Suppose we know C'_{j-1} and we want to add p_j

- If p_j is inside C'_{j-1}, then $C'_j = C'_{j-1}$
- If p_j is outside C'_{j-1}, then C'_j will have p_j on its boundary (and also p of course!)

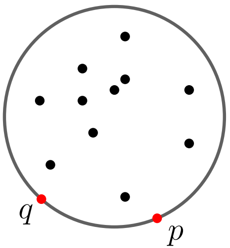
Adding a point



Adding a point

How do we find the smallest enclosing circle of $p_1 \ldots, p_{j-1}$ with p and p_j on the boundary?

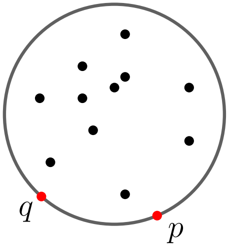
We study the *new(!)* geometric problem of computing the smallest enclosing circle with two given points on its boundary



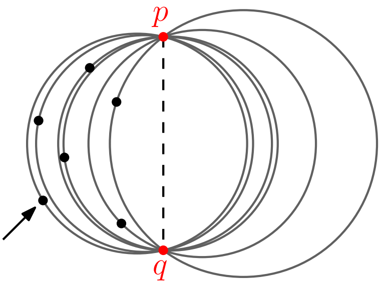
Smallest enclosing circle with two points

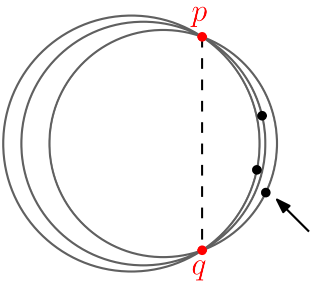
Given a set P of points and two special points p and q, determine the smallest enclosing circle of P that must have p and q on the boundary

Question: How do we solve it?

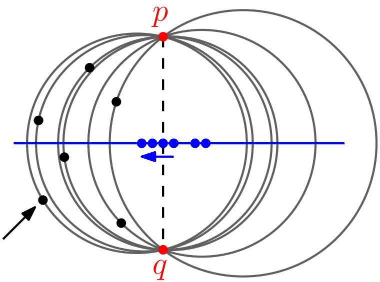


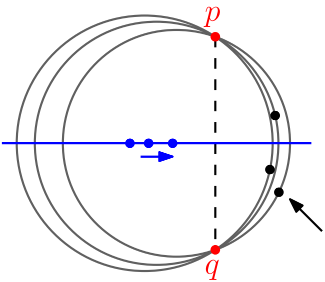
Two points known





Two points known





Algorithm: two points known

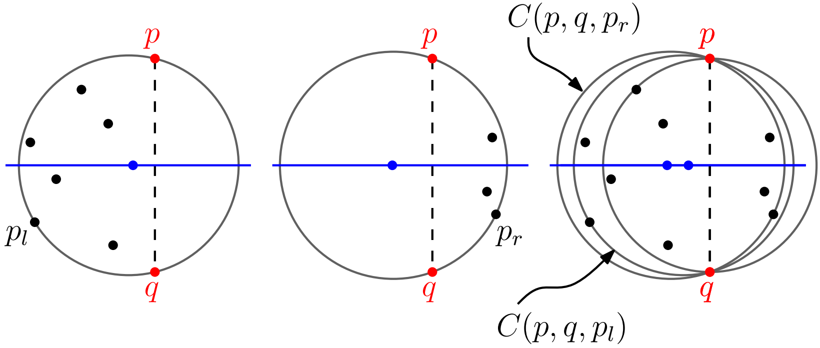
Assume w.lo.g. that p and q lie on a vertical line. Let ℓ be the line through p and q and let ℓ' be their bisector

For all points left of ℓ, find the one that, together with p and q, defines a circle whose center is leftmost $\longrightarrow p_l$

For all points right of ℓ, find the one that, together with p and q, defines a circle whose center is rightmost $\longrightarrow p_r$

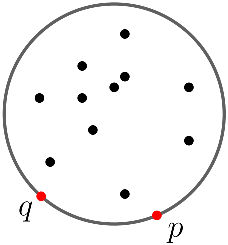
Decide if $C(p,q,p_l)$ or $C(p,q,p_r)$ or $C(p,q)$ is the smallest enclosing circle

Two points known



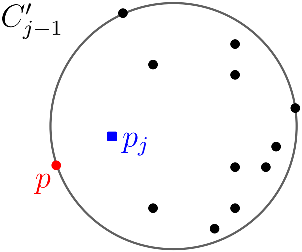
Analysis: two points known

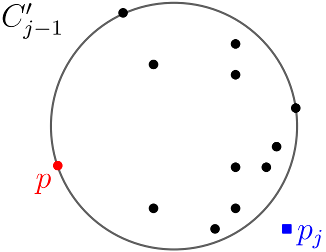
Smallest enclosing circle for n points with two points already known takes $O(n)$ time, worst case



Algorithm: one point known

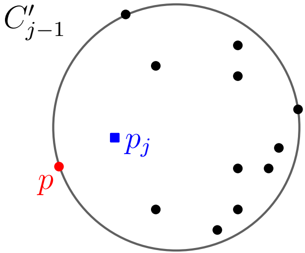
- Use a random order for $p_1, \ldots, p_n$; start with $C_1 = C(p, p_1)$
- **for** $j \leftarrow 2$ **to** n **do**
  If p_j in or on C_{j-1} then $C_j = C_{j-1}$; otherwise, solve smallest enclosing circle for $p_1, \ldots, p_{j-1}$ with two points known (p and p_j)

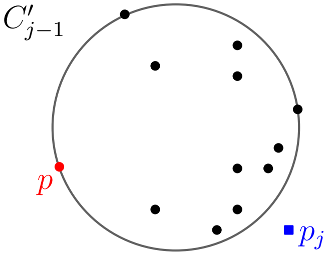




Analysis: one point known

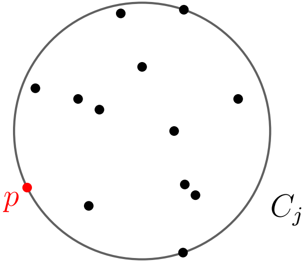
If only one point is known, we used randomized incremental construction, so we need an *expected time analysis*

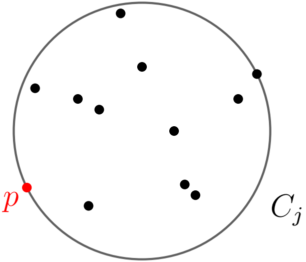




Analysis: one point known

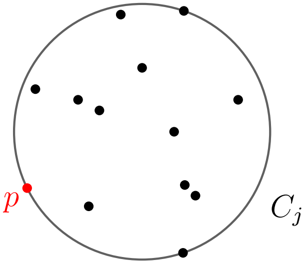
Backwards analysis: Consider the situation *after* adding p_j, so we have computed C_j





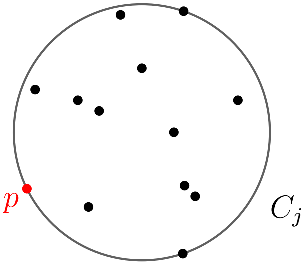
Analysis: one point known

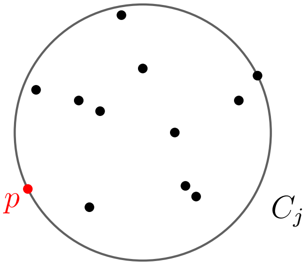
The probability that the j-th addition was expensive is the same as the probability that the smallest enclosing circle changes (decreases in size) if we remove a random point from the j points



Analysis: one point known

This probability is $2/j$ in the left situation and $1/j$ in the right situation





Analysis: one point known

The expected time for the j-th addition of a point is

$$\frac{j-2}{j} \cdot \Theta(1) + \frac{2}{j} \cdot \Theta(j) = O(1)$$

or

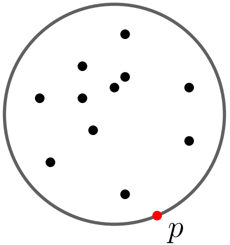
$$\frac{j-1}{j} \cdot \Theta(1) + \frac{1}{j} \cdot \Theta(j) = O(1)$$

The expected running time of the algorithm for n points is:

$$\Theta(n) + \sum_{j=2}^{n} \Theta(1) = \Theta(n)$$

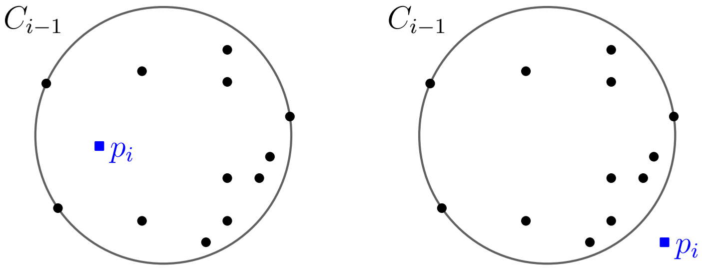
Analysis: one point known

Smallest enclosing circle for n points with one point already known takes $\Theta(n)$ time, expected



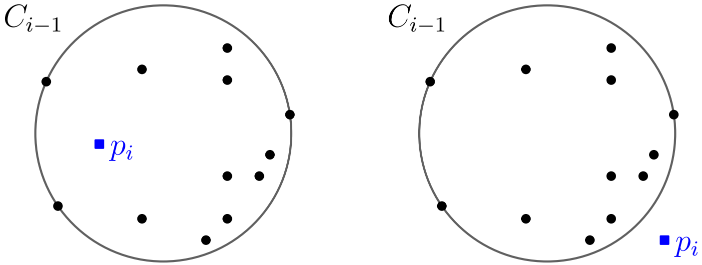
Algorithm: smallest enclosing circle

- Use a random order for $p_1, \ldots, p_n$; start with $C_2 = C(p_1, p_2)$
- **for** $i \leftarrow 3$ **to** n **do**
  If p_i in or on C_{i-1} then $C_i = C_{i-1}$; otherwise, solve smallest enclosing circle for $p_1, \ldots, p_{i-1}$ with one point known (p_i)



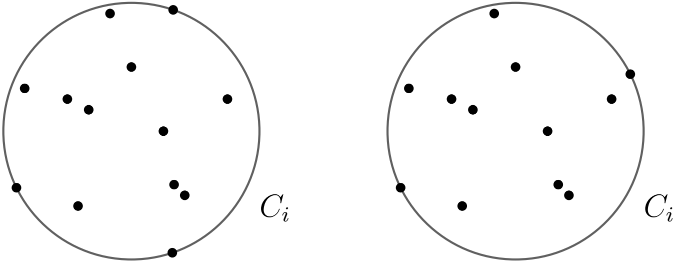
Analysis: smallest enclosing circle

For smallest enclosing circle, we used randomized incremental construction, so we need an *expected time analysis*



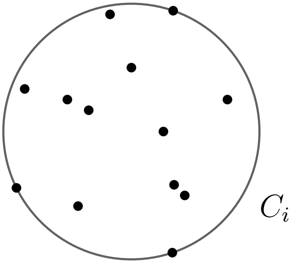
Analysis: smallest enclosing circle

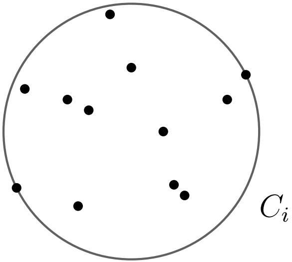
Backwards analysis: Consider the situation *after* adding p_i, so we have computed C_i



Analysis: smallest enclosing circle

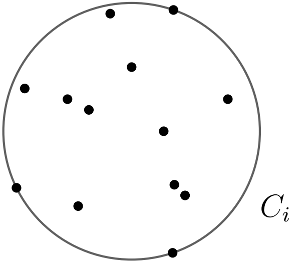
The probability that the i-th addition was expensive is the same as the probability that the smallest enclosing circle changes (decreases in size) if we remove a random point from the i points

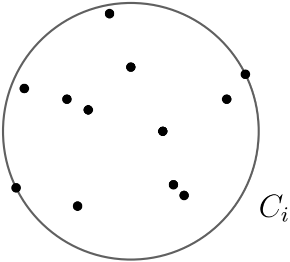




Analysis: smallest enclosing circle

This probability is $3/i$ in the left situation and $2/i$ in the right situation





Analysis: smallest enclosing circle

The expected time for the i-th addition of a point is

$$\frac{i-3}{i} \cdot \Theta(1) + \frac{3}{i} \cdot \Theta(i) = O(1)$$

or

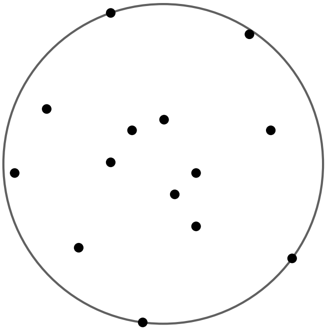
$$\frac{i-2}{i} \cdot \Theta(1) + \frac{2}{i} \cdot \Theta(i) = O(1)$$

The expected running time of the algorithm for n points is:

$$\Theta(n) + \sum_{i=3}^{n} \Theta(1) = \Theta(n)$$

Result: smallest enclosing circle

Theorem The smallest enclosing circle for n points in plane can be computed in $O(n)$ expected time



When does it work?

Randomized incremental construction algorithms of this sort (compute an 'optimal' thing) work if:

1.) The test whether the next input object violates the current optimum must be possible and fast

2.) If the next input object violates the current optimum, finding the new optimum must be an *easier* problem than the general problem

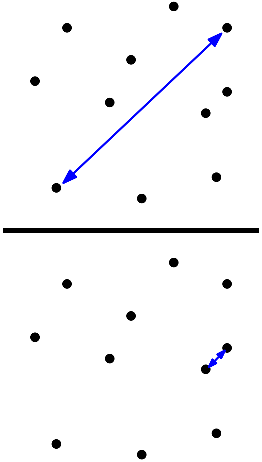
3.) The thing must already be defined by $O(1)$ of the input objects

4.) Ultimately: the analysis must work out

Diameter, closest pair

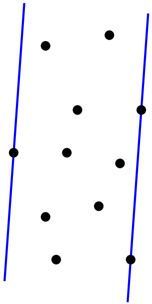
Diameter: Given a set of n points in the plane, compute the two points furthest apart

Closest pair: Given a set of n points in the plane, compute the two points closest together



Width

Width: Given a set of n points in the plane, compute the smallest distance between two parallel lines that contain the points (narrowest strip)

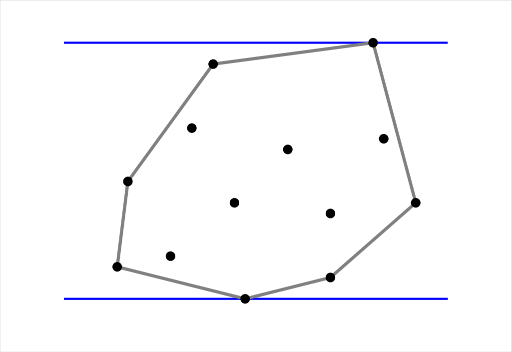


Rotating callipers

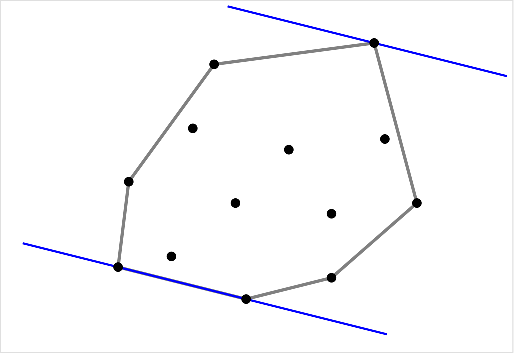
The width can be computed using the rotating callipers algorithm

- Compute the convex hull
- Find the highest and lowest point on it; they define two horizontal lines that enclose the points
- Rotate the lines together while proceeding along the convex hull

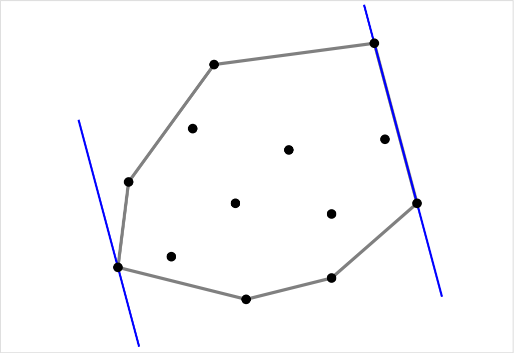
Rotating callipers



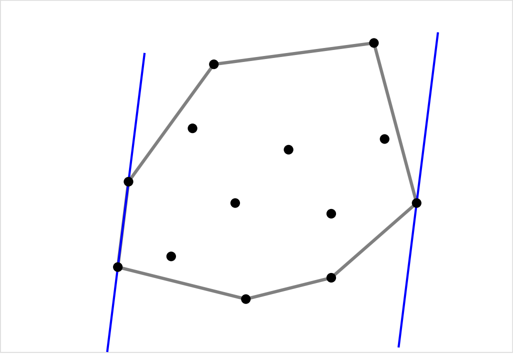
Rotating callipers



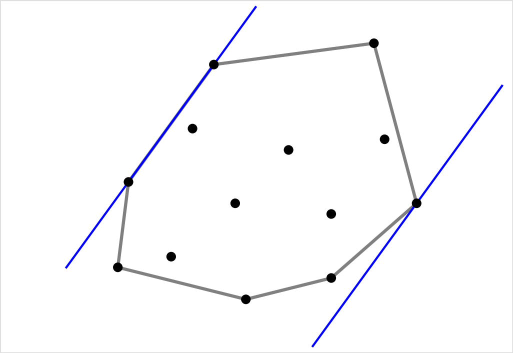
Rotating callipers



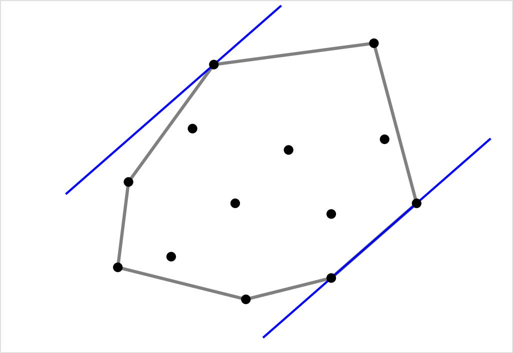
Rotating callipers



Rotating callipers



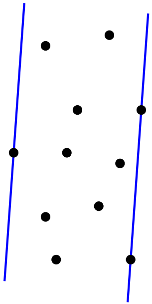
Rotating callipers



Width

Property: The width is always determined by three points of the set

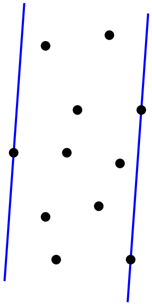
Theorem: The rotating callipers algorithm determines the width (and the diameter) in $O(n \log n)$ time



Width by RIC?

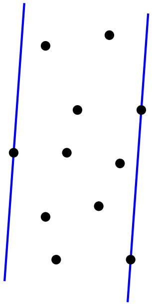
Property: The width is always determined by three points of the set

We can maintain the two lines defining the width to have a fast test for violation

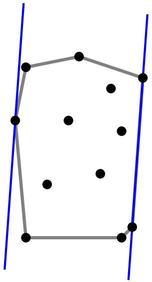


Adding a point

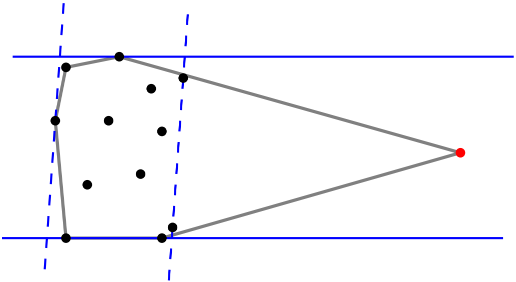
Question: How about adding a point? If the new point lies inside the narrowest strip we are fine, but what if it lies outside?



Adding a point

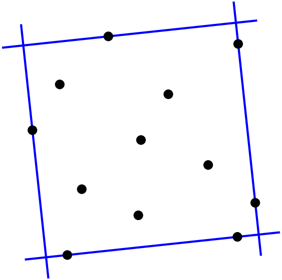


Adding a point



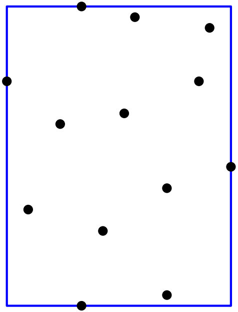
Width

A good reason to be very suspicious of randomized incremental construction as a working approach is *non-uniqueness* of a solution



Minimum bounding box

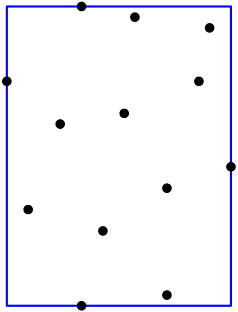
Question: Can we compute the minimum axis-parallel bounding box by randomized incremental construction?



Minimum bounding box

Yes, in $O(n)$ expected time

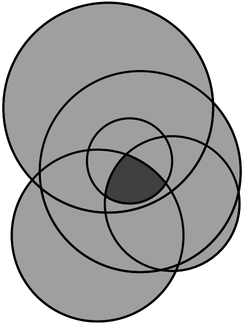
... but a normal incremental algorithm does it in $O(n)$ worst case time



Lowest point in circles

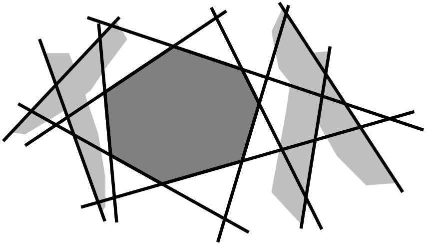
Problem 1: Given n disks in the plane, can we compute the lowest point in their common intersection efficiently by randomized incremental construction?

Problem 2: Given n disks in the plane, can we compute the lowest point in their union efficiently by randomized incremental construction?



Half-plane intersection

Problem: Given a set of n half-planes, can we decide efficiently if their intersection is empty?



Red-blue separation

Problem: Given a set of n red and blue points in the plane, can we decide efficiently if they have a separating line?