# Properties of Localization Using Distance-Differences

Xiaochun Xu and Sartaj Sahni
Computer and Information Science and Engineering Department
University of Florida
Gainesville, FL 32611
Email: {xxu,sahni}@cise.ufl.edu
Nageswara S. V. Rao
Computer Science and Mathematics Division
Oak Ridge National Laboratory
Oak Ridge, TN 37831
Email: raons@ornl.gov

***Abstract*— We study several basic properties related to the task of localizing a source using distance-difference measurements to it. These properties enable minimalistic realizations of localization systems. We establish conditions for the unique identification of a source in Euclidean plane, and derive minimum number of sensors needed for unique source identification within the Euclidean plane and a polygonal monitoring region. Compared to four possible intersections of two hyperbolas, this task leads to at most 2 intersections, which correspond to potential source estimates.**

## I. Introduction

The Difference of Time-of-Arrival (DTOA) localization problem deals with estimating the location of a source using distance-difference measurements from multiple sensors. This classical problem has been extensively studied in applications in aerospace systems [1], [2], wireless communication networks [3], and wireless sensor networks [4], [5]. There are two basic formulations of the DTOA localization problem: (i) the distance-differences to a source are measured from known sensor locations, and the problem is to estimate the location of the source; and (ii) a device (i.e., a mobile node) receives distance-differences from beacon nodes with known locations, and the problem is to estimate the location of the device, that is self-localization. The classic DTOA localization methods include two general approaches: (i) linear algebraic solution which typically involves matrix inversion and solution to a quadratic equation [6], [7], [2], and (ii) application of general intersection method of hyperbolic curves [8].

The renewed interest in this problem is in part due to the need for minimalistic implementations suitable for nodes with limited computational resources and networks with limited number of sensors. In terms of computation, the computational geometry method for DTOA localization in Euclidean plane [10], [11], [12] offers efficient computation. This method employs a binary search on a distance-difference curve in $R^2$ using a second distance-difference as the objective function. To support the binary search, this method establishes the unimodality of the directional derivative of the objective function within each of a small number of suitably decomposed regions of $R^2$ [12]. However, despite the extensive literature on DTOA localization, several basic aspects needed for minimalistic network realizations do not seem to be reported.

In this paper, we present a number of results that establish basic properties of DTOA localization. We first consider the unique identification of a source and establish the following:

1) DTOA localization uniquely identifies a source in Euclidean plane $R^2$ iff the sensors do not lie on a hyperbola[1].
2) At least four sensors are necessary for unique localization of a source in Euclidean plane, and it is sufficient to place the four sensors at the corners of a parallelogram to achieve this.
3) A minimal sensor set to achieve unique source identification (i.e., a sensor set none of whose proper subsets is also a uniquely identifying sensor set) has between 4 and 6 sensors.
4) Three sensors are sufficient to uniquely identify any source in a monitoring region bounded by a polygon. These sensors, however, must be placed outside the polygon.

We then consider the computational aspects of DTOA localization that utilizes the intersection of hyperbolas corresponding to distance-difference measurements. In general, two hyperbolas may have four intersection points, but we show that two hyperbolas that correspond to distance-differences to a source that have a common focus may have at most 2 intersections. We also show that when non-collinear sensors are used, at most 2 points can have the same DTOA values. These results establish that the DTOA problem is more structured and easier in this sense compared to computing intersection points of hyperbolas.

This paper is organized as follows. In Section II, we present some fundamental properties and definitions. Properties of

[1]For convenience, in this paper, the term hyperbola is used to refer to even a portion of a hyperbola.

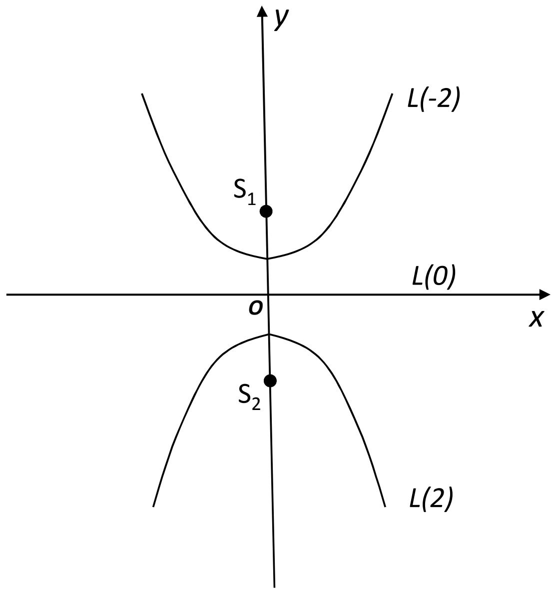


Fig. 1. Examples of the locus $L_{12}$

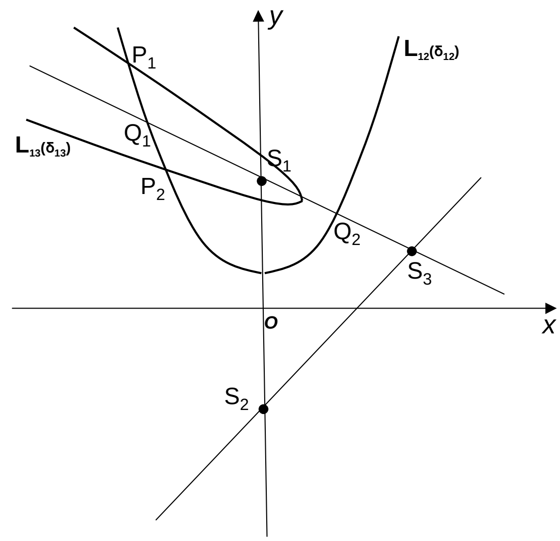


Fig. 2. Three non-collinear sensors $S_1$, $S_2$, and $S_3$ form a triangle and two hyperbolas $L_{12}(\delta_{12})$ and $L_{13}(\delta_{13})$ intersect each other at $P_1$ and $P_2$.

sensor sets that uniquely identify all sources in Euclidean space are developed in Section III. Our detailed analysis of Section IV establishes the bound on the the number of intersections of two DTOA hyperbolas. In Section V we show that at most 2 points can have the same set of DTOA values. The minimum number of sensors needed to uniquely identify all sources in a bounding polygon is derived in Section VI. Finally, we conclude in Section VII.

## II. Preliminaries and Definitions

Let $S_i$ = $(x_i, y_i)$, $1 \leq i \leq k$, be the locations of $k$ sensors in Euclidean space $R^2$. These locations are assumed to be distinct. For any point $P$ =$(x,y)$ in $R^2$, the distance, $d(P,S_i)$, between $P$ and $S_i$ is $\sqrt{(x-x_i)^2+(y-y_i)^2}$. A signal originating at $P$ at time 0 arrives at $S_i$ at time proportional to $d(P,S_i)$. For simplicity, we assume that the arrival time is $d(P,S_i)$. The difference, $\Delta_{ij}$, in the time of arrival (DTOA) at $S_i$ and $S_j$ is given by

$$\Delta_{ij}(P) = d(P,S_i) - d(P,S_j).$$

From the triangle inequality, it follows that $|\Delta_{ij}(P)| \leq d(S_i,S_j)$. Furthermore, the locus, $L_{ij}(\delta)$, of points defined by

$$L_{ij}(\delta) = \{P|\Delta_{ij}(P) = \delta\}$$

is a hyperbola[2] (see Figure 1).

In this paper, we consider the *DTOA localization* problem of estimating the location of a source $S$ from the measurements of $\Delta_{ij}(S)$, $1 \leq i < j \leq k$. When $\Delta_{ij}(P) = \Delta_{ij}(Q)$ for every $i,j \in \{1,2,...,k\}$, the points $P$ and $Q$ are indistinguishable. Actually, since $\Delta_{ij}(P) = \Delta_{1j}(P) - \Delta_{1i}(P)$, for all $i$ and $j$, $P$ and $Q$ are indistinguishable iff $\Delta_{1j}(P) = \Delta_{1j}(Q)$ for every $j \in \{2,...,k\}$. So, the set of sensor locations (also referred to as the sensor set) $SS = \{S_1, S_2, \cdots, S_k\}$ can uniquely identify every source $S$ in Euclidean space $R^2$ iff for every pair $P$ and $Q$ of distinct points in Euclidean space $R^2$, we have $\Delta_{1j}(P) \neq \Delta_{1j}(Q)$ for at least one $j \in \{2,3,\cdots,k\}$. A sensor set that can uniquely identify (localize) every possible point in Euclidean space is called an *identifying sensor set*, *ISS*. Two points that are indistinguishable are *duals*.

The DTOA method localizes the source by determining the common intersections of the hyperbolas[3] $L_{1j}(\Delta_{1j}(S))$, $2 \leq j \leq k$. When these hyperbolas have more than one common intersection, the source is not uniquely localized. Figure 2 gives an example of two hyperbolas $L_{12}(\delta_{12})$ and $L_{13}(\delta_{13})$ that intersect at two distinct locations $P_1$ and $P_2$. So, using $L_{12}$ and $L_{13}$ alone, we are unable to uniquely localize the source. We are able only to assert that the source location is either $P_1$ or $P_2$. To uniquely identify the source using the DTOA method, the hyperbolas $L_{1j}$, $2 \leq j \leq k$ should have exactly one common intersection. Alternatively, these hyperbolas should have exactly one common intersection inside a region in which the source is known to lie.

## III. Properties of Identifying Sensor Sets

In this section, we establish, in Theorem 1 a necessary and sufficient condition for a sensor set $SS$ to be an *ISS*. Theorem 2 shows that every *ISS* has at least 4 sensors and Theorem 4 shows that every *ISS* with more than 6 sensors has a subset of size at most 6 that is an *ISS*.

*Theorem 1:* The sensor set $SS = \{S_1, \cdots, S_k\}$ is an *ISS* iff no hyperbola passes through all points of $SS$.

*Proof:*

We first show that if $SS$ is an *ISS*, then no hyperbola may pass through all points of $SS$. By contradiction, suppose there exists a hyperbola, say $L$, that passes through all points of in $SS$. Let $P_1$ and $P_2$ be the two foci of $L$. From the definition of a hyperbola, it follows that $d(P_1,S_i) - d(P_2,S_i) = d(P_1,S_j) - d(P_2,S_j)$, $1 \leq i < j \leq k$. So, $\Delta_{ij}(P_1) = d(P_1,S_i) - d(P_1,S_j) =$

[2]Strictly speaking, $L_{ij}(\delta)$ is one branch of a hyperbola and $L_{ij}(-\delta)$ is the other branch. As mentioned earlier, for convenience, in this paper, we use the term hyperbola to refer to one branch of a hyperbola.

[3]A point in $R^2$ is a common intersection of a set of hyperbolas iff this point is on each of the hyperbolas

$d(P_2,S_i) - d(P_2,S_j) = \Delta_{ij}(P2)$, $1 \le i < j \le k$. Hence, $P_1$ and $P_2$ are indistinguishable and $SS$ is not an $ISS$, a contradiction.

Next, we show that if $SS$ is not an $ISS$, then at least one hyperbola passes through all points of $SS$. Let $P_1$ and $P_2$ be two different points that are indistinguishable. So, $\Delta_{1j}(P_1) = d(P_1,S_1) - d(P_1,S_j) = d(P_2,S_1) - d(P_2,S_j) = \Delta_{1j}(P_2)$, $2 \le j \le k$. Hence, $d(P_1,S_1) - d(P_2,S_1) = d(P_1,S_j) - d(P_2,S_j)$, $2 \le j \le k$. Therefore there is a hyperbola with $P_1$ and $P_2$ as as its foci that passes through all points of $SS$. ■

*Theorem 2:* If $SS$ is an $ISS$, then $|SS| \ge 4$ and there exist $ISS$s that have exactly 4 sensors.

*Proof:*

We first prove that 3 sensors are not sufficient to constitute an $ISS$ and so, $|SS| \ge 4$ whenever $SS$ is an $ISS$. Let $SS = \{S_1,S_2,S_3\}$. When $S_1$, $S_2$, and $S_3$ are collinear, the straight line through these three sensors is a trivial hyperbola through the points of $SS$. From Theorem 1, it follows that $SS$ is not an $ISS$. When $S_1$, $S_2$, and $S_3$ are not collinear, they define a nontrivial triangle as shown in Figure 2. Clearly, there exists a negative constant, $\delta_{12}$, such that the hyperbola $L_{12}(\delta_{12})$ intersects the line $S_1S_3$ at two distinct points $Q_1$ and $Q_2$. Observe that the hyperbola $L_{13}(-d(S_1,S_3))$ is actually a ray that originates at $S_1$ and intersects $L_{12}(\delta_{12})$ at $Q_1$ only. Let $\delta_{13}$ be a negative constant slightly greater than $-d(S_1,S_3)$. The hyperbola $L_{13}(\delta_{13})$ intersects $L_{12}(\delta_{12})$ at two distinct points $P_1$ and $P_2$ (see Figure 2). So, $P_1$ and $P_2$ are indistinguishable and $SS$ is not an $ISS$.

Next, we show that whenever $SS = \{S_1,S_2,S_3,S_4\}$ are the corners of a parallelogram with side length $> 0$, $SS$ is an $ISS$. We show this by proving that no 4 distinct points of a hyperbola define the corners of a parallelogram. The result then follows from Theorem 1.

Consider the hyperbola $L$ of Figure 3. Let $S_1$, $S_2$, $S_3$, and $S_4$ be 4 points on this hyperbola. The case shown in Figure 3 has $S_1$ and $S_4$ on one part (arm) of the hyperbola and $S_2$ and $S_3$ on the second part. (There are two other cases for the location of the 4 points–exactly 3 points on one part of $L$ and 4 points on one part of $L$.) Let $Q_1$ and $Q_2$, respectively, be the intersections of the line segments $\overline{S_1S_2}$ and $\overline{S_3S_4}$ with the $x$-axis, which is the semimajor axis of $L$. If the 4 identified points on $L$ are the corners of a parallelogram, $\overline{S_1S_2}$ and $\overline{S_3S_4}$ are parallel and of equal length. However, if these segments are parallel, $d(S_1,Q_1) < d(S_4,Q_2)$ and $d(S_2,Q_1) < d(S_3,Q_2)$. So, $d(S_1,S_2) = d(S_1,Q_1) + d(S_2,Q_1) < d(S_4,Q_2) + d(S_3,Q_2)$ = $d(S_3,S_4)$. So, $\overline{S_1S_2}$ and $\overline{S_3S_4}$ cannot be parallel and of equal length. The remaining two cases are similar. ■

*Corollary 1:* An infinite number of hyperbolas pass through any 3 non-collinear sensors in Euclidean space $R^2$.

*Corollary 2:* Whenever $SS$ contains the corners of a parallelogram with side length $> 0$, $SS$ is an $ISS$. In particular, whenever 4 sensors of $SS$ are at the 4 corners of a square with side length $> 0$, $SS$ is an $ISS$.

An $ISS$ is a *minimal ISS* (*MISS*) iff no proper subset of the $ISS$ is also an $ISS$. Theorem 4 establishes an upper bound of 6 on the size of an $MISS$. To prove this theorem, we need to use Bezout's bound on the number of intersections of curves in Euclidean space.

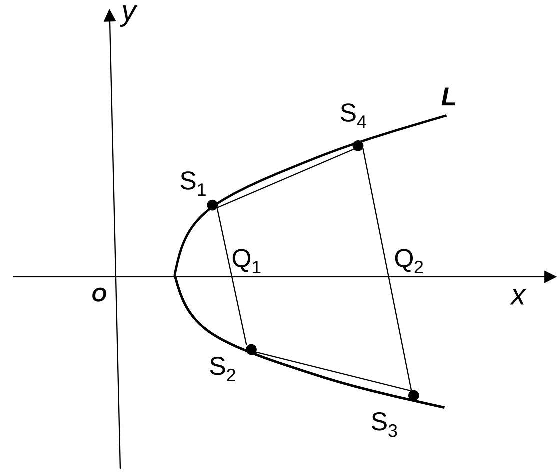


Fig. 3. A hyperbola $L$ that passes through $S_i$ $(1 \le i \le 4)$.

*Theorem 3:* **[Bezout's Theorem [13]]:** Let $C_1$ and $C_2$ be curves of degree $m$ and $n$, respectively, in Euclidean space $R^2$. If $C_1$ and $C_2$ have no curves in common, then the number of intersections of $C_1$ and $C_2$ is at most $mn$.

*Corollary 3:* Two hyperbolas in Euclidean space $R^2$ have at most 4 intersections.

*Lemma 1:* At most 1 hyperbola may pass through any set of 5 or more distinct points.

*Proof:* Consider any set $SS$ with 5 or more points. If two hyperbolas pass through the points of $SS$, then these two hyperbolas intersect at the points of $SS$ and so have more than 4 intersections. This violates Corollary 3. Hence, at most 1 hyperbola may pass through the points of $SS$. ■

*Theorem 4:* Every $SS$ that is a $MISS$ satisfies $4 \le |SS| \le 6$.

*Proof:* $4 \le |SS|$ follows from Theorem 2 and the fact that a $MISS$ is an $ISS$. $|SS| \le 6$ may be shown by contradiction. Suppose that $|SS| > 6$. Let $SS'$ be a subset of $SS$ such that $|SS'| = 5$. From Lemma 1, $SS'$ has at most 1 hyperbola passing through its 5 points. If no hyperbola passes through these points, then $SS'$ is an $ISS$ (Theorem 1) and $SS$ cannot be an $MISS$. So, we may assume that exactly one hyperbola passes through $SS'$. Since $SS$ is an $ISS$, $SS$ contains at least one point $S_i$ that does not lie on this hyperbola. Hence, there is no hyperbola that passes through the 6 points $SS' \bigcup \{S_i\}$. From Theorem 1, it follows that $SS' \bigcup \{S_i\} \subset SS$ is an $ISS$. This contradicts the assumption that $SS$ is an $MISS$.

■

## IV. Number of Intersections of $L_{12}$ and $L_{13}$

Although two hyperbolas in Euclidean space may have up to 4 intersections (Corollary 3), two DTOA hyperbolas $L_{12}$ and $L_{13}$ may have no more than 2 intersections when $S_1$, $S_2$, and $S_3$ are non-collinear. Without loss of generality (w.l.o.g), we choose our coordinate system as in Figure 4. The features of this choice are (a) $\overline{S_1S_2}$ falls on the $y$-axis, (b) the midpoint of $\overline{S_1S_2}$ is the origin $O$ of the coordinate system, and (c) $S_3$ lies on the right side of the $y$-axis. We see that $\overline{S_1S_2}$, $\overline{S_2S_3}$, and $\overline{S_1S_3}$ partition the Euclidean space $R^2$ into seven regions (a)-(g). At most one intersection of $L_{12}$ and $L_{13}$ lies in the union of regions (a), (b), (f), and (g) and at most one intersection lies in

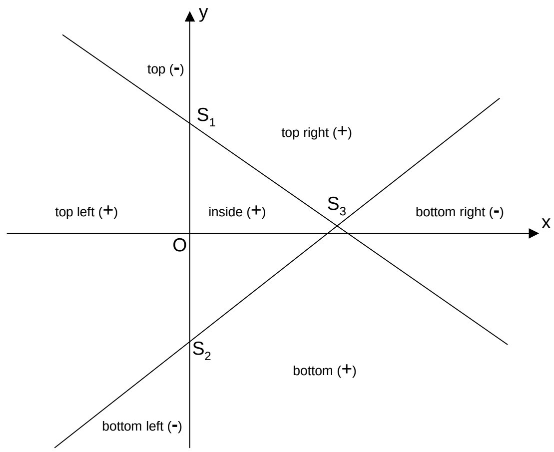


Fig. 4. Regions of monitoring area: (a) top left, (b) inside, (c) bottom right, (d) top, (e) bottom left, (f) bottom, and (g) top right. The sign of the directional derivative for each region is also given.

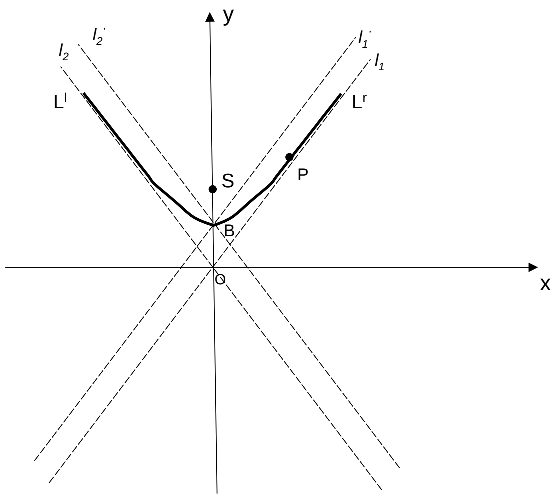


Fig. 5. A hyperbola $L = L^l \bigcup L^r$ with focus $S$ and semimajor axis $y$-axis. The asymptotes of $L$ are shown by two broken lines $l_1$ and $l_2$ through the origin $O$. The broken lines $l_1{}'$ and $l_2{}'$ through the vertex $B$ are parallel to $l_1$ and $l_2$, respectively.

the union of regions (c), (d), and (e). To prove these assertions, we need a result from [12] that establishes the monotonicity of the directional derivative of $\Delta_{13}(P)$ along the hyperbola $L_{12}(\Delta_{12}(P))$ within each of the 7 regions of Figure 4.

*Theorem 5:* **[X. Xu, N. S. V. Rao, and S. Sahni [12]]** For any point $P$ in Euclidean space $R^2$, the directional derivative of $\Delta_{13}(P)$ along the hyperbola $L_{12}(\Delta_{12}(P))$ is monotone in each of seven regions specified by three non-collinear sensors, as shown in Figure 4. The directional derivative is positive in regions (a), (b), (f), and (g), and is negative in regions (c), (d), and (e).

In the following, we use $L^l$ and $L^r$ to refer to the two symmetric parts (arms) of the hyperbola $L$ (see Figure 5). The two parts $L^l$ and $L^r$ intersect only at the vertex $B$. $l_1$ and $l_2$ are the two asymptotes of the hyperbola and $l_1'$ and $l_2'$ are lines that intersect at the vertex $B$ and are parallel to these asymptotes. From our choice of coordinate system, it follows that the asymptotes intersect at $O$.

*Lemma 2:* 1) $L^r(L^l)$ strictly lies between $l_1(l_2)$ and $l_1{}'(l_2{}')$.

2) The shortest Euclidean distance between a point $P$ on $L^r$ ($L^l$) and the asymptote $l_1$ ($l_2$) decreases monotonically as $P$ gets farther from the vertex $B$.

3) The shortest Euclidean distance between a point $P$ on $L^r$ ($L^l$) and the line $l_1'$ ($l_2'$) increases monotonically as $P$ gets farther from the vertex $B$.

*Proof:* Follows from the definition of a hyperbola, its asymptotes, and the lines $l_1'$ and $l_2'$. ■

In Theorem 6, we show that when $S_1$ is closer to the source $S$ than are $S_2$ and $S_3$, $L_{12}(\Delta_{12}(S))$ and $L_{13}(\Delta_{13}(S))$ have at most 2 intersections including the source $S$. This restriction on the source being closer to $S_1$ than the remaining two sensors is removed in Theorem 7. We often use $L_{ij}$ as an abbreviation for $L_{ij}(\Delta_{ij}(S))$.

*Theorem 6:* When $S_1$ is closer to the source $S$ than are $S_2$ and $S_3$, $L_{12}$ and $L_{13}$ have at most 2 intersections.

*Proof:*

Let $P_i = (x_i, y_i)$, $1 \le i \le m$ be intersections of $L_{12}$ and $L_{13}$. From the definition of a hyperbola, it follows that $\Delta_{12}(P_i) = \Delta_{12}(P_{i'})$ and $\Delta_{13}(P_i) = \Delta_{13}(P_{i'})$ for $1 \le i < i' \le m$.

There are 4 possible cases for the relationship between the line $\overline{S_2S_3}$ and the hyperbola $L_{12}$–(1) the line is below $L^r_{12}$, (2) the line intersects $L^l_{12}$, (3) the line intersects $L^r_{12}$ and $\angle S_3S_1S_2 \ge 90$, and (4) the line intersects $L^r_{12}$ and $\angle S_3S_1S_2 < 90$. These 4 cases are shown in Figures 6-9, respectively. We show below that $L_{12}$ and $L_{13}$ have at most 2 intersections in each of these cases.

Case 1: $\overline{S_2S_3}$ lies below $L_{12}$

When $\overline{S_2S_3}$ lies below $L_{12}$, $L_{12}$ must lie wholly within regions (a) top left, (b) inside, (d) top, and (g) top right, (Figure 6). $\Delta_{13}$, from Theorem 5, monotonically increases in regions (a), (b), and (g) and monotonically decreases in (d). So, if no component of $L_{12}$ is in region (d), then $\Delta_{13}$ monotonically increases along all of $L_{12}$ and the value of $\Delta_{13}$ for each point $P$ on $L_{12}$ is unique. Hence, $L_{12}$ and $L_{13}$ have only 1 intersection. If region (d) contains a portion of $L_{12}$, then when one moves the point $P$ from left to right along $L_{12}$, (d) is the first region to be visited. So, when moving from left to right along $L_{12}$, $\Delta_{13}$ monotonically decreases while we are moving along the portion of $L_{12}$ that is inside region (d) and then monotonically increases for the remainder of $L_{12}$. Hence $L_{12}$ has at most 2 distinct points for any given value of $\Delta_{13}$. So, $L_{12}$ and $L_{13}$ have at most 2 intersections.

Case 2: $\overline{S_2S_3}$ intersects $L^l_{12}$

When $\overline{S_2S_3}$ intersects $L^l_{12}$, $\angle S_3S_2S_1 > 90$ (Figure 7). So, $L_{12}$ cannot have a component in either of the regions (c) (bottom right) and (f) (bottom). Additionally, $L_{12}$ cannot have a component in region (d) (top). To see this, observe that $L^r_{12}$ is wholly to the right of the $y$-axis while region (d) is wholly to the left of this axis. So, no portion of $L^r_{12}$ is in region (d). To see that no portion of $L^l_{12}$ is in region (d) either, note that $L^l_{12}$ is below $l_2'$ (Lemma 2). Since, $\overline{S_2S_3}$ intersects $L^l_{12}$ and $l_2$ is strictly below $L^l_{12}$ (Lemma 2), $\overline{S_2S_3}$ intersects the asymptote $l_2$. Now, since $l_2'$ is parallel to $l_2$, $\overline{S_2S_3}$ also intersects $l_2'$. which implies that the slope of $\overline{S_2S_3}$ is less than that of $l_2'$. Hence, the slope of $\overline{S_1S_3}$ is less than that of $l_2'$. From this, the fact that $L^l_{12}$ lies below $l_2'$, and the fact that the intersection (vertex $B$ of $L_{12}$) of $L^l_{12}$ and $l_2'$ is below $S_1$, it follows that no portion

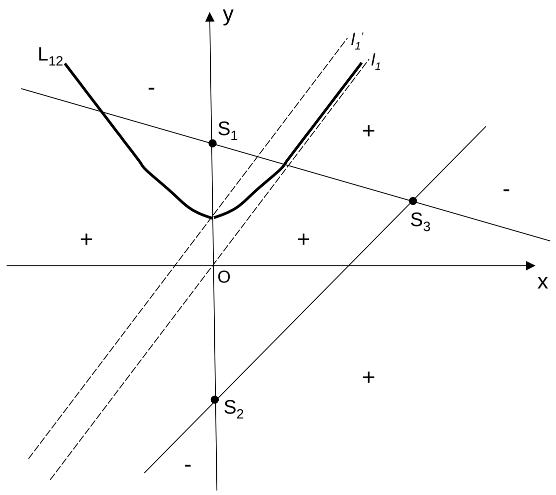


Fig. 6. Case 1: $\overline{S_2S_3}$ lies below $L_{12}$

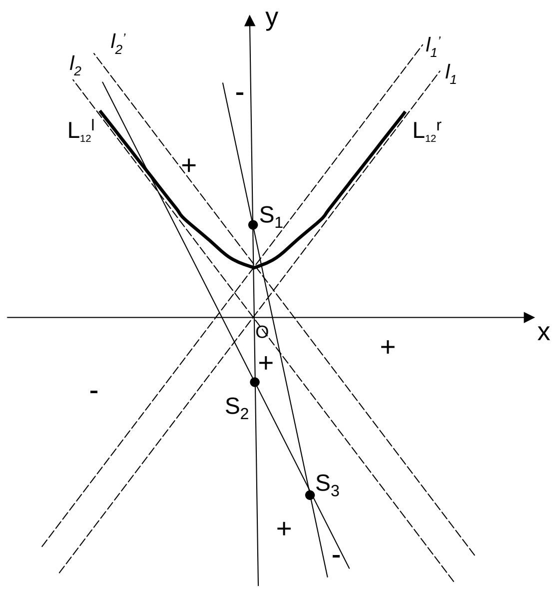


Fig. 7. Case 2: $\overline{S_2S_3}$ intersects $L_{12}^l$

of $L_{12}^r$ is inside the top region (d).

Consequently, as one moves from left to right along $L_{12}$, the region (e) (i.e., bottom left) is the first region to be visited. $\Delta_{13}$ monotonically decreases inside this region and monotonically increases in the remaining regions that $L_{12}$ is in. Hence $L_{12}$ has at most 2 distinct points for any given value of $\Delta_{13}$. So, $L_{12}$ and $L_{13}$ have at most 2 intersections.

Case 3: $\overline{S_2S_3}$ intersects $L_{12}^r$ and $\angle S_3S_1S_2 \geq 90$

In this case, region (e) (bottom left) lies entirely below $L_{12}$ (Figure 8). Hence, no portion of $L_{12}$ is in region (e). Since $\angle S_3S_1S_2 \geq 90$, $\theta < 90$ (see Figure 8). Hence, $d(P,S_1) > d(P,S_3)$ for every point $P$ inside region (c) (bottom right). Since, by assumption, $S_1$ is closer to the source $S$ than is $S_3$, no portion of $L_{13}$ is in region (c). Hence, $L_{12}$ and $L_{13}$ have no intersection in region (c).

If $L_{12}$ has an overlap with region (d) (top), then region (d) is the first region encountered as we move from left to right along $L_{12}$ and if $L_{12}$ overlaps with region (c) (bottom right), region (c) is the last region encountered as we move

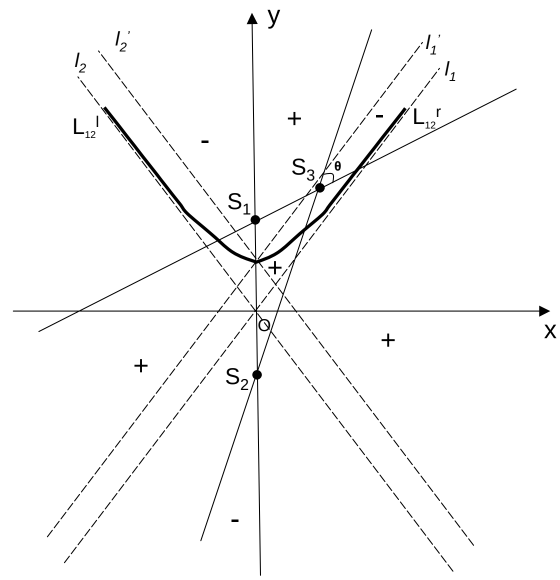


Fig. 8. Case 3: $\overline{S_2S_3}$ intersects $L_{12}^r$ and $\angle S_3S_1S_2 \geq 90$.

from left to right along $L_{12}$. $\Delta_{13}$ monotonically decreases in region (d), $L_{12}$ and $L_{13}$ do not intersect in region (c), and $\Delta_{13}$ monotonically increases in the remaining regions that $L_{12}$ may overlap. So, $L_{12}$ and $L_{13}$ have at most 2 intersections.

Case 4: $\overline{S_2S_3}$ intersects $L_{12}^r$ and $\angle S_3S_1S_2 < 90$

As in Case 3, no portion of $L_{12}$ is in region (e) (bottom left). Further, $L_{13}$ may overlap with either region (c) (bottom right) or region (d) (top) but not both. To see this, suppose that $L_{13}$ overlaps with region (c). For this to happen, $L_{13}^r$ must cross $\overline{S_2S_3}$. Using an argument similar to that used in Case 2, we may show that the slope of $\overline{S_2S_3}$ is greater than that of $L_{13}^r$. Furthermore, the remaining portion of $L_{13}^r$ once after crossing $\overline{S_2S_3}$ lies strictly below $\overline{S_2S_3}$. So, no portion of $L_{13}^r$ is in region (d). Since $L_{13}^l$ is to the left of $S_1S_3$, no portion of $L_{13}^l$ is in region (d) either. So, $L_{13}$ may overlap only one of the regions (c) and (d). Therefore, $L_{12}$ and $L_{13}$ cannot have an intersection in both region (c) and region (d). Finally, if a portion of $L_{12}$ is in region (d), region (d) is the first region encountered as we move along $L_{12}$ from left to right and if a portion of $L_{12}$ is in region (c), then region (c) is the last region encountered. $\Delta_{13}$ monotonically decreases as we move from left to right along $L_{12}$ inside regions (c) and (d) and monotonically increases in the remaining regions that $L_{12}$ overlaps. So, $L_{12}$ and $L_{13}$ have at most 2 intersections.

■

*Theorem 7:* $L_{12}$ and $L_{13}$ have at most 2 intersections.

*Proof:*

Since, $\Delta_{23}(P) = \Delta_{13}(P) - \Delta_{12}(P)$ for every point $P$, the hyperbola pairs $(L_{12}, L_{13})$, $(L_{12}, L_{23})$, and $(L_{13}, L_{23})$ have the same set of intersections. Suppose, w.l.o.g., that the source is closer to $S_2$ than to $S_1$ and $S_3$. It follows from Theorem 6 that $L_{21}$ and $L_{23}$ have at most 2 intersections. Hence, $L_{12}$ and $L_{13}$ have at most 2 intersections. ■

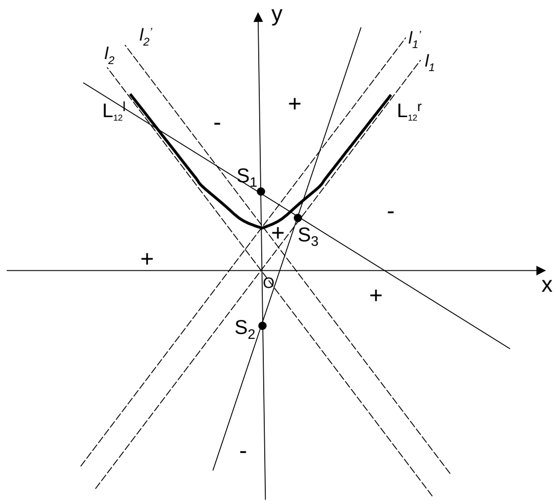


Fig. 9. Case 4: $\overline{S_2S_3}$ intersects $L^r_{12}$ and $\angle S_3S_1S_2 < 90$.

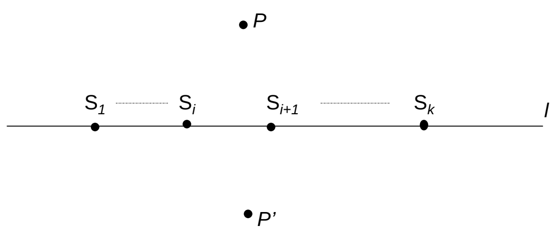


Fig. 10. Collinear sensors

## V. Indistinguishable Points

When *SS* is not an *ISS*, there is at least one pair of distinct points that are indistinguishable. That is, there are distinct points $P_1$ and $P_2$ for which $\Delta_{ij}(P_1) = \Delta_{ij}(P_2)$, $1 \le i < j \le k$ (or equivalently, $\Delta_{1j}(P_1) = \Delta_{1j}(P_2)$, $2 \le j \le k$). $P_1$ and $P_2$ are dual points. When *SS* is an *ISS*, no point $P$ has a dual. In this section, we first show that the indistinguishable relation is an equivalence relation. Then, we show that each point $P$ may have at most 1 dual point.

*Theorem 8:* The *indistinguishable* relation is an equivalence relation on $R^2$.

*Proof:*

A relation is an equivalence relation iff it is reflexive, symmetric, and transitive. Reflexivity is immediate as a point is indistinguishable from itself. Also, if $P_1$ and $P_2$ are indistinguishable then so also are $P_2$ and $P_1$. So, the relation is symmetric. For any three points $P_1$, $P_2$, and $P_3$ such that $P_1$ and $P_2$ are indistinguishable and $P_2$ and $P_3$ are indistinguishable, we have $\Delta_{ij}(P_1) = d(P_1,S_i) - d(P_1,S_j) = d(P_2,S_i) - d(P_2,S_j) = \Delta_{ij}(P_2)$ and $\Delta_{ij}(P_2) = d(P_2,S_i) - d(P_2,S_j) = d(P_3,S_i) - d(P_3,S_j) = \Delta_{ij}(P_3)$, $1 \le i < j \le k$. So, $\Delta_{ij}(P_1) = d(P_1,S_i) - d(P_1,S_j) = d(P_3,S_i) - d(P_3,S_j) = \Delta_{ij}(P3)$, $1 \le i < j \le k$. Hence, the *indistinguishable* relation is transitive.

■

Clearly, the *indistinguishable* relation partitions Euclidean space $R^2$ into a collection of disjoint equivalence classes. If *SS* is an *ISS*, then each equivalence class is of unit cardinality; otherwise, the cardinality of at least one equivalence class is more than 1.

When $k = 2$, each equivalence class corresponds to a hyperbola with foci $S_1$ and $S_2$ and vice verse. The cardinality of each equivalence class in this case is infinite. When $k > 2$ and the sensors are collinear (Figure 10), each point on the line segment $\overline{S_1S_k}$, exclusive of $S_1$ and $S_k$, defines an equivalence class of unit cardinality because no such point has a dual. All points on the line $l$ that runs through the collinear sensors and that are to the left (right) of $S_1(S_k)$, inclusive, form an equivalence class of infinite cardinality. For each point $P$ not on the line $l$, has a single dual point $P'$ that is the reflection of $P$ with respect to $l$. Point $P$ and its dual $P'$ define an equivalence class of cardinality 2.

When the sensors are not collinear (this can happen only when $k > 2$), Theorem 9 establishes that the cardinality of each equivalence class is at most 2.

*Theorem 9:* When the sensors are not collinear, the cardinality of each equivalence class defined by the indistinguishable relation is at most 2.

*Proof:*

We prove this by contradiction. Let *SS* be the sensor set. Suppose there is an equivalence class whose cardinality is more than 2. Let $P_1$, $P_2$, and $P_3$ be any three points in this equivalence class. Since $P_1$ and $P_2$ are indistinguishable, from the proof of Theorem 1, it follows that there is a hyperbola $L_{12}$, whose foci are $P_1$ and $P_2$, that passes through the points of *SS*. Similarly, there is a hyperbola $L_{13}$, whose foci are $P_1$ and $P_3$, that passes through the points of *SS*. $L_{12}$ and $L_{13}$ intersect at at least the points of *SS*, which are more than 2 in number. This contradicts Theorem 7, which states that these two hyperbola may have at most two intersections.

■

## VI. ISSs for Polygonal Regions

Although 4 properly positioned sensors are required to uniquely identify a source in Euclidean space (Theorem 2), in many real-world applications, the monitoring region is bounded by a polygon and 3 sensors suffice. We assume that the sensors are restricted to be placed on or inside the bounding polygon. As an aside, we note that when the monitoring region is a simple line segment, say $\overline{S_iS_j}$, then two sensors placed at $S_i$ and $S_j$, respectively, are sufficient to uniquely identify any source on this segment. To see this, observe that as we move $P$ from $S_i$ to $S_j$ along the line segment $\overline{S_iS_j}$, $\Delta_{ij}(P)$ varies monotonically from $-d(S_i,S_j)$ to $d(S_i,S_j)$. Hence, there is no pair of indistinguishable points on this segment.

*Lemma 3:* Every non-degenerate simple polygon has an *MISS* whose size is 3.

*Proof:*

**Case 1:** The simple polygon is convex.

Let $S_1$ and $S_2$ be the end points of an edge of the polygon. Let $S_3$ be any other point on this edge. Note that the 3 chosen points are collinear and the entire convex polygon lies on one side of the edge that these 3 points lie on. From the discussion preceding Theorem 9, it follows that the dual of every point of the polygon that is not on this edge is on the other side of this edge. Points on the edge either have no dual or have dual(s) outside the polygon. Hence every point in or on the polygon is uniquely identifiable and $\{S_1,S_2,S_3\}$ is a size 3 *MISS* for the polygon.

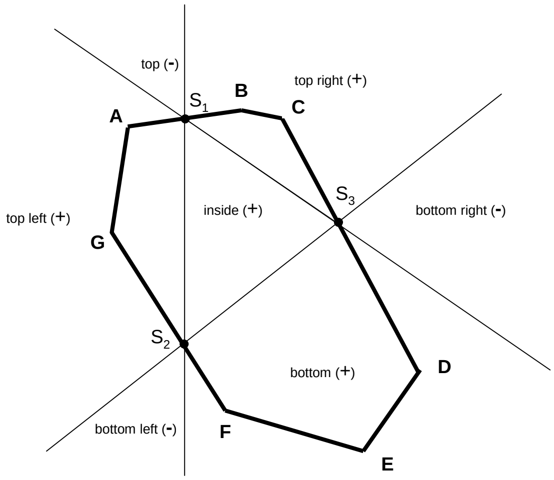


Fig. 11. Sensors $S_1$, $S_2$, and $S_3$ on the boundary of a convex polygon. The 7 planar regions induced by these 3 sensors are (a) top left, (b) inside, (c) bottom right, (d) top, (e) bottom left, (f) bottom, and (g) top right. The sign of the directional derivative for each region is also shown.

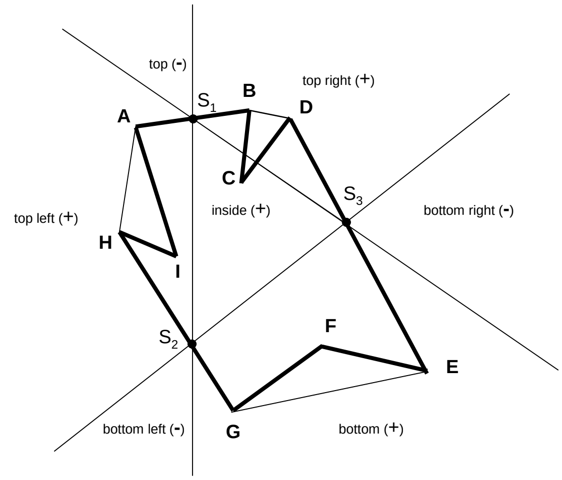


Fig. 12. A concave polygon, its bounding convex polygon, and three sensors $S_1$, $S_2$, and $S_3$ placed on the common boundary of the concave and convex polygons

An alternative construction for a size 3 *MISS* is to consider any 3 non-collinear points $S_1$, $S_2$, and $S_3$ that are on the boundary of the polygon (Figure 11). Now, the entire convex polygon must be contained in the union of four regions: (a) top left, (b) inside, (f) bottom, and (g) top right. From Theorem 5, the directional derivative of $\Delta_{13}$ along $L_{12}$ increases monotonically in each of these four regions. Further, the intersection of $L_{12}$ and the convex polygon is a continuous curve $C$ that is limited to these four regions (see Theorem 6). Since, $\Delta_{13}$ is monotonically increasing along $C$, $L_{12}$ and $L_{13}$ have at most one intersection on $C$. Hence, every point in or on the convex polygon is uniquely identifiable.

**Case 2:** The simple polygon is concave.

We start with a a minimum bounding convex polygon of the concave polygon (Figure 12). Let $S_1$, $S_2$, and $S_3$ be any three points on the intersection of the boundary of these concave and convex polygons. From Case 1, it follows that every point in and on the boundary of the convex bounding polygon, and so every point in and on the boundary of the concave polygon, is uniquely identifiable.

■

In Lemma 3, we prove that by choosing 3 sensor locations on the boundary of a simple polygon, an *SS* of size 3 uniquely identifies any source $S$ on or inside a simple polygon. We show in Lemma 4 when a sensor is placed strictly inside a simple polygon, 3 sensors are not sufficient to uniquely identify every point in or on the polygon.

*Lemma 4:* Let *SS* be an *ISS* set for a non-degenerate simple polygon. If at least one location of *SS* is inside the polygon, $|SS| \geq 4$.

*Proof:*

Suppose that *SS* is an *ISS* and that $|SS| = 3$. W.l.o.g, assume $S_1$ lies inside the simple polygon as shown in Figure 13. Note that a portion of the simple polygon must lie inside the top region. We may choose two negative constants $\delta_{12}$ and $\delta_{13}$, such that $L_{12}(\delta_{12})$ and $L_{13}(\delta_{13})$ intersect at two distinct points $P_1$ in the top region and $P_2$ in the top left region. Since both

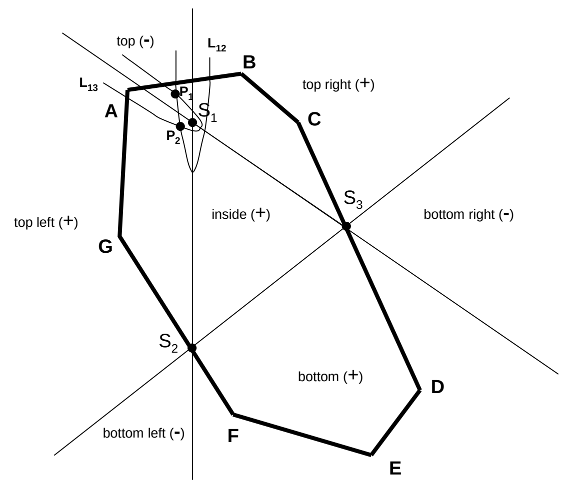


Fig. 13. $S_1$ lies inside a simple polygon while $S_2$ and $S_3$ are on the boundary. $P_1$ in the top region is a dual point of $P_2$ which lies in the top left region.

$P_1$ and $P_2$ are inside the simple polygon and $P_1$ is the dual of $P_2$, *SS* is not an *ISS* for the points of the simple polygon.

■

*Theorem 10:* 3 sensors can uniquely identify any source in or on a non-degenerate simple polygon iff the sensors are on the common boundary of the given polygon and its minimum bounding convex polygon. In case the 3 boundary sensors are collinear, 2 must be at the end points of an edge of the bounding convex polygon and the third at an in-between point.

*Proof:*

Follows from Lemmas 3 and 4.

■

## VII. Conclusions

In this paper, we studied the impact of sensor deployment on the uniqueness of sournce estimate in Euclidean plane as well as in a simple polygon. We derived necessary and sufficient conditions for each case. A tight bound on the size of a minimal identifying sensor set in $R^2$ was given. We re-investigated the number of intersections of two hyperbolas having a common focus, and showed it to be at most 2. Specifically, at most one intersection lies in the union of inside region, top left region, top right region, and bottom region, while at most one intersection lies in the union of top region, bottom left region, and bottom right region. Each sensor deployment corresponds to an equivalence relation on $R^2$. For each identifying sensor set, each equivalence class is of unit cardinality. For each non-identifying sensor set, at least one equivalence class is of greater than unit cardinality.

There are several future directions to be considered. It would be interesting to study the effect of randomness in distance-differences, which could be due to measurement errors or due to the underlying process. In particular, if would be interesting to investicate the effects on both uniqueness and mimimality results presented in this paper. Applications of these methods to practical radiation detection systems would be of future interest.

## ACKNOWLEDGMENTS

This work is funded by the SensorNet program at Oak Ridge National Laboratory (ORNL) through Office of Naval Research. ORNL is managed by UT-Battelle, LLC for U.S. Department of Energy under Contract No. DE-AC05-00OR22725.

## REFERENCES


[1] R. Schmidt, "A new approach to geometry of range difference location," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 8, no. 3, 1972.

[2] G. Mellen, M. Pachter, and J. Raquet, "Closed-form solution for determining emitter location using time difference of arrival measurements," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 39, no. 3, pp. 1056–1058, 2003.

[3] A. H. Sayed, A. Tarighat, and N. Khajehnouri, Network-based wireless location, *IEEE Signal Processing Magazine,* pp. 24-40, July 2005.

[4] F. Zhao and L. Guibas, Wireless Sensor Networks, Elsevier, 2004.

[5] B. Krishnamachari, Ed., Networking Wireless Sensors, Cambridge University Press, 2005.

[6] H. C. Schau and A. Z. Robinson, "Passive source localization employing intersecting spherical surfaces from time-of-arrival differences," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 35, no. 8, pp. 1223–1225, 1987.

[7] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Trans. on Image Processing*, vol. 42, no. 8, pp. 1905–1915, 1994.

[8] B. T. Fang, "Simple solutions for hyperbolic and related position fixes," *IEEE Transactions on Systems, Man and Cybernetics-B*, vol. 26, no. 9, pp. 748–753, 2005.

[9] F. Gustafsson and F. Gunnarson, Positioning using time-difference of arrival measurements, *IEEE International Conference on Acoustics, Speech, and Signal Processing(ICASSP'03)*, 2003.

[10] N. S. V. Rao, "Identification of simple product-form plumes using networks of sensors with random errors," in *International Conference on Information Fusion*, 2006.

[11] N. S. V. Rao, X. Xu, and S. Sahni, "A Computational Geometry Method for DTOA Triangulation," submitted to *International Conference on Information Fusion*, 2007.

[12] X. Xu, N. S. V. Rao, and S. Sahni, "A computational geometric method for triangulation using differences of distances," *ACM Transactions on Sensor Networks*, to appear.

[13] F. Kirwan, "Complex Algebraic Curves," United Kingdom: Cambridge University Press, 1992.