Subset & Permutation Problems

- Subset problem of size $n$.
  - Nonsystematic search of the space for the answer takes $O(p^{2^n})$ time, where $p$ is the time needed to evaluate each member of the solution space.

- Permutation problem of size $n$.
  - Nonsystematic search of the space for the answer takes $O(pn!)$ time, where $p$ is the time needed to evaluate each member of the solution space.

- Backtracking and branch and bound perform a systematic search; often taking much less time than taken by a nonsystematic search.

Tree Organization Of Solution Space

- Set up a tree structure such that the leaves represent members of the solution space.
- For a size $n$ subset problem, this tree structure has $2^n$ leaves.
- For a size $n$ permutation problem, this tree structure has $n!$ leaves.
- The tree structure is too big to store in memory; it also takes too much time to create the tree structure.
- Portions of the tree structure are created by the backtracking and branch and bound algorithms as needed.

Subset Problem

- Use a full binary tree that has $2^n$ leaves.
- At level $i$ the members of the solution space are partitioned by their $x_i$ values.
- Members with $x_i = 1$ are in the left subtree.
- Members with $x_i = 0$ are in the right subtree.
- Could exchange roles of left and right subtree.
Subset Tree For n = 4

Permutation Problem

- Use a tree that has $n!$ leaves.
- At level $i$ the members of the solution space are partitioned by their $x_i$ values.
- Members (if any) with $x_i = 1$ are in the first subtree.
- Members (if any) with $x_i = 2$ are in the next subtree.
- And so on.

Permutation Tree For n = 3

Backtracking

- Search the solution space tree in a depth-first manner.
- May be done recursively or use a stack to retain the path from the root to the current node in the tree.
- The solution space tree exists only in your mind, not in the computer.
Backtracking Depth-First Search

Tree structure with nodes labeled with $x_1=1$, $x_1=0$, $x_2=1$, and $x_2=0$. The nodes are connected in a depth-first manner, with branches indicating the progression of the search.
Backtracking Depth-First Search

O(2^n) Subset Sum & Bounding Functions

{10, 5, 2, 1}, c = 14

Each forward and backward move takes $O(1)$ time.

Bounding Functions

- When a node that represents a subset whose sum equals the desired sum $c$, terminate.
- When a node that represents a subset whose sum exceeds the desired sum $c$, backtrack. I.e., do not enter its subtrees, go back to parent node.
- Keep a variable $r$ that gives you the sum of the numbers not yet considered. When you move to a right child, check if current subset sum + $r$ $\geq$ c. If not, backtrack.

Backtracking

- Space required is $O(\text{tree height})$.
- With effective bounding functions, large instances can often be solved.
- For some problems (e.g., 0/1 knapsack), the answer (or a very good solution) may be found quickly but a lot of additional time is needed to complete the search of the tree.
- Run backtracking for as much time as is feasible and use best solution found up to that time.
Branch And Bound

• Search the tree using a breadth-first search (FIFO branch and bound).
• Search the tree as in a bfs, but replace the FIFO queue with a stack (LIFO branch and bound).
• Replace the FIFO queue with a priority queue (least-cost (or max priority) branch and bound). The priority of a node $p$ in the queue is based on an estimate of the likelihood that the answer node is in the subtree whose root is $p$.

Branch And Bound

• Space required is $O$ (number of leaves).
• For some problems, solutions are at different levels of the tree (e.g., 16 puzzle).

FIFO branch and bound finds solution closest to root. Backtracking may never find a solution because tree depth is infinite (unless repeating configurations are eliminated).

Least-cost branch and bound directs the search to parts of the space most likely to contain the answer. So it could perform better than backtracking.