Rank

Rank of an element is its position in ascending key order.

\[ [2,6,7,8,10,15,18,20,25,30,35,40] \]

\[ \text{rank}(2) = 0 \]
\[ \text{rank}(15) = 5 \]
\[ \text{rank}(20) = 7 \]

Selection Problem

- Given \( n \) unsorted elements, determine the \( k \)'th smallest element. That is, determine the element whose rank is \( k-1 \).

- Applications
  - Median score on a test.
    - \( k = \text{ceil}(n/2) \).
  - Median salary of Computer Scientists.
  - Identify people whose salary is in the bottom 10\%. First find salary at the 10\% rank.
Selection By Sorting

- Sort the $n$ elements.
- Pick up the element with desired rank.
- $O(n \log n)$ time.

Divide-And-Conquer Selection

- Small instance has $n \leq 1$. Selection is easy.
- When $n > 1$, select a pivot element from out of the $n$ elements.
- Partition the $n$ elements into 3 groups left, middle and right as is done in quick sort.
- The rank of the pivot is the location of the pivot following the partitioning.
- If $k-1 = \text{rank}(\text{pivot})$, pivot is the desired element.
- If $k-1 < \text{rank}(\text{pivot})$, determine the $k$’th smallest element in left.
- If $k-1 > \text{rank}(\text{pivot})$, determine the $(k-\text{rank}(\text{pivot})-1)$’th smallest element in right.
D&C Selection Example

Find $k$th element of:

$$a = [3, 2, 8, 0, 1, 10, 1, 2, 9, 7, 1]$$

Use 3 as the pivot and partition.

$$a = [1, 2, 4, 0, 2, 3, 10, 11, 9, 7, 8]$$

$\text{rank}(\text{pivot}) = 5$. So pivot is the 6’th smallest element.

D&C Selection Example

$$a = [1, 2, 4, 0, 2, 3, 10, 11, 9, 7, 8]$$

- If $k = 6$ ($k-1 = \text{rank}(\text{pivot})$), pivot is the element we seek.
- If $k < 6$ ($k-1 < \text{rank}(\text{pivot})$), find $k$’th smallest element in left partition.
- If $k > 6$ ($k-1 > \text{rank}(\text{pivot})$), find $(k-\text{rank}(\text{pivot})-1)$’th smallest element in right partition.
Time Complexity

- Worst case arises when the partition to be searched always has all but the pivot.
  - $O(n^2)$
- Expected performance is $O(n)$.
- Worst case becomes $O(n)$ when the pivot is chosen carefully.
  - Partition into $n/9$ groups with 9 elements each (last group may have a few more)
  - Find the median element in each group.
  - pivot is the median of the group medians.
  - This median is found using select recursively.

Closest Pair Of Points

- Given $n$ points in 2D, find the pair that are closest.
Applications

• We plan to drill holes in a metal sheet.
• If the holes are too close, the sheet will tear during drilling.
• Verify that no two holes are closer than a threshold distance (e.g., holes are at least 1 inch apart).

Air Traffic Control

• 3D -- Locations of airplanes flying in the neighborhood of a busy airport are known.
• Want to be sure that no two planes get closer than a given threshold distance.
Simple Solution

- For each of the \( \frac{n(n-1)}{2} \) pairs of points, determine the distance between the points in the pair.
- Determine the pair with the minimum distance.
- \( O(n^2) \) time.

Divide-And-Conquer Solution

- When \( n \) is small, use simple solution.
- When \( n \) is large
  - Divide the point set into two roughly equal parts \( A \) and \( B \).
  - Determine the closest pair of points in \( A \).
  - Determine the closest pair of points in \( B \).
  - Determine the closest pair of points such that one point is in \( A \) and the other in \( B \).
  - From the three closest pairs computed, select the one with least distance.
Example

- Divide so that points in $A$ have $x$-coordinate $\leq$ that of points in $B$.

Example

- Find closest pair in $A$.
- Let $d_1$ be the distance between the points in this pair.
Example

A \hspace{3cm} B

• Find closest pair in B.
• Let $d_2$ be the distance between the points in this pair.

Example

A \hspace{3cm} B

• Let $d = \min\{d_1, d_2\}$.
• Is there a pair with one point in A, the other in B and distance $< d$?
Example

- Candidates lie within $d$ of the dividing line.
- Call these regions $R_A$ and $R_B$, respectively.

Example

- Let $q$ be a point in $R_A$.
- $q$ need be paired only with those points in $R_B$ that are within $d$ of $q$.y.
Example

• Points that are to be paired with q are in a d x 2d rectangle of $R_B$ (comparing region of q).
• Points in this rectangle are at least d apart.

Example

• So the comparing region of q has at most 6 points.
• So number of pairs to check is $\leq 6| R_A | = O(n)$. 
**Time Complexity**

- Create a sorted by $x$-coordinate list of points.
  - $O(n \log n)$ time.
- Create a sorted by $y$-coordinate list of points.
  - $O(n \log n)$ time.
- Using these two lists, the required pairs of points from $R_A$ and $R_B$ can be constructed in $O(n)$ time.
- Let $n < 4$ define a small instance.

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**Time Complexity**

- Let $t(n)$ be the time to find the closest pair (excluding the time to create the two sorted lists).
- $t(n) = c$, $n < 4$, where $c$ is a constant.
- When $n \geq 4$,
  $$t(n) = t(\text{ceil}(n/2)) + t(\text{floor}(n/2)) + an,$$
  where $a$ is a constant.
- To solve the recurrence, assume $n$ is a power of 2 and use repeated substitution.
- $t(n) = O(n \log n)$.  