## Divide-And-Conquer Sorting

- Small instance.
  - n <= 1 elements.
  - n <= 10 elements.
  - We'll use n <= 1 for now.
- Large instance.
  - Divide into k >= 2 smaller instances.
  - k = 2, 3, 4, … ?
  - What does each smaller instance look like?
  - Sort smaller instances recursively.
  - How do you combine the sorted smaller instances?

## Insertion Sort


- k = 2
- First n - 1 elements (a[0:n-2]) define one of the smaller instances; last element (a[n-1]) defines the second smaller instance.
- a[0:n-2] is sorted recursively.
- a[n-1] is a small instance.

## Insertion Sort


- Combining is done by inserting a[n-1] into the sorted a[0:n-2] .
- Complexity is $O(n^2)$.
- Usually implemented nonrecursively.

## Selection Sort


- k = 2
- To divide a large instance into two smaller instances, first find the largest element.
- The largest element defines one of the smaller instances; the remaining n-1 elements define the second smaller instance.

## Selection Sort


- The second smaller instance is sorted recursively.
- Append the first smaller instance (largest element) to the right end of the sorted smaller instance.
- Complexity is $O(n^2)$.
- Usually implemented nonrecursively.

## Bubble Sort

- Bubble sort may also be viewed as a $k = 2$ divide-and-conquer sorting method.
- Insertion sort, selection sort and bubble sort divide a large instance into one smaller instance of size $n - 1$ and another one of size 1.
- All three sort methods take $O(n^2)$ time.

## Divide And Conquer

- Divide-and-conquer algorithms generally have best complexity when a large instance is divided into smaller instances of approximately the same size.
- When $k = 2$ and $n = 24$, divide into two smaller instances of size 12 each.
- When $k = 2$ and $n = 25$, divide into two smaller instances of size 13 and 12, respectively.

## Merge Sort

- $k = 2$
- First ceil(n/2) elements define one of the smaller instances; remaining floor(n/2) elements define the second smaller instance.
- Each of the two smaller instances is sorted recursively.
- The sorted smaller instances are combined using a process called merge.
- Complexity is $O(n \log n)$.
- Usually implemented nonrecursively.

## Merge Two Sorted Lists

- A = (2, 5, 6)
  B = (1, 3, 8, 9, 10)
  C = ()
- Compare smallest elements of A and B and merge smaller into C.
- A = (2, 5, 6)
  B = (3, 8, 9, 10)
  C = (1)

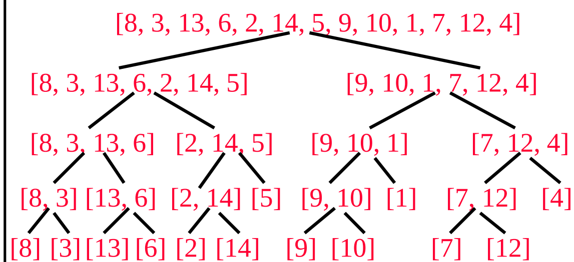
## Merge Two Sorted Lists

- A = (5, 6)
  B = (3, 8, 9, 10)
  C = (1, 2)
- A = (5, 6)
  B = (8, 9, 10)
  C = (1, 2, 3)
- A = (6)
  B = (8, 9, 10)
  C = (1, 2, 3, 5)

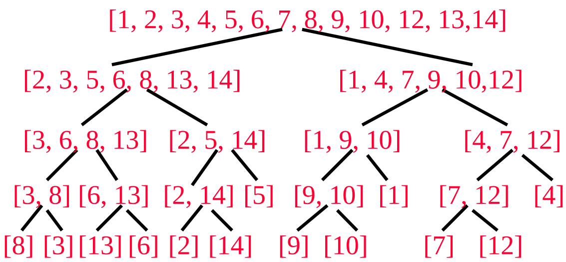
## Merge Two Sorted Lists

- A = ()
  B = (8, 9, 10)
  C = (1, 2, 3, 5, 6)
- When one of A and B becomes empty, append the other list to C.
- O(1) time needed to move an element into C.
- Total time is O(n + m), where n and m are, respectively, the number of elements initially in A and B.

## Merge Sort



## Merge Sort




## Time Complexity

- Let t(n) be the time required to sort n elements.
- t(0) = t(1) = c, where c is a constant.
- When n > 1,

  t(n) = t(ceil(n/2)) + t(floor(n/2)) + dn,

  where d is a constant.
- To solve the recurrence, assume n is a power of 2 and use repeated substitution.
- t(n) = O(n log n).

## Merge Sort

- Downward pass over the recursion tree.
  - Divide large instances into small ones.
- Upward pass over the recursion tree.
  - Merge pairs of sorted lists.
- Number of leaf nodes is n.
- Number of nonleaf nodes is n-1.

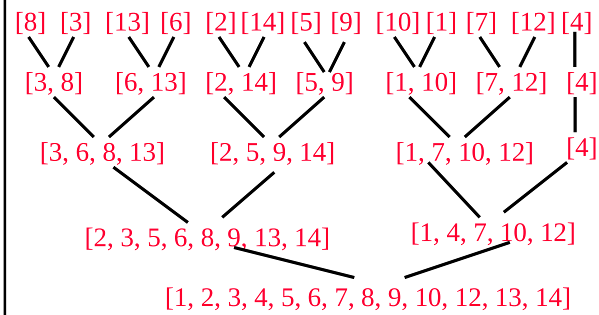
## Time Complexity

- Downward pass.
  - O(1) time at each node.
  - O(n) total time at all nodes.
- Upward pass.
  - O(n) time merging at each level that has a nonleaf node.
  - Number of levels is O(log n).
  - Total time is O(n log n).

## Nonrecursive Version

- Eliminate downward pass.
- Start with sorted lists of size 1 and do pairwise merging of these sorted lists as in the upward pass.

## Nonrecursive Merge Sort




## Complexity

- Sorted segment size is 1, 2, 4, 8, …
- Number of merge passes is ceil($\log_2 n$).
- Each merge pass takes O(n) time.
- Total time is O(n log n).
- Need O(n) additional space for the merge.
- Merge sort is slower than insertion sort when n <= 15 (approximately). So define a small instance to be an instance with n <= 15.
- Sort small instances using insertion sort.
- Start with segment size = 15.

## Natural Merge Sort

- Initial sorted segments are the naturally ocurring sorted segments in the input.
- Input = [8, 9, 10, 2, 5, 7, 9, 11, 13, 15, 6, 12, 14].
- Initial segments are:

  [8, 9, 10] [2, 5, 7, 9, 11, 13, 15] [6, 12, 14]
- 2 (instead of 4) merge passes suffice.
- Segment boundaries have a[i] > a[i+1].

# Quick Sort

- Small instance has n <= 1. Every small instance is a sorted instance.
- To sort a large instance, select a pivot element from out of the n elements.
- Partition the n elements into 3 groups left, middle and right.
- The middle group contains only the pivot element.
- All elements in the left group are <= pivot.
- All elements in the right group are >= pivot.
- Sort left and right groups recursively.
- Answer is sorted left group, followed by middle group followed by sorted right group.

# Example

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 2 | 8 | 5 | 11 | 10 | 4 | 1 | 9 | 7 | 3 |

Use 6 as the pivot.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 4 | 1 | 3 | 6 | 7 | 9 | 10 | 11 | 8 |

Sort left and right groups recursively.

# Choice Of Pivot

- Pivot is leftmost element in list that is to be sorted.
  - When sorting a[6:20], use a[6] as the pivot.
  - Text implementation does this.
- Randomly select one of the elements to be sorted as the pivot.
  - When sorting a[6:20], generate a random number r in the range [6, 20]. Use a[r] as the pivot.

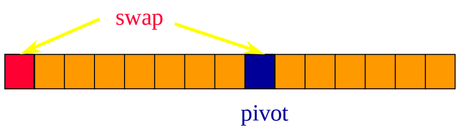
# Choice Of Pivot

- Median-of-Three rule. From the leftmost, middle, and rightmost elements of the list to be sorted, select the one with median key as the pivot.
  - When sorting a[6:20], examine a[6], a[13] ((6+20)/2), and a[20]. Select the element with median (i.e., middle) key.
  - If a[6].key = 30, a[13].key = 2, and a[20].key = 10, a[20] becomes the pivot.
  - If a[6].key = 3, a[13].key = 2, and a[20].key = 10, a[6] becomes the pivot.

## Choice Of Pivot

- If a[6].key = 30, a[13].key = 25, and a[20].key = 10, a[13] becomes the pivot.

- When the pivot is picked at random or when the median-of-three rule is used, we can use the quick sort code of the text provided we first swap the leftmost element and the chosen pivot.




## Partitioning Into Three Groups

- Sort a = [6, 2, 8, 5, 11, 10, 4, 1, 9, 7, 3].
- Leftmost element (6) is the pivot.
- When another array b is available:
  - Scan a from left to right (omit the pivot in this scan), placing elements <= pivot at the left end of b and the remaining elements at the right end of b.
  - The pivot is placed at the remaining position of the b.

## Partitioning Example Using Additional Array

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 6 | 2 | 8 | 5 | 11 | 10 | 4 | 1 | 9 | 7 | 3 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| b | 2 | 5 | 4 | 1 | 3 | 6 | 7 | 9 | 10 | 11 | 8 |

Sort left and right groups recursively.

## In-place Partitioning

- Find leftmost element (bigElement) > pivot.
- Find rightmost element (smallElement) < pivot.
- Swap bigElement and smallElement provided bigElement is to the left of smallElement.
- Repeat.

## In-Place Partitioning Example

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 6 | 2 | 8 | 5 | 11 | 10 | 4 | 1 | 9 | 7 | 3 |
| a | 6 | 2 | 3 | 5 | 11 | 10 | 4 | 1 | 9 | 7 | 8 |
| a | 6 | 2 | 3 | 5 | 1 | 10 | 4 | 11 | 9 | 7 | 8 |
| a | 6 | 2 | 3 | 5 | 1 | 4 | 10 | 11 | 9 | 7 | 8 |

bigElement is not to left of smallElement, terminate process. Swap pivot and smallElement.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 4 | 2 | 3 | 5 | 1 | 6 | 10 | 11 | 9 | 7 | 8 |

## Complexity

- $O(n)$ time to partition an array of $n$ elements.
- Let $t(n)$ be the time needed to sort $n$ elements.
- $t(0) = t(1) = c$, where $c$ is a constant.
- When $t > 1$,

  $t(n) = t(|left|) + t(|right|) + dn$,

  where $d$ is a constant.
- $t(n)$ is maximum when either $|left| = 0$ or $|right| = 0$ following each partitioning.

## Complexity

- This happens, for example, when the pivot is always the smallest element.
- For the worst-case time,

  $t(n) = t(n-1) + dn$, $n > 1$
- Use repeated substitution to get $t(n) = O(n^2)$.
- The best case arises when $|left|$ and $|right|$ are equal (or differ by 1) following each partitioning.
- For the best case, the recurrence is the same as for merge sort.

## Complexity Of Quick Sort

- So the best-case complexity is $O(n \log n)$.
- Average complexity is also $O(n \log n)$.
- To help get partitions with almost equal size, change in-place swap rule to:
  - Find leftmost element (bigElement) >= pivot.
  - Find rightmost element (smallElement) <= pivot.
  - Swap bigElement and smallElement provided bigElement is to the left of smallElement.
- $O(n)$ space is needed for the recursion stack. May be reduced to $O(\log n)$ (see Exercise 19.22).

## Complexity Of Quick Sort

- To improve performance, define a small instance to be one with n <= 15 (say) and sort small instances using insertion sort.

## java.util.arrays.sort

- Arrays of a primitive data type are sorted using quick sort.
  - n < 7 => insertion sort
  - 7 <= n <= 40 => median of three
  - n > 40 => pseudo median of 9 equally spaced elements
    - divide the 9 elements into 3 groups
    - find the median of each group
    - pivot is median of the 3 group medians

## java.util.arrays.sort

- Arrays of a nonprimitive data type are sorted using merge sort.
  - n < 7 => insertion sort
  - skip merge when last element of left segment is <= first element of right segment
- Merge sort is stable (relative order of elements with equal keys is not changed).
- Quick sort is not stable.