Algorithm Design Methods

- Greedy method.
- Divide and conquer.
- Dynamic Programming.
- Backtracking.
- Branch and bound.

Some Methods Not Covered

- Linear Programming.
- Integer Programming.
- Simulated Annealing.
- Neural Networks.
- Genetic Algorithms.
- Tabu Search.
Optimization Problem

A problem in which some function (called the optimization or objective function) is to be optimized (usually minimized or maximized) subject to some constraints.

Machine Scheduling

Find a schedule that minimizes the finish time.
- optimization function … finish time
- constraints
  - each job is scheduled continuously on a single machine for an amount of time equal to its processing requirement
  - no machine processes more than one job at a time
Bin Packing

Pack items into bins using the fewest number of bins.
- optimization function \( \text{\text{number of bins}} \)
- constraints
  - each item is packed into a single bin
  - the capacity of no bin is exceeded

Min Cost Spanning Tree

Find a spanning tree that has minimum cost.
- optimization function \( \text{\text{sum of edge costs}} \)
- constraints
  - must select \( n-1 \) edges of the given \( n \) vertex graph
  - the selected edges must form a tree
Feasible And Optimal Solutions

A feasible solution is a solution that satisfies the constraints.

An optimal solution is a feasible solution that optimizes the objective/optimization function.

Greedy Method

• Solve problem by making a sequence of decisions.
• Decisions are made one by one in some order.
• Each decision is made using a greedy criterion.
• A decision, once made, is (usually) not changed later.
Machine Scheduling

LPT Scheduling.

- Schedule jobs one by one and in decreasing order of processing time.
- Each job is scheduled on the machine on which it finishes earliest.
- Scheduling decisions are made serially using a greedy criterion (minimize finish time of this job).
- LPT scheduling is an application of the greedy method.

LPT Schedule

- LPT rule does not guarantee minimum finish time schedules.
- \( \frac{(LPT \text{ Finish Time})}{(Minimum \text{ Finish Time})} \leq \frac{4}{3} - \frac{1}{(3m)} \)
  where \( m \) is number of machines
- Minimum finish time scheduling is NP-hard.
- In this case, the greedy method does not work.
- The greedy method does, however, give us a good heuristic for machine scheduling.
Container Loading

- Ship has capacity $c$.
- $m$ containers are available for loading.
- Weight of container $i$ is $w_i$.
- Each weight is a positive number.
- Sum of container weights $> c$.
- Load as many containers as is possible without sinking the ship.

Greedy Solution

- Load containers in increasing order of weight until we get to a container that doesn’t fit.
- Does this greedy algorithm always load the maximum number of containers?
- Yes. May be proved using a proof by induction (see text).
Container Loading With 2 Ships

Can all containers be loaded into 2 ships whose capacity is $c$ (each)?

- Same as bin packing with 2 bins.
  - Are 2 bins sufficient for all items?
- Same as machine scheduling with 2 machines.
  - Can all jobs be completed by 2 machines in $c$ time units?
- NP-hard.

0/1 Knapsack Problem
0/1 Knapsack Problem

• Hiker wishes to take \( n \) items on a trip.
• The weight of item \( i \) is \( w_i \).
• The items are to be carried in a knapsack whose weight capacity is \( c \).
• When sum of item weights \( \leq c \), all \( n \) items can be carried in the knapsack.
• When sum of item weights \( > c \), some items must be left behind.
• Which items should be taken/left?

0/1 Knapsack Problem

• Hiker assigns a profit/value \( p_i \) to item \( i \).
• All weights and profits are positive numbers.
• Hiker wants to select a subset of the \( n \) items to take.
  - The weight of the subset should not exceed the capacity of the knapsack. (constraint)
  - Cannot select a fraction of an item. (constraint)
  - The profit/value of the subset is the sum of the profits of the selected items. (optimization function)
  - The profit/value of the selected subset should be maximum. (optimization criterion)
0/1 Knapsack Problem

Let \( x_i = 1 \) when item \( i \) is selected and let \( x_i = 0 \) when item \( i \) is not selected.

\[
\text{maximize } \sum_{i=1}^{n} p_i x_i
\]

subject to \( \sum_{i=1}^{n} w_i x_i \leq c \)

and \( x_i = 0 \) or 1 for all \( i \)

Greedy Attempt 1

Be greedy on capacity utilization.

- Select items in increasing order of weight.

\( n = 2, \) \( c = 7 \)
\( w = [3, 6] \)
\( p = [2, 10] \)

only item 1 is selected
profit (value) of selection is 2
not best selection!
Greedy Attempt 2

Be greedy on profit earned.

- Select items in decreasing order of profit.

\( n = 3, \ c = 7 \)
\( w = [7, 3, 2] \)
\( p = [10, 8, 6] \)

only item 1 is selected
profit (value) of selection is 10
not best selection!

Greedy Attempt 3

Be greedy on profit density \((p/w)\).

- Select items in decreasing order of profit density.

\( n = 2, \ c = 7 \)
\( w = [1, 7] \)
\( p = [10, 20] \)

only item 1 is selected
profit (value) of selection is 10
not best selection!
Greedy Attempt 3

Be greedy on profit density \((p/w)\).
- Works when selecting a fraction of an item is permitted
- Select items in decreasing order of profit density, if next item doesn’t fit take a fraction so as to fill knapsack.

\[ n = 2, \ c = 7 \]
\[ w = [1, 7] \]
\[ p = [10, 20] \]

item 1 and 6/7 of item 2 are selected

0/1 Knapsack Greedy Heuristics

- Select a subset with \(\leq k\) items.
- If the weight of this subset is \(> c\), discard the subset.
- If the subset weight is \(\leq c\), fill as much of the remaining capacity as possible by being greedy on profit density.
- Try all subsets with \(\leq k\) items and select the one that yields maximum profit.
0/1 Knapsack Greedy Heuristics

• (best value - greedy value)/(best value) <= 1/(k+1)

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Number of solutions (out of 600) within x% of best.

0/1 Knapsack Greedy Heuristics

• First sort into decreasing order of profit density.
• There are $O(n^k)$ subsets with at most $k$ items.
• Trying a subset takes $O(n)$ time.
• Total time is $O(n^{k+1})$ when $k > 0$.
• (best value - greedy value)/(best value) <= 1/(k+1)