Initializing A Max Heap

input array = [-, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

Start at rightmost array position that has a child.
Index is \( n/2 \).
Initializing A Max Heap

Move to next lower array position.
Initializing A Max Heap

[Diagram of a max heap with numbers 1 to 11, where 1 is at the root, 2 and 7 are children, and 9, 10, 4, 8, 5, 11, 6, 3 are further down the tree.]

Initializing A Max Heap

[Same diagram as above.]
Find a home for 2.
Initializing A Max Heap

Done, move to next lower array position.

Find home for 1.
Initializing A Max Heap

Find home for 1.

Initializing A Max Heap

Find home for 1.
Initializing A Max Heap

Find home for 1.

Done.
**Time Complexity**

Height of heap = $h$.

Number of subtrees with root at level $j$ is $\leq 2^{j-1}$.

Time for each subtree is $O(h-j+1)$.

**Complexity**

Time for level $j$ subtrees is $\leq 2^{j-1}(h-j+1) = t(j)$.

Total time is $t(1) + t(2) + \ldots + t(h-1) = O(n)$. 
Leftist Trees

Linked binary tree.
Can do everything a heap can do and in the same asymptotic complexity.
Can meld two leftist tree priority queues in $O(\log n)$ time.

Extended Binary Trees

Start with any binary tree and add an external node wherever there is an empty subtree.
Result is an extended binary tree.
A Binary Tree

An Extended Binary Tree

number of external nodes is $n + 1$
The Function \( s() \)

For any node \( x \) in an extended binary tree, let \( s(x) \) be the length of a shortest path from \( x \) to an external node in the subtree rooted at \( x \).

\( \text{s()} \) Values Example
Properties Of $s()$

If $x$ is an external node, then $s(x) = 0$.

Otherwise,

$$s(x) = \min \{ s(\text{leftChild}(x)), \ s(\text{rightChild}(x)) \} + 1$$
Height Biased Leftist Trees

A binary tree is a (height biased) leftist tree iff for every internal node $x$,
$s(\text{leftChild}(x)) \geq s(\text{rightChild}(x))$
Leftist Trees--Property 1

In a leftist tree, the rightmost path is a shortest root to external node path and the length of this path is $s(\text{root})$. 

A Leftist Tree

Length of rightmost path is 2.
Leftist Trees—Property 2

The number of internal nodes is at least

\[ 2^{s(\text{root})} - 1 \]

Because levels 1 through \( s(\text{root}) \) have no external nodes.
So, \( s(\text{root}) \leq \log(n+1) \)

A Leftist Tree

Levels 1 and 2 have no external nodes.
Leftist Trees—Property 3

Length of rightmost path is $O(\log n)$, where $n$ is the number of nodes in a leftist tree.

Follows from Properties 1 and 2.

Leftist Trees As Priority Queues

Min leftist tree … leftist tree that is a min tree.

Used as a min priority queue.

Max leftist tree … leftist tree that is a max tree.

Used as a max priority queue.
A Min Leftist Tree

Some Min Leftist Tree Operations

put()
remove()
meld()
initialize()
put() and remove() use meld().
Create a single node min leftist tree.
Create a single node min leftist tree.
Meld the two min leftist trees.

Remove Min
Remove Min

Remove the root.

Remove Min

Remove the root.
Meld the two subtrees.
Traverse only the rightmost paths so as to get logarithmic performance.

Meld right subtree of tree with smaller root and all of other tree.
Meld Two Min Leftist Trees

Meld right subtree of tree with smaller root and all of other tree.

Meld Two Min Leftist Trees

Meld right subtree of tree with smaller root and all of other tree.
Meld Two Min Leftist Trees

Meld right subtree of tree with smaller root and all of other tree.

Right subtree of 6 is empty. So, result of melding right subtree of tree with smaller root and other tree is the other tree.

Meld Two Min Leftist Trees

Make melded subtree right subtree of smaller root.

Swap left and right subtree if $s(\text{left}) < s(\text{right})$. 
Meld Two Min Leftist Trees

Make melded subtree right subtree of smaller root.

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Meld Two Min Leftist Trees

Make melded subtree right subtree of smaller root.

Swap left and right subtree if $s(\text{left}) < s(\text{right})$. 
Meld Two Min Leftist Trees

Initializing In O(n) Time

- create \( n \) single node min leftist trees and place them in a FIFO queue
- repeatedly remove two min leftist trees from the FIFO queue, meld them, and put the resulting min leftist tree into the FIFO queue
- the process terminates when only 1 min leftist tree remains in the FIFO queue
- analysis is the same as for heap initialization