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Lecture MAX k-FUNCTION SAT

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Problem 1 *Given n Boolean variables $x_1, x_2, \ldots, x_n$ and m functions $f_1, \ldots, f_m$ each of which is a function of k of the boolean variables, find a truth assignment to $x_1, \ldots, x_n$ that maximizes the number of functions satisfied. Here k is a fixed constant (not part of input).*

Lemma 1 *There exists a constant k for which there is a polynomial-time reduction from* ***SAT*** *to MAX k-FUNCTION SAT that transforms a boolean formula ϕ to an instance I of MAX k-FUNCTION SAT such that:*

- *If ϕ is satisfiable, $OPT(I) = m$ and*
- *If ϕ is not satisfiable, then $OPT(I) < \frac{1}{2}m$*

Proof: Note that a E3SAT formula φ on m clause can be seen as a set of m k-functions where $k = 3$. Hence, the Theorem 3 on GAP-MAX-E3SAT$_{1,s}$ is a stronger result than this lemma. In other words, the proof presented here is actually a subpart of the proof of the Theorem 3.

We first show that there is a polynomial time reduction from an instance G of SAT to an instance I of MAX k-FUNCTION SAT. Since SAT $\in$ NP, there is a verifier $V \in PCP(clogn, q)$ for SAT where c, q are fixed constants. Given a proof π, for each possible random string r define boolean function f_r as the restriction of acceptance/rejection of V to the corresponding q bits. Set $k = q$ and I consists of f_r for all possible random strings r. Each function consists of at most q bits (variable). Moreover, there are at most n^c different random string r, we have at most $poly(n)$ number of functions f_r. Therefore, I is an instance of MAX k-FUNCTION SAT with size at most $poly(n)$. Second, we prove that the reduction satisfies the two conditions:

- *Completeness.* If $G \in SAT$, there exists a proof π such that Prob$[V(G, \pi) = YES] = 1$. Thus, π is also a truth assignment satisfies all m k-functions.
- *Soundness.* If $G \in SAT$, for every proof π, Prob$[V(G, \pi) = YES] < \frac{1}{2}$. Thus there are always less than $\frac{1}{2}m$ satisfied functions i.e. $OPT(I) < \frac{1}{2}m$.

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References

[1] V.V. Vazirani, Approximation Algorithms, Springer, 2001.