Topological Relationships Between Complex Lines and Complex Regions

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Abstract. Topological relationships between spatial objects in the two-dimensional space have been investigated for a long time in a number of disciplines like artificial intelligence, cognitive science, linguistics, and robotics. In the context of spatial databases and geographical information systems, as predicates they especially support the design of suitable query languages for spatial data retrieval and analysis. But so far, they have only been defined for simplified abstractions of spatial objects like continuous lines and simple regions. With the introduction of *complex spatial data types* in spatial data models and extensions of commercial database systems, an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. This paper first introduces a formally defined, conceptual model of general and versatile spatial data types for *complex lines* and *complex regions*. Based on the well known 9-intersection model, it then formally determines the complete set of mutually exclusive topological relationships between complex lines and complex regions. Completeness and mutual exclusion are shown by a proof technique called *proof-by-constraint-and-drawing*.

Keywords: Topological predicate, topological constraint rule, proof-by-constraint-and-drawing, complex spatial data type, 9-intersection model.

1 Introduction

For a long time, the study of topological relationships between objects in two-dimensional space has been a multi-disciplinary research issue involving disciplines like artificial intelligence, cognitive science, geographical information systems (GIS), linguistics, psychology, robotics, spatial database systems, and qualitative spatial reasoning. From a database and GIS perspective, their development has been stimulated by the necessity of formally defined topological predicates as filter conditions for spatial selections and spatial joins in spatial query languages and as a support for spatial data retrieval and analysis tasks.

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Topological relationships like *overlap*, *inside*, or *meet* describe purely qualitative properties that characterize the relative positions of spatial objects and are preserved under affine transformations. A restriction and shortcoming of current topological models is that topological relationships have so far only been determined for simplified abstractions of spatial objects like simple lines and simple regions. Simple lines are one-dimensional continuous features embedded in the plane with two end points, and simple regions are two-dimensional point sets topologically equivalent to a closed disc. Unfortunately, these simple geometric structures are insufficient to cover the variety and complexity of geographic reality. Universal and versatile type specifications are needed for (more) complex spatial objects that can be leveraged in many different applications. With regard to *complex lines*, we permit arbitrary, finite collections of one-dimensional curves, i.e., spatially embedded networks possibly consisting of several disjoint connected components, as line objects (e.g., to model the ramifications of the Nile Delta). With regard to *complex regions*, the two main extensions relate to separations of the exterior (holes) and to separations of the interior (multiple components). For example, countries (like Italy) can be made up of multiple components (like the mainland and the offshore islands) and can have holes (like the Vatican). Hence, a first goal of this paper is to give a formal definition of spatial data types for complex lines and complex regions.

With the integration of *complex spatial data types* into spatial type systems from a formal perspective as well as into GIS and spatial extension packages of commercial database systems from an application perspective, an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. This is of interest simply from a theoretical point of view but has especially impact on the aforementioned disciplines and on spatial selections and spatial joins. Hence, a second goal is to explore and derive the possible topological relationships between all combinations of complex spatial data types. In this paper, we show the derivation mechanism for complex lines and complex regions on the basis of the well known 9-intersection model. For this purpose, we draw up collections of constraints specifying conditions for valid topological relationships and satisfying the properties of *completeness* and *exclusiveness*. The property of completeness ensures a full coverage of all topological situations on the basis of the 9-intersection model. The property of exclusiveness ensures that two different relationships cannot hold for the same two spatial objects.

The remainder of the paper is organized as follows: Section 2 discusses related work on complex lines, complex regions, and topological relationships. Section 3 formalizes the spatial concepts of complex lines and complex regions. Section 4 explains our general strategy, called the *Proof-By-Constraint-And-Drawing Method*, for deriving topological relationships from the 9-intersection model. As an example, Section 5 identifies all topological relationships between complex lines and complex regions. Finally, Section 6 draws some conclusions and discusses future work.

2 Related Work

In the past, numerous data models and query languages for spatial data have been proposed with the aim of formulating and processing spatial queries in databases and GIS [7]. *Spatial data types* (see [10] for a survey) like *point*, *line*, or *region* provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. A few models [1,8,9,10] have been developed towards complex spatial objects. All these approaches allow multiple object components. In some approaches object components are allowed to intersect [1,9]. Some approaches are based on a finite geometric domain [8,10] whereas we define our data types in the infinite Euclidean plane.

Topological predicates have so far only been determined for *simple* object structures like continuous lines and simple regions. An important approach for characterizing them rests on the so-called 9-*intersection model* [3], which leverages point set theory and point set topology [6] as its formal framework. For example, a complete collection of 19 mutually exclusive topological relationships has been determined between a simple line and a simple region [4]. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object A with the corresponding components of another object B. Each intersection is tested with regard to the topologically invariant criteria of emptiness and non-emptiness. The topological relationship between two spatial objects A and B can be expressed by evaluating the well known intersection matrix in Table 1. For this matrix $2^9 = 512$ different configurations are possible from which only a certain subset makes sense depending on the *definition* and *combination* of spatial data types. For each combination of spatial types this means that each of its predicates is associated with a unique intersection matrix so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness.

Table 1. The 9-intersection matrix

$$\begin{pmatrix} A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ \partial A \cap B^\circ \neq \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^- \cap B^\circ \neq \varnothing & A^- \cap \partial B \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

Surprisingly, topological predicates have so far not been defined on complex spatial objects. So far, two works [2,5] have given a definition of topological relationships between two more complex regions. But either their region definition only allows sets of disjoint simple regions without holes [2] or only single simple regions with holes [5]. The results are also problematic in the sense that they either depend on the number of components or on the number of holes.

3 Complex Lines and Complex Regions

This section defines the underlying spatial data model for our topological predicates. We strive for a very general, abstract definition of complex lines and com-

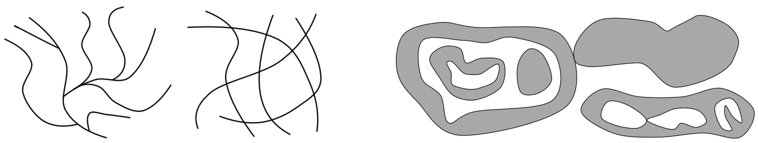

Fig. 1. Examples of a complex line object (a) and a complex region object (b)

plex regions (see Figure 1) in the Euclidean plane $\mathbb{R}^2$. Our formal framework are basic concepts of point set theory and point set topology [6]. The task is to determine those point sets that are admissible for complex line (Section 3.1) and complex region (Section 3.2) objects. The definitions we give contribute to an "unstructured" object definition which solely determines the point set of a line or region. Due to space restrictions, we do not identify structural components. But a complex line represents a spatially embedded network possibly consisting of several connected components, and a complex region represents a multi-part region possibly with holes. For both spatial data types we specify the topological notions of *boundary*, *interior*, *exterior*, and *closure* since these notions are later needed for the specification of topological relationships.

3.1 Complex Lines

Before we start with a definition for complex lines (Figure 1a), we need a few definitions of some well-known and needed topological concepts. We assume the existence of the Euclidean distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $d(p, q) = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. With the notion of distance, we can now proceed to define what is meant by a *neighborhood* of a point in $\mathbb{R}^2$.

Definition 1. *Let $q \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}^+$. The set $N_\epsilon(q) = \{p \in \mathbb{R}^2 \mid d(p, q) < \epsilon\}$ is called the* open neighborhood of radius ϵ and center q. *Any open neighborhood with center q is denoted by $N(q)$.* □

We are now able to define the notion of a *continuous mapping* which preserves neighborhood relations between mapped points in two spaces of the plane. Hence, the property of continuity of this mapping ensures the maintenance of the closure and connectivity of the mapping domain for its image. These mappings are also called *topological transformations* and include translation, rotation, and scaling.

Definition 2. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}^2$. Then f is said to be* continuous at a point $x_0 \in X$ *if, given an arbitrary number $\epsilon > 0$, there exists a number $\delta > 0$ (usually depending on ϵ) such that for every $x \in N_\delta(x_0) \cap X$ we obtain that $f(x) \in N_\epsilon(f(x_0))$. The mapping f is said to be* continuous on X *if it is continuous at every point of X.* □

For a function $f : X \to Y$ and a set $A \subseteq X$ we introduce the notation $f(A) = \{f(x) \mid x \in A\}$. Definition 2 enables us to give an unstructured definition for complex lines as the union of the images of a finite number of continuous mappings.

Definition 3. *The spatial data type line is defined as*

$$\begin{aligned} \mathit{line} = \{L \subset \mathbb{R}^2 \mid & \text{(i)} \quad L = \textstyle\bigcup_{i=1}^{n} f_i([0,1]) \text{ with } n \in \mathbb{N}_0 \\ & \text{(ii)} \ \ \forall\, 1 \leq i \leq n : f_i : [0,1] \to \mathbb{R}^2 \text{ is a continuous mapping} \\ & \text{(iii)} \ \forall\, 1 \leq i \leq n : |f_i([0,1])| > 1\} \end{aligned}$$

We call a value of this type complex line *and the image of a continuous mapping* continuous line*.* □

The first condition also allows a line object to be the empty point set ($n = 0$ in Definition 3). The third condition avoids degenerate line objects consisting only of a single point.

The boundary of a complex line L is the set of its end points minus those end points that are shared by several continuous lines. The shared points belong to the interior of a complex line. Based on Definition 3, let $E(L) = \bigcup_{i=1}^{n}\{f_i(0), f_i(1)\}$ be the set of end points of all continuous lines. We obtain

$$\begin{aligned} \partial L = E(L) - \{p \in E(L) \mid & \mathit{card}(\{f_i \mid 1 \leq i \leq m \ \wedge \ f_i(0) = p\}) + \\ & \mathit{card}(\{f_i \mid 1 \leq i \leq m \ \wedge \ f_i(1) = p\}) \neq 1\} \end{aligned}$$

Let $L \neq \varnothing$. It is possible that ∂L is empty (e.g., if L is a closed continuous line). The closure $\overline{L}$ of L is the set of all points of L including the end points. Therefore $\overline{L} = L$ holds. For the interior of L we obtain $L^\circ = \overline{L} - \partial L = L - \partial L \neq \varnothing$, and for the exterior we get $L^- = \mathbb{R}^2 - L$, since $\mathbb{R}^2$ is the embedding space.

3.2 Complex Regions

Regions are embedded into the two-dimensional Euclidean space $\mathbb{R}^2$ and modeled as special infinite point sets. We briefly introduce some needed concepts from point set topology in $\mathbb{R}^2$.

Definition 4. *Let $X \subseteq \mathbb{R}^2$ and $q \in \mathbb{R}^2$. q is an* interior point *of X if there exists a neighborhood N such that $N(q) \subseteq X$. q is an* exterior point *of X if there exists a neighborhood N such that $N(q) \cap X = \varnothing$. q is a* boundary point *of X if q is neither an interior nor exterior point of X. q is a* closure point *of X if q is either an interior or boundary point of X.*

The set of all interior points of X is called the interior *of X and is denoted by X°. The set of all exterior points of X is called the* exterior *of X and is denoted by X^-. The set of all boundary points of X is called the* boundary *of X and is denoted by ∂X. The set of all closure points of X is called the* closure *of X and is denoted by $\overline{X}$.*

A point q is a limit point *of X if for every neighborhood $N(q)$ holds that $(N - \{q\}) \cap X \neq \varnothing$. X is called an* open set *in $\mathbb{R}^2$ if $X = X^\circ$. X is called a* closed set *in $\mathbb{R}^2$ if every limit point of X is a point of X.* □

It follows from the definition that every interior point of X is a limit point of X. Thus, limit points need not be boundary points. The converse is also true. A boundary point of X need not be a limit point; it is then called an *isolated* point of X. For the closure of X we obtain that $\overline{X} = \partial X \cup X^\circ$.

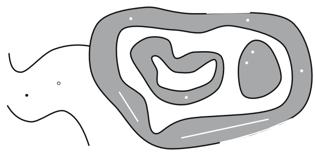

Fig. 2. Examples of possible geometric anomalies of a region object

It is obvious that arbitrary point sets do not necessarily form a region. But open and closed point sets in $\mathbb{R}^2$ are also inadequate models for complex regions since they can suffer from undesired geometric anomalies (Figure 2). A complex region defined as an open point set runs into the problem that it may have missing lines and points in the form of cuts and punctures. At any rate, its boundary is missing. A complex region defined as a closed point set admits isolated or dangling point and line features. *Regular closed* point sets [12] avoid these anomalies.

Definition 5. *Let $X \subseteq \mathbb{R}^2$. X is called a* regular closed set *if, and only if, $X = \overline{X^\circ}$.* □

The effect of the *interior* operation is to eliminate dangling points, dangling lines, and boundary parts. The effect of the *closure* operation is to eliminate cuts and punctures by appropriately supplementing points and to add the boundary.

For the specification of the *region* data type, definitions are needed for bounded and connected sets.

Definition 6. *(i) Two sets $X, Y \subseteq \mathbb{R}^2$ are said to be* separated *if, and only if, $X \cap \overline{Y} = \varnothing = \overline{X} \cap Y$. A set $X \subseteq \mathbb{R}^2$ is* connected *if, and only if, it is not the union of two non-empty separated sets. (ii) Let $q = (x, y) \in \mathbb{R}^2$. Then the* length *or* norm *of q is defined as $||q|| = \sqrt{x^2 + y^2}$. (iii) A set $X \subseteq \mathbb{R}^2$ is said to be* bounded *if there exists a number $r \in \mathbb{R}^+$ such that $||q|| < r$ for every $q \in X$.* □

We are now able to give an unstructured type definition for complex regions:

Definition 7. *The spatial data type region is defined as*

$$
\begin{aligned}
\mathit{region} = \{R \subset \mathbb{R}^2 \mid\ & (i) \quad R \text{ is regular closed} \\
& (ii) \quad R \text{ is bounded} \\
& (iii) \quad \text{The number of connected sets of } R \text{ is finite}\}
\end{aligned}
$$

We call a value of this type complex region. □

A region object can also be the empty object (empty set). Let $F = \bigcup_{i=1}^{n} F_i$ be a non-empty region with faces $\{F_1, \ldots, F_n\}$. Then the boundary of F is given as $\partial F = \bigcup_{i=1}^{n} \partial F_i$ $(\neq \varnothing)$, and the interior of F is given as $F^\circ = \bigcup_{i=1}^{n} F_i^\circ = F - \partial F$ $(\neq \varnothing)$. Further, we obtain $\overline{F} = \partial F \cup F^\circ = F$ and $F^- = \mathbb{R}^2 - \overline{F} = \mathbb{R}^2 - F$ $(\neq \varnothing)$.

4 The Proof-by-Constraint-and-Drawing Method

An apparently promising approach to deriving topological relationships is to leverage the component view of a spatial object. But research on region objects in this direction reveals that considering components leads to rather complicated and impractical models. We demonstrate this by first considering two simple regions A and B with n and m holes, respectively. If we take into account the regions A and B *without* holes and call them A^* and B^*, respectively, the total number of topological relationships that can be specified between A^* and its holes with B^* and its holes amounts to $(n + m + 2)^2$ [5]. It has also been shown in [5] that this number can be reduced to $mn + m + n + 1$. The problems of this approach are the dependency on the number of holes and the resulting large number of topological relationships.

We are confronted with a similar problem if we take another strategy and have a look on the topological relationships between two complex regions A and B with n and m faces, respectively, possibly with holes. Each face of A is in relationship with any face of B. This gives us a total of $8^{n \cdot m}$ possible topological configurations if we take the eight topological relationships between two simple regions with holes, as they are specified in [11], as the basis. As a result, the total number of relationships between the faces of two complex regions depends on the numbers of faces, is therefore not bounded by a constant, and increases in an exponential way. This approach is obviously not manageable and thus not acceptable.

Hence, the comparison of structural components of the objects with respect to their topological relationships does not seem to be an adequate and efficient method. Often, such a detailed investigation is not desired and thus even unnecessary. For instance, if two regions intersect (according to some definition), the number of intersecting face pairs, as long as it is greater than 0, is irrelevant since it does not influence the fact of intersection. Consequently, the analysis of topological relationships between two complex spatial objects requires a more general strategy.

Our strategy is simple and yet very general and expressive. Instead of applying the 9-intersection model to point sets belonging to simple spatial objects, we extend it to point sets belonging to complex spatial objects. Due to the special features of the objects (point, linear, areal properties), the embedding space (here: $\mathbb{R}^2$), the relation between the objects and the embedding space (e.g., it makes a difference whether we consider a point in $\mathbb{R}$ or in $\mathbb{R}^2$), and the employed spatial data model (e.g., discrete, continuous), a number of topological configurations cannot exist and have to be excluded. For each pair of complex spatial data types, our goal is to determine topological constraints that have to be satisfied. These serve as criteria for excluding all impossible configurations. The approach taken employs a proof technique which we call *Proof-By-Constraint-And-Drawing*. It starts with the 512 possible matrices and is a two-step process:

(i) For each type combination we give the formalization of a collection of topological *constraint* rules for existing relationships in terms of the nine intersections. For each constraint rule we give reasons for its validity, correctness, and meaningfulness. The evaluation of each constraint rule gradually reduces the set of the currently valid matrices by all those matrices not fulfilling the constraint rule under consideration.

(ii) The existence of topological relationships given by the remaining matrices is verified by realizing prototypical spatial configurations in $\mathbb{R}^2$, i.e., these configurations can be *drawn* in the plane.

Still open issues relate to the evaluation order, completeness, and minimality of the collection of constraint rules. Each constraint rule is a predicate that is matched with all intersection matrices under consideration. All constraint rules must be satisfied together so that they represent a conjunction of predicates. To say it in other words, constraint rules are all formulated in conjunctive normal form. Since the conjunction (logical *and*) operator is commutative and associative, the *evaluation order* of the constraint rules is irrelevant; the final result is always the same.

The *completeness* of the collection of constraints is directly ensured by the second step of the two-step process provided that spatial configurations for all remaining matrices can be drawn.

The aspect of *minimality* addresses the possible redundancy of constraint rules. Redundancy can arise for two reasons. First, several constraint rules may be correlated in the sense that one of them is more general than the others, i.e., it eliminates at least the matrices excluded by all the other, covered constraints. This can be easily checked by analyzing the constraint rules themselves and searching for the most non-restrictive and common constraint rule. Even then the same matrix can be excluded by several constraint rules simultaneously. Second, a constraint rule can be covered by some combination of other constraint rules. This can be checked by a comparison of the matrix collection fulfilling all n constraint rules with the matrix collection fulfilling $n-1$ constraint rules. If both collections are equal, then the omitted constraint rule was implied by the combination of the other constraint rules and is therefore redundant. In this paper, we are not so much interested in the aspect of minimality since our goal is to identify the topologically invalid intersection matrices (predicates) so that the valid matrices remain.

5 Topological Relationships for the Complex Line/Complex Region Case

Leveraging the proof technique developed in the last section, we develop constraint rules for the identification of all topological relationships between a complex line and a complex region. In the following, we assume that A is a non-empty object of type *line* and B is a non-empty object of type *region*.

Lemma 1. *The exteriors of a complex line and a complex region always intersect with each other, i.e.,*

$$A^- \cap B^- \neq \varnothing$$

Proof. We know that $\overline{A} \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $\overline{A} = \mathbb{R}^2$, or (ii) $\overline{B} = \mathbb{R}^2$, or (iii) $\overline{A} \cup \overline{B} = \mathbb{R}^2$. The situations are all impossible, since A, B, and hence $A \cup B$ are bounded, but the Euclidean plane $\mathbb{R}^2$ is unbounded. □

Lemma 2. *The interior of a complex region always intersects the exterior of a complex line, i.e.,*

$$A^- \cap B^\circ \neq \varnothing$$

Proof. Assuming that this constraint rule is wrong. Then $A^- \cap B^\circ = \varnothing$, and we can conclude that $A \supseteq B^\circ$. From this we obtain that $\forall\, p \in B^\circ\ \exists\, \epsilon \in \mathbb{R}^+ : N_\epsilon(p) \subseteq B^\circ \Rightarrow N_\epsilon(p) \subseteq A$. This leads to a contradiction since $\forall\, p \in B^\circ\ \forall\, \epsilon \in \mathbb{R}^+ : N_\epsilon(p) \nsubseteq A$. □

Intuitively, a line object as a one-dimensional, linear entity cannot cover a region object, which is a two-dimensional, areal entity.

Lemma 3. *The interior or the exterior of a complex line intersects the boundary of a complex region, i.e.,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing$$

Proof. We know that $\partial B \neq \varnothing$ and that hence $\mathbb{R}^2 \cap \partial B \neq \varnothing$. Since $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$, we obtain that $(A^\circ \cup \partial A \cup A^-) \cap \partial B \neq \varnothing$. This leads to $A^\circ \cap \partial B \neq \varnothing \ \vee\ \partial A \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing$. Since ∂A is a finite point set and ∂B is an infinite point set, either $\partial A \subset \partial B$ or $\partial A \cap \partial B = \varnothing$. This means that the constraint rule $A^\circ \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing$ must hold. □

Lemma 4. *The interior of a complex line intersects at least one part of a complex region, i.e.,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee\ A^\circ \cap B^\circ \neq \varnothing \ \vee\ A^\circ \cap B^- \neq \varnothing$$

Proof. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $\partial B \cup B^\circ \cup B^- = \mathbb{R}^2$. Since only non-empty object parts of both objects are taken into account, we obtain $A^\circ \cap \mathbb{R}^2 = A^\circ \cap (\partial B \cup B^\circ \cup B^-) \neq \varnothing$. This statement is equivalent to the constraint rule. □

Lemma 5. *If the boundary of a complex line intersects the interior of a complex region, also its interior intersects the interior of the complex region, i.e.,*

$$\begin{aligned} &(\partial A \cap B^\circ \neq \varnothing \Rightarrow A^\circ \cap B^\circ \neq \varnothing) \\ \Leftrightarrow\ &(\partial A \cap B^\circ = \varnothing \ \vee\ A^\circ \cap B^\circ \neq \varnothing) \end{aligned}$$

Proof. Without loss of generality, let $p \in \partial A \cap B^\circ$. Since $p \in B^\circ$, an $\epsilon \in \mathbb{R}^+$ exists such that $N_\epsilon(p) \subset B^\circ$. Fixing such an ϵ, and because a continuous curve with an infinite number of points starts in p, we obtain that $N_\epsilon(p) \cap A^\circ \neq \varnothing$. This leads to the conclusion that $A^\circ \cap B^\circ \neq \varnothing$. □

Lemma 6. *If the boundary of a complex line intersects the exterior of a complex region, also its interior intersects the exterior of the complex region, i.e.,*

$$
\begin{aligned}
&(\partial A \cap B^- \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing)\\
\Leftrightarrow\ &(\partial A \cap B^- = \varnothing \ \vee\ A^\circ \cap B^- \neq \varnothing)
\end{aligned}
$$

Proof. The argumentation is analogous to the argumentation for the constraint rule in Lemma 5. □

Lemma 7. *If the boundary of a complex line intersects the boundary of a complex region, also its exterior intersects the boundary of the complex region, i.e.,*

$$
\begin{aligned}
&(\partial A \cap \partial B \neq \varnothing \Rightarrow A^- \cap \partial B \neq \varnothing)\\
\Leftrightarrow\ &(\partial A \cap \partial B = \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing)
\end{aligned}
$$

Proof. The boundary of a region B is a line object L whose components are all closed curves. Hence, this line object only consists of interior points ($L = L^\circ$). Without loss of generality, let P be an endpoint of the boundary of A located on L. From P exactly one curve of A starts or ends. Either P divides a curve of L into two subcurves, or P is endpoint of more than one curve of L. Hence, in P at least two curves of L end. Since the curve of A can coincide with at most one of the curves of L, at least one of the curves of L must be situated in the exterior of A. □

An evaluation of all 512 3×3-intersection matrices against these seven constraint rules with the aid of a simple test program reveals that 43 matrices satisfy these rules and thus represent the possible topological relationships between a complex line and a complex region. The matrices and their geometric interpretations are shown in Table 2. Between a *simple* line and a *simple* region we can distinguish 19 topological relationships [3]. These topological predicates are contained in the 43 general ones and correspond to the matrices 2-4, 7, 11-13, 15-17, 28, 30, 31, 35-37, 39, 41, and 42, respectively.

Finally, we can summarize our result as follows:

Theorem 1. *Based on the 9-intersection model, 43 different topological relationships can be identified between a complex line object and a complex region object.*

Proof. The argumentation is based on the *Proof-By-Constraint-And-Drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 1 to 7, reduce the number of the 512 possible intersection matrices to 43 matrices. The ability to draw prototypes of the corresponding 43 topological configurations proves that the constraint rules are complete. □

Table 2 in the Appendix shows for each topological predicate a prototypical configuration as a drawing.

6 Conclusions and Future Work

In this paper we have given a very general definition of spatial data types for complex lines and complex regions in the two-dimensional Euclidean space on the basis of point set theory and point set topology. Further, we have developed a proof technique called *Proof-By-Constraint-And-Drawing* which enables the derivation of a complete collection of mutually exclusive topological relationships between all combinations of complex spatial data types. We have demonstrated this mechanism by deriving all 43 topological relationships between a complex line and a complex region.

Future work will relate to the derivation of topological predicates for all other combinations of complex spatial data types. Further, the efficient implementation of the large numbers of predicates that have to be expected will be another topic.

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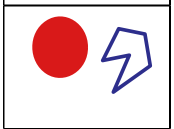
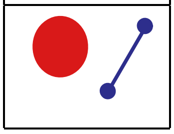
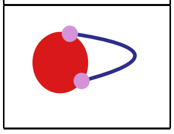
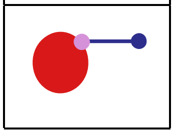
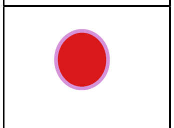
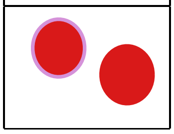
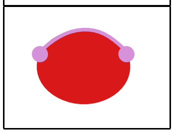
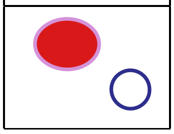
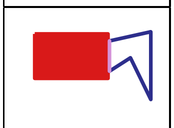
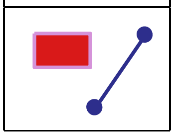
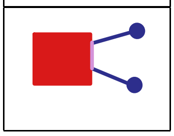
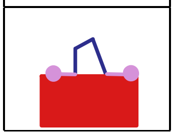
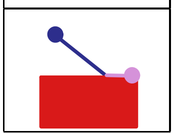
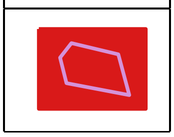
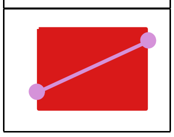
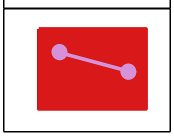
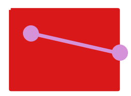
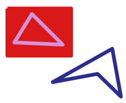
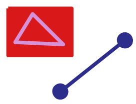
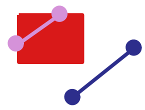
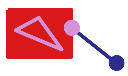
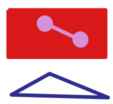
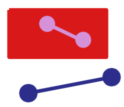
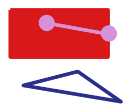
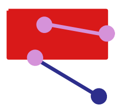
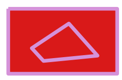
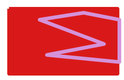
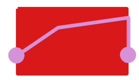
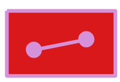
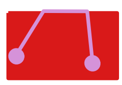
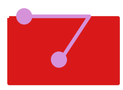
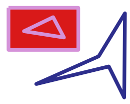
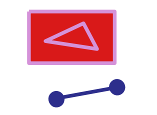
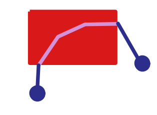
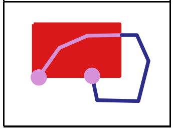
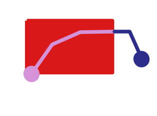
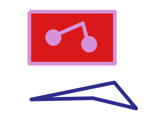
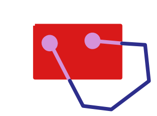
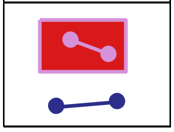
Appendix

Table 2. The 43 topological relationships between a complex line and a complex region

Matrix 1	Matrix 2	Matrix 3	Matrix 4
$\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&0&1 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&1&1 \\ 1&1&1 \end{pmatrix}$
			
Matrix 5	Matrix 6	Matrix 7	Matrix 8
$\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0 \\ 0&1&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 1&0&1 \end{pmatrix}$
			
Matrix 9	Matrix 10	Matrix 11	Matrix 12
$\begin{pmatrix} 0&1&1 \\ 0&0&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&1 \\ 0&0&1 \\ 1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&1 \\ 0&0&1 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&1 \\ 0&1&0 \\ 1&1&1 \end{pmatrix}$
			
Matrix 13	Matrix 14	Matrix 15	Matrix 16
$\begin{pmatrix} 0&1&1 \\ 0&1&1 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 1&1&1 \end{pmatrix}$
			

Matrix 17	Matrix 18	Matrix 19	Matrix 20
$\begin{pmatrix} 1&0&0\\ 1&1&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 0&0&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 0&0&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 0&1&0\\ 1&1&1 \end{pmatrix}$
			
Matrix 21	Matrix 22	Matrix 23	Matrix 24
$\begin{pmatrix} 1&0&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 1&0&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 1&0&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1\\ 1&1&0\\ 1&1&1 \end{pmatrix}$
			
Matrix 25	Matrix 26	Matrix 27	Matrix 28
$\begin{pmatrix} 1&0&1\\ 1&1&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0\\ 0&0&0\\ 1&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0\\ 0&0&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0\\ 0&1&0\\ 1&1&1 \end{pmatrix}$
			
Matrix 29	Matrix 30	Matrix 31	Matrix 32
$\begin{pmatrix} 1&1&0\\ 1&0&0\\ 1&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0\\ 1&0&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0\\ 1&1&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&1\\ 0&0&0\\ 1&0&1 \end{pmatrix}$
			

Matrix 33	Matrix 34	Matrix 35	Matrix 36
$\begin{pmatrix}1&1&1\\0&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&0\\1&1&1\end{pmatrix}$
			

Matrix 37	Matrix 38	Matrix 39	Matrix 40
$\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&1\\1&0&1\end{pmatrix}$
			

Matrix 41	Matrix 42	Matrix 43
$\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$