Topological Relationships Between Map Geometries

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Abstract. The importance of topological relationships between spatial objects is recognized in many disciplines. In the field of spatial databases, topological relationships have played an important role, providing mechanisms for spatial constraint specification and spatial indexing techniques. The use of spatial data in a map form has become popular, resulting in spatial data models such as *topologies* or, more generally, *map geometries*, which model collections of spatial objects that satisfy certain topological constraints. However, the topological relationships between map geometries remain unexplored. In this paper, we identify the set of valid topological relationships between map geometries, and then provide a mechanism by which they can be directly implemented on top of existing systems by using topological relationships between regions.

1 Introduction

The study of topological relationships between objects in space has lately received much attention from a variety of research disciplines including robotics, geography, cartography, artificial intelligence, cognitive science, computer vision, image databases, spatial database systems, and geographical information systems (GIS). In the areas of databases and GIS, the motivation for formally defining topological relationships between spatial objects has been driven by a need for querying mechanisms that can filter these objects in spatial selections or spatial joins based on their topological relationships to each other, as well as a need for appropriate mechanisms to aid in spatial data analysis and spatial constraint specification.

A current trend in spatial databases and GIS is the use of *topologically integrated spatial data models*, or what are often referred to as *topologies*. We will refer to these models as *map geometries* to avoid confusion with other terms. A map geometry, in this sense, is a collection of spatial objects that satisfy certain topological constraints; specifically, spatial data objects are only allowed to *meet* or be *disjoint*. Typically, spatial data arranged in a map geometry can be viewed from two levels: the object level, in which spatial objects are allowed to overlap, and the topological level, in which *topological primitives* consisting of polygons,

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lines, and nodes form a map geometry in the plane. We use the term map geometry to refer to the topological level. For instance, consider a region representing the United States, and a region representing a high temperature zone that intersects both the US and the Pacific Ocean. These two objects arranged in a map geometry would result in three topological primitives, the polygon consisting of the intersection of both regions, and the polygons representing the remainder of the US and the high temperature zone.

Currently, map geometries are used in spatial databases to enforce topological constraints between points, lines, and regions, and ensure spatial data quality standards when manipulating them. However, there is currently no method of performing topological data analysis between map geometries. For instance, consider a user who creates a map geometry modelling a swamp that is broken into regions based on pollution levels. For example, a section that is not very polluted will be represented as a region, as will be an area that is highly polluted. Once the map geometry is created, it would be interesting to query the database to find related map geometries. For instance, if someone has already completed a survey of a more specific part of the swamp, there may be map geometry in the database that consists of a more detailed view of a region in the original map geometry. Furthermore, map geometries may exist of the same swamp, but broken into regions differently than the original map. Such map geometries may offer information as to pollution sources. Finally, it may be useful to discover which other map geometries overlap the original one, or are adjacent to it.

Although topological queries over map geometries allow many useful queries, we currently do not know the possible topological relationships between map geometries. Because map geometries are more general than complex regions, it follows that there are more possible relationships between them than there are between complex regions. Furthermore, it is unclear how such topological relationships can be implemented. Existing techniques of computing topological relationships could be extended in order to compute the new relationships, but this requires modification of database internals. If the topological relationships between map geometries could be computed based upon topological relationships between their components (i.e., their topological primitives) that correspond to existing spatial data types, then they could be directly used by existing spatial database systems. Therefore, the first goal of this paper is to discover the complete set of valid topological relationships between map geometries. We use the spatial data type of *spatial partitions* to represent map geometries, and derive the topological relationships based on their definition. Note that spatial partitions only represent map geometries consisting of regions. We leave the treatment of map geometries containing point and line features to future work. The second goal of this paper is to characterize topological relationships between map geometries based on their components. In this case, the components of spatial partitions are complex regions. Thus, by characterizing the new topological relationships as thus, we provide a method to directly use them in spatial databases in which topological predicates between complex regions are already implemented.

The remainder of this paper is structured as follows: in Section 2, we examine research related to maps, spatial objects, and topological relationships. We then present the definition for the type of spatial partitions in Section 3. The topological relationships between map geometries are derived and presented in Section 4. We then characterize the topological relationships between map geometries using topological relationships between complex regions, and show to integrate them into existing databases in Section 5. Finally, we draw some conclusions and consider topics for future work in Section 6.

2 Related Work

The work in this paper deals with defining topological relationships between map geometries modeled by spatial partitions. The research related to this falls into two general categories: research exploring spatial data types and map geometries in general, and research into the topological relationships between spatial data types. The spatial data types that have received the most attention in the spatial database and GIS literature are types for simple and complex points, lines, and regions. Simple lines are continuous, one-dimensional features embedded in the plane with two endpoints; simple regions are two dimensional point sets that are topologically equivalent to a closed disc. Increased application requirements and a lack of closure properties of the simple spatial types lead to the development of the complex spatial types. In [13], the authors formally define complex data types, such as complex points (a single point object consisting of a collection of simple points), complex lines (which can represent networks such as river systems), and complex regions that are made up of multiple faces and holes (i.e., a region representing Italy, its islands, and the hole representing the Vatican).

Map geometries have been studied extensively in the literature. In [6, 7, 10, 16], a map is not defined as a data type itself, but as a collection of spatial regions that satisfy some topological constraints. Because these map types are essentially collections of more basic spatial types, it is unclear how topological constraints can be enforced. Other approaches to defining map geometries center around raster or tessellation models [11, 15]. However, these lack generality in that they are restricted to the tessellation scheme in use. Topologies, such as those used in GIS and spatial systems, provide the most general map geometries available [8, 9]; however, these approaches all center around *implementation models*, and do not provide a formal mathematical model upon which we can base our work. This paper is based on the formal model of spatial partitions presented in [5] in which map geometries are defined as a complete partition of the Euclidean plane into regions which are complex regions such that each region has a unique *label* and regions are allowed to share common boundaries or be disjoint. This model, formally presented in Section 3, has a formal, mathematical definition and provides very few restrictions as to its generality.

Topological relationships indicate qualitative properties of the positions of spatial objects that are preserved under continuous transformations such as translation, rotation, and scaling. Quantitative measures such as distance or

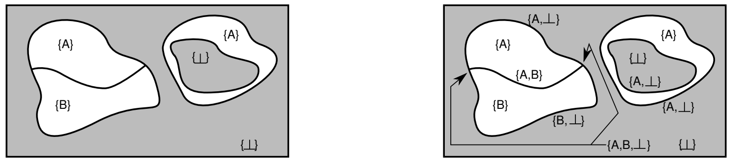
$$\begin{pmatrix} A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ \partial A \cap B^\circ \neq \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^- \cap B^\circ \neq \varnothing & A^- \cap \partial B \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

Fig. 1. The 9-intersection matrix for spatial objects A and B

size measurements are deliberately excluded in favor of modeling notions such as connectivity, adjacency, disjointedness, inclusion, and exclusion. Attempts to model and rigorously define the relationships between certain types of spatial objects have lead to the development of two popular approaches: the *9-intersection model* [3], which is developed based on point set theory and point set topology, and the *RCC model* [12], which utilizes spatial logic. The 9-intersection model characterizes the topological relationship between two spatial objects by evaluating the non-emptiness of the intersection between all combinations of the interior ($^\circ$), boundary (∂) and exterior ($^-$) of the objects involved. A unique 3×3 matrix, termed the *9-intersection matrix* (9IM), with values filled as illustrated in Figure 1 describes the topological relationship between a pair of objects:

Various models of topological predicates using both *component derivations*, in which relationships are derived based on the interactions of all components of spatial objects, and *composite derivations*, in which relationships model the global interaction of two objects, exist in the literature. Examples of component derivations can be found in [4, 1]. In [4], the authors define topological relationships between regions with holes in which each of the relationships between all faces and holes are calculated. Given two regions, R and S, containing m and n holes respectively, a total of $(n+m+2)^2$ topological predicates are possible. It is shown that this number can be reduced $mn+m+n+1$; however, the total number of predicates between two objects depends on the number of holes the objects contain. Similarly, in [1], predicates between complex regions without holes are defined based on the topological relationship of each face within one region with all other faces of the same region, all faces of the other region, and the entire complex regions themselves. Given regions S and R with m and n faces respectively, a matrix is constructed with $(m+n+2)^2$ entries that represent the topological relationships between S and R and each of their faces.

The most basic example of a composite derivation model (in which the global interaction of two spatial objects is modeled) is the derivation of topological predicates between simple spatial objects in [3]. This model has been used as the basis for modeling topological relationships between object components in the component models discussed above. In [13], the authors apply an extended 9-intersection model to point sets belonging to complex points, lines, and regions. Based on this application, the authors are able to construct a composite derivation model for complex data types and derive a complete and finite set of topological predicates between them, thus resolving the main drawback of the component derivation model. We use a composite derivation model in this paper.



a **b**

Fig. 2. Figure *a* shows a spatial partition with two regions and annotated with region labels. Figure *b* shows the same spatial partition with its region and boundary labels (arrows are used to indicate points and are not part of the spatial partition). Note that labels are modeled as sets of attributes in spatial partitions.

3 Data Model: Spatial Partitions

In this section, we review the definition of spatial partitions upon which we base our model of topological relationships between map geometries. A spatial partition, in two dimensions, is a subdivision of the plane into pairwise disjoint *regions* such that each region is associated with a *label* or *attribute* having simple or complex structure, and these regions are separated from each other by *boundaries*. The label of a region describes the thematic data associated with the region. All points within the spatial partition that have an identical label are part of the same region. Topological relationships are implicitly modeled among the regions in a spatial partition. For instance, neglecting common boundaries, the regions of a partition are always disjoint; this property causes spatial partitions to have a rather simple structure. Note that the *exterior* of a spatial partition (i.e., the unbounded face) is always labeled with the $\perp$ symbol. Figure 2a depicts an example spatial partition consisting of two regions.

We stated above that each region in a spatial partition is associated with a single attribute or label. A spatial partition is modeled by mapping Euclidean space to such labels. Labels themselves are modeled as sets of attributes. The regions of the spatial partition are then defined as consisting of all points which contain an identical label. Adjacent regions each have different labels in their interior, but their common boundary is assigned the label containing the labels of both adjacent regions. Note that labels are not relevent in determining topological relationships, but are required by the type definition of spatial partitions. Figure 2b shows an example spatial partition complete with boundary labels.

3.1 Notation

We now briefly summarize the mathematical notation used throughout the following sections. The application of a function $f : A \rightarrow B$ to a set of values $S \subseteq A$ is defined as $f(S) := \{f(x) | x \in S\} \subseteq B$. In some cases we know that $f(S)$ returns a singleton set, in which case we write $f[S]$ to denote the single element, i.e. $f(S) = \{y\} \Longrightarrow f[S] = y$. The inverse function $f^{-1} : B \rightarrow 2^A$ of f is defined as $f^{-1}(y) := \{x \in A | f(x) = y\}$. It is important to note that f^{-1} is a total function and that f^{-1} applied to a set yields a set of sets. We define the range function of a function $f : A \to B$ that returns the set of all elements that f returns for an input set A as $rng(f) := f(A)$.

Let (X, T) be a topological space [2] with topology $T \subseteq 2^x$, and let $S \subseteq X$. The *interior* of S, denoted by S°, is defined as the union of all open sets that are contained in S. The *closure* of S, denoted by $\overline{S}$ is defined as the intersection of all closed sets that contain S. The *exterior* of S is given by $S^- := (X - S)^\circ$, and the *boundary* or *frontier* of S is defined as $\partial S := \overline{S} \cap \overline{X - S}$. An open set is *regular* if $A = \overline{A}^\circ$ [14]. In this paper, we deal with the topological space $\mathbb{R}^2$.

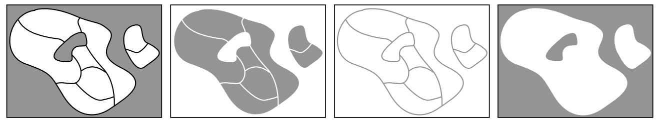
A *partition* of a set S, in set theory, is a complete decomposition of the set S into non-empty, disjoint subsets $\{S_i | i \in I\}$, called blocks: (i) $\forall i \in I : S_i \neq \emptyset$, (ii) $\bigcup_{i \in I} S_i = S$, and (iii) $\forall i, j \in I, i \neq j : S_i \cap S_j = \emptyset$, where I is an index set used to name different blocks. A partition can equivalently be regarded as a total and surjective function $f : S \to I$. However, a spatial partition cannot be defined simply as a set-theoretic partition of the plane, that is, as a partition of $\mathbb{R}^2$ or as a function $f : \mathbb{R}^2 \to I$, for two reasons: first, f cannot be assumed to be total in general, and second, f cannot be uniquely defined on the borders between adjacent subsets of $\mathbb{R}^2$.

3.2 The Definition of Spatial Partitions

In [5], spatial partitions have been defined in several steps. First a *spatial mapping* of type A is a total function $\pi : \mathbb{R}^2 \to 2^A$. The existence of an undefined element $\bot_A$ is required to represent undefined labels (i.e., the exterior of a partition). Definition 1 identifies the different components of a partition within a spatial mapping. The labels on the borders of regions are modeled using the power set 2^A; a *border* of π (Definition 1(ii)) is a block that is mapped to a subset of A containing two or more elements, as opposed to a *region* of π (Definition 1(i)) which is a block mapped to a singleton set. The *interior* of π (Definition 1(iii)) is defined as the union of π's regions. The *boundary* of π (Definition 1(iv)) is defined as the union of π's borders. The *exterior* of π (Definition 1(v)) is the block mapped $\bot_A$. It is also useful to note the *exterior boundary* of a spatial partition as the union of all borders that carry the label $\bot_A$. As an example, let π be the spatial partition in Figure 2 of type $X = \{A, B, \bot\}$. In this case, $rng(\pi) = \{\{A\}, \{B\}, \{\bot\}, \{A, B\}, \{A, \bot\}, \{B, \bot\}, \{A, B, \bot\}\}$. Therefore, the regions of π are the blocks labeled $\{A\}$, $\{B\}$, and $\{\bot\}$ and the boundaries are the blocks labeled $\{A, B\}$, $\{A, \bot\}$, $\{B, \bot\}$, and $\{A, B, \bot\}$. Figure 3 provides a pictorial example of the interior, exterior, and boundary of a more complex example map (note that the borders and boundary consist of the same points, but the boundary is a single point set whereas the borders are a set of point sets).

Definition 1. Let π be a spatial mapping of type A

(i) $\rho(\pi) := \pi^{-1}(rng(\pi) \cap \{X \in 2^A |\ |X| = 1\})$ (*regions*)
(ii) $\omega(\pi) := \pi^{-1}(rng(\pi) \cap \{X \in 2^A |\ |X| > 1\})$ (*borders*)
(iii) $\pi^\circ := \bigcup_{r \in \rho(\pi) | \pi[r] \neq \{\bot_A\}} r$ (*interior*)
(iv) $\partial\pi := \bigcup_{b \in \omega(\pi)} b$ (*boundary*)
(v) $\pi^- := \pi^{-1}(\{\bot_A\})$ (*exterior*)



a b c d

Fig. 3. Figure a shows a spatial partition π with two disconnected faces, one containing a hole. The interior (π°), boundary ($\partial\pi$), and exterior (π^-) of the partition are shown if Figures b c, and d, respectively. Note that the labels have been omitted in order to emphasize the components of the spatial partition.

A *spatial partition* of type A is then defined as a spatial mapping of type A whose regions are regular open sets [14] and whose borders are labeled with the union of labels of all adjacent regions. Since we are using spatial partitions to model map geometries, we use the terms 'spatial partition' and 'map geometry' interchangeably for the remainder of the paper.

Definition 2. A *spatial partition* of type A is a spatial mapping π of type A with:

(i) $\forall r \in \rho(\pi) : r = \overline{r}^\circ$

(ii) $\forall b \in \omega(\pi) : \pi[b] = \{\pi[[r]] | r \in \rho(\pi) \wedge b \subseteq \partial r\}$

4 Topological Relationships

In this section, we describe a method for deriving the topological relationships between a given pair of map geometries. We begin by describing various approaches to the problem, then outline our chosen method, and finally derive the actual relationships based on this method.

The goal of this paper is to define a complete, finite set of topological predicates between map geometries; therefore, we employ a method similar to that found in [13], in which the 9-intersection model is extended to describe complex points, lines, and regions. In Section 3, we defined a point set topological model of map geometries in which we identified the interior, exterior, and boundary point sets belonging to maps. Based on this model, we extend the 9-intersection model to apply to the point sets belonging to map objects. However, due to the spatial features of map geometries, the embedding space ($\mathbb{R}^2$), and the interaction of map geometries with the embedding space, some topological configurations are impossible and must be excluded. Therefore, we must identify topological constraints that must be satisfied in order for a given topological configuration to be valid. Furthermore, we must identify these constraints such that all invalid topological configurations are excluded, and the complete set of valid configurations remains. We achieve this through a proof technique called *Proof-by-Constraint-and-Drawing*, in which we begin with the total set of 512 possible 9-intersection matrices, and determine the set of valid configurations by first providing a collection of topological constraint rules that invalidate impossible topological configurations, and second, validating all matrices that satisfy *all* constraint rules by providing a prototypical spatial configuration (i.e., the configurations can be drawn in the embedding space). Completeness is achieved because all topological configurations are either eliminated by constraint rules, or are proven to be possible through the drawing of a prototype. The remainder of this section contains the constraints, and the prototypical drawings of map geometries are shown in Table 1.

We identify eight constraint rules that 9IMs for map geometries must satisfy in order to be valid. Each constraint rule is first written in sentences and then expressed mathematically. Some mathematical expressions are written in two equivalent expressions so that they may be applied to the 9-intersection matrix more easily. Following each rule is the rationale explaining why the rule is correct. In the following, let π and σ be two spatial partitions.

Lemma 1. *Each component of a map geometry intersects at least one component of the other map geometry:*

$$\begin{array}{l}(\forall C_\pi \in \{\pi^\circ, \partial\pi, \pi^-\} :\ C_\pi \cap \sigma^\circ \neq \varnothing \ \vee\ C_\pi \cap \partial\sigma \neq \varnothing \ \vee\ C_\pi \cap \sigma^- \neq \varnothing) \\ \wedge\ (\forall C_\sigma \in \{\sigma^\circ, \partial\sigma, \sigma^-\} :\ C_\sigma \cap \pi^\circ \neq \varnothing \ \vee\ C_\sigma \cap \partial\pi \neq \varnothing \ \vee\ C_\sigma \cap \pi^- \neq \varnothing)\end{array}$$

Proof. Because spatial mappings are defined as total functions, it follows that $\pi^\circ \cup \partial\pi \cup \pi^- = \mathbb{R}^2$ and that $\sigma^\circ \cup \partial\sigma \cup \sigma^- = \mathbb{R}^2$. Thus, each part of π must intersect at least one part of σ, and vice versa. □

Lemma 2. *The exteriors of two map geometries always intersect:*

$$\pi^- \cap \sigma^- \neq \varnothing$$

Proof. The closure of each region in a map geomtry corresponds to a complex region as defined in [13]. Since complex regions are closed under the union operation, it follows that the union of all regions that compose a map geometry is a complex region, whose boundary is defined by a Jordan curve. Therefore, every spatial partition has an exterior. Furthermore, in [5], the authors prove that spatial partitions are closed under intersection. Thus, the intersection of any two spatial partitions is a spatial partition that has an exterior. Therefore, the exteriors of any two spatial partitions intersect, since their intersection contains an exterior. □

Lemma 3. *If the boundary of a map geometry intersects the interior of another map geometry, then their interiors intersect:*

$$\begin{array}{l}((\partial\pi \cap \sigma^\circ \neq \varnothing \Rightarrow \pi^\circ \cap \sigma^\circ \neq \varnothing)\ \wedge\ (\pi^\circ \cap \partial\sigma \neq \varnothing \Rightarrow \pi^\circ \cap \sigma^\circ \neq \varnothing)) \\ \Leftrightarrow ((\partial\pi \cap \sigma^\circ = \varnothing\ \vee\ \pi^\circ \cap \sigma^\circ \neq \varnothing)\ \wedge\ (\pi^\circ \cap \partial\sigma = \varnothing\ \vee\ \pi^\circ \cap \sigma^\circ \neq \varnothing))\end{array}$$

Proof. Assume that a boundary b of partition π intersects the interior of partition σ but their interiors do not intersect. In order for this to be true, the label of the regions on either side of b must be labeled with the empty label. According to the definition of spatial partitions, a boundary separates two regions with different labels; thus, this is impossible and we have a proof by contradiction. □

Lemma 4. *If the boundary of a map geometry intersects the exterior of a second map geometry, then the interior of the first map geometry intersects the exterior of the second:*

$$\begin{aligned} &((\partial\pi \cap \sigma^- \neq \varnothing \Rightarrow \pi^\circ \cap \sigma^- \neq \varnothing) \ \wedge \ (\pi^- \cap \partial\sigma \neq \varnothing \Rightarrow \pi^- \cap \sigma^\circ \neq \varnothing)) \\ \Leftrightarrow &((\partial\pi \cap \sigma^- = \varnothing \ \vee \ \pi^\circ \cap \sigma^- \neq \varnothing) \ \wedge \ (\pi^- \cap \partial\sigma = \varnothing \ \vee \ \pi^- \cap \sigma^\circ \neq \varnothing)) \end{aligned}$$

Proof. This proof is similar to the previous proof. Assume that the boundary b of partition π intersects the exterior of partition σ but the interior of π does not intersect the exterior of σ. In order for this to be true, the label of the regions on either side of b must be labeled with the empty label. According to the definition of spatial partitions, a boundary separates two regions with different labels; thus, this is impossible and we have a proof by contradiction. □

Lemma 5. *If the boundaries of two map geometries are equivalent, then their interiors intersect:*

$$\begin{aligned} &(\partial\pi = \partial\sigma \Rightarrow \pi^\circ \cap \sigma^\circ \neq \varnothing) \Leftrightarrow (c \Rightarrow d) \Leftrightarrow (\neg c \ \vee \ d) \text{ where} \\ &c = \partial\pi \cap \partial\sigma \neq \varnothing \ \wedge \ \pi^\circ \cap \partial\sigma = \varnothing \ \wedge \ \partial\pi \cap \sigma^\circ = \varnothing \\ &\qquad \wedge \ \partial\pi \cap \sigma^- = \varnothing \ \wedge \ \pi^- \cap \partial\sigma = \varnothing \\ &d = \pi^\circ \cap \sigma^\circ \neq \varnothing \end{aligned}$$

Proof. Assume that two spatial partitions have an identical boundary, but their interiors do not intersect. The only configuration which can accommodate this situation is if one spatial partition's interior is equivalent to the exterior of the other spatial partition. However, according to Lemma 2, the exteriors of two partitions always intersect. If a partition's interior is equivalent to another partition's exterior, then their exteriors would not intersect. Therefore, this configuration is not possible, and the interiors of two partitions with equivalent boundaries must intersect. □

Lemma 6. *If the boundary of a map geometry is completely contained in the interior of a second map geometry, then the boundary and interior of the second map geometry must intersect the exterior of the first, and vice versa:*

$$\begin{aligned} &(\partial\pi \subset \sigma^\circ \Rightarrow \pi^- \cap \partial\sigma \neq \varnothing \ \wedge \ \pi^- \cap \sigma^\circ \neq \varnothing) \Leftrightarrow (\neg c \ \vee \ d) \text{ where} \\ &c = \partial\pi \cap \sigma^\circ \neq \varnothing \ \wedge \ \partial\pi \cap \partial\sigma = \varnothing \ \wedge \ \partial\pi \cap \sigma^- = \varnothing \\ &d = \pi^- \cap \partial\sigma \neq \varnothing \ \wedge \ \pi^- \cap \sigma^\circ \neq \varnothing \end{aligned}$$

Proof. If the boundary of spatial partition π is completely contained in the interior of spatial partition σ, it follows from the Jordan Curve Theorem that the boundary of σ is completely contained in the exterior of π. By Lemma 4, it then follows that the interior of σ intersects the exterior of π. □

Lemma 7. *If the boundary of one map geometry is completely contained in the interior of a second map geometry, and the boundary of the second map geometry is completely contained in the exterior of the first, then the interior of the first map geometry cannot intersect the exterior of the second and the interior of the second map geometry must intersect the exterior of the first and vice versa:*

$$
\begin{aligned}
&((\partial\pi \subset \sigma^\circ \ \wedge \ \pi^- \supset \partial\sigma) \Rightarrow (\pi^\circ \cap \sigma^- = \varnothing \ \wedge \ \pi^- \cap \sigma^\circ \neq \varnothing)) \\
&\Leftrightarrow (c \Rightarrow d) \Leftrightarrow (\neg c \ \vee \ d) \text{ where} \\
&c = \partial\pi \cap \sigma^\circ \neq \varnothing \ \wedge \ \partial\pi \cap \partial\sigma = \varnothing \ \wedge \ \partial\pi \cap \sigma^- = \varnothing \\
&\qquad \wedge \ \pi^\circ \cap \partial\sigma = \varnothing \ \wedge \ \pi^- \cap \partial\sigma \neq \varnothing \\
&d = \pi^\circ \cap \sigma^- = \varnothing \ \wedge \ \pi^- \cap \sigma^\circ \neq \varnothing
\end{aligned}
$$

Proof. We construct this proof in two parts. According to Lemma 6, if $\partial\pi \subseteq \sigma^\circ$, then $\pi^- \cap \sigma^\circ \neq \varnothing$. Now we must prove that π° cannot intersect σ^-. Since $\partial\pi \subset \sigma^\circ$, it follows that π° intersects σ°. Therefore, the only configuration where $\pi^\circ \cap \sigma^- \neq \varnothing$ can occur is if σ contains a hole that is contained by π. However, in order for this configuration to exist, the $\partial\sigma$ would have to intersect the interior or the boundary of π. Since the lemma specifies the situation where $\pi^- \supset \partial\sigma$, this configuration cannot exist; thus, the interior of π cannot intersect the exterior of σ. □

Lemma 8. *If the boundary of a map geometry is completely contained in the exterior of a second map geometry and the boundary of the second map geometry is completely contained in the exterior of the first, then the interiors of the map geometries cannot intersect:*

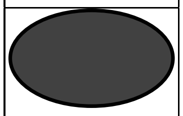
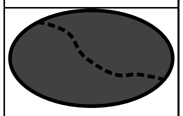
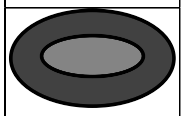
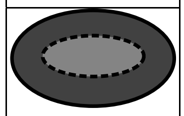
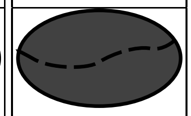
$$
\begin{aligned}
&((\partial\pi \subset \sigma^- \ \wedge \ \pi^- \supset \partial\sigma) \Rightarrow (\pi^\circ \cap \sigma^\circ = \varnothing)) \\
&\Leftrightarrow (c \Rightarrow d) \Leftrightarrow (\neg c \ \vee \ d) \text{ where} \\
&c = \partial\pi \cap \sigma^\circ = \varnothing \ \wedge \ \partial\pi \cap \partial\sigma = \varnothing \ \wedge \ \partial\pi \cap \sigma^- \neq \varnothing \\
&\qquad \wedge \ \pi^\circ \cap \partial\sigma = \varnothing \ \wedge \ \pi^- \cap \partial\sigma \neq \varnothing \\
&d = \pi^\circ \cap \sigma^\circ = \varnothing
\end{aligned}
$$

Proof. The lemma states that the interiors of two disjoint maps do not intersect. Without loss of generality, consider two map geometries that each consist of a single region. We can consider these map geometries as complex region objects. If two complex regions are disjoint, then their interiors do not intersect. We can reduce any arbitrary map to a complex region by computing the spatial union of its regions. It follows that because the interiors of two disjoint regions do not intersect, the interiors of two disjoint maps do not intersect. □

Using a simple program to apply these eight constraint rules reduces the 512 possible matrices to 49 valid matrices that represent topological relationships between two maps geometries. The matrices and their validating prototypes are depicted in Table 1. Finally, we summarize our results as follows:

Theorem 1. *Based on the 9-intersection model for spatial partitions, 49 different topological relationships exist between two map geometries.*

Proof. The argumentation is based on the *Proof-by-Constraint-and-Drawing* method. The constraint rules, whose correctness has been proven in Lemmas 1 to 8, reduce the 512 possible 9-intersection matrices to 49 matrices. The ability to draw prototypes of the corresponding 49 topological configurations in Table 1 proves that the constraint rules are complete. □

Matrix 1	Matrix 2	Matrix 3	Matrix 4	Matrix 5
				
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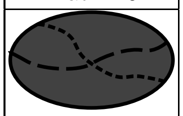
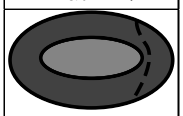
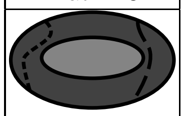
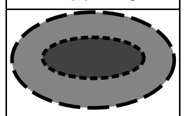
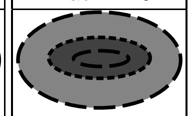
Matrix 6	Matrix 7	Matrix 8	Matrix 9	Matrix 10
				
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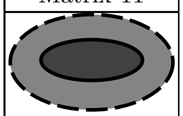
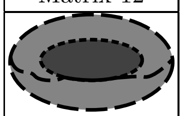
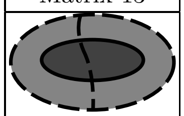
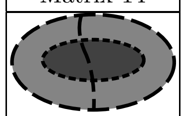
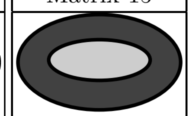
Matrix 11	Matrix 12	Matrix 13	Matrix 14	Matrix 15
				
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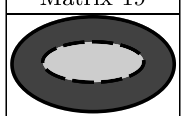
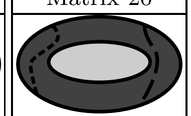
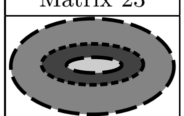
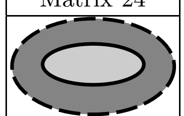
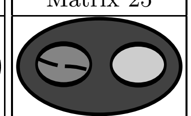
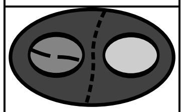
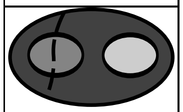
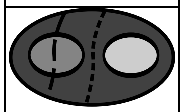
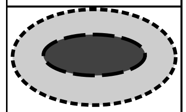
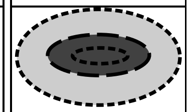
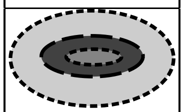
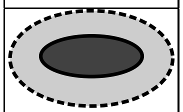
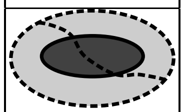
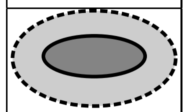
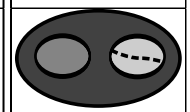
Matrix 16	Matrix 17	Matrix 18	Matrix 19	Matrix 20
				
$\begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&1&0\\1&0&1\end{pmatrix}$
Matrix 21	Matrix 22	Matrix 23	Matrix 24	Matrix 25
				
$\begin{pmatrix}1&0&1\\1&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&0&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&1\\1&0&1\end{pmatrix}$

Table 1. The first 40 valid matrices and protoypical drawings representing the possible topological relationships between maps. Each drawing shows the interaction of two maps, one map is medium-grey and has a dashed boundary, the other is light-grey and has a dotted boundary. Overlapping map interiors are dark-grey, and overlapping boundaries are drawn as a solid line. For reference, the figure for matrix 41 shows two disjoint maps and the figure for matrix 1 shows two equal maps.

Matrix 26	Matrix 27	Matrix 28	Matrix 29	Matrix 30
				
$\begin{pmatrix}1&1&1\\0&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&0&0\\1&1&1\end{pmatrix}$
Matrix 31	Matrix 32	Matrix 33	Matrix 34	Matrix 35
				
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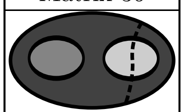
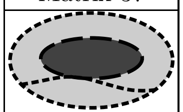
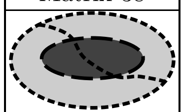
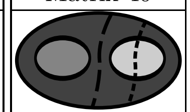
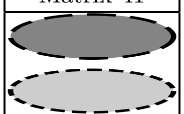
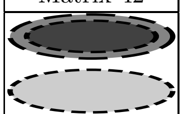
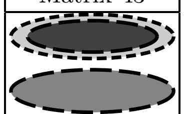
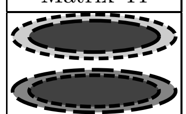
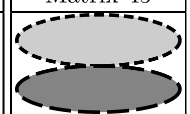
Matrix 36	Matrix 37	Matrix 38	Matrix 39	Matrix 40
				
$\begin{pmatrix}1&1&1\\0&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix}$
Matrix 41	Matrix 42	Matrix 43	Matrix 44	Matrix 45
				
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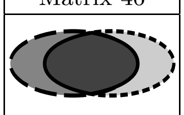
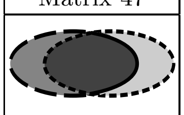
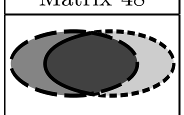
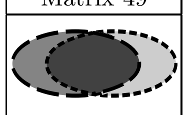
Matrix 46	Matrix 47	Matrix 48	Matrix 49	
				
$\begin{pmatrix}1&0&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	

Table 1. *(Continued)*

5 Computing Topological Relationships

Map geometries are currently implemented in many spatial systems as GIS topologies, but no mechanism to compute topological relationships between pairs of map geometries exists. However, map geometries, as considered in this paper, are composed of complex regions, and the topological predicates between complex regions are implemented in many spatial systems. Therefore, we now derive a method to determine the topological relationship between two map geometries based on the topological relationships between their component regions.

Recall that topological relationships based on the 9-intersection model are each defined by a unique 9IM. Given two map geometries, our approach is to derive the values in the 9IM representing their topological relationship by examining the *local* interactions between their component regions. We begin by discussing the properties of the 9IM, and then use these properties to show how we can derive a 9IM representing the topological relationship between two map geometries from the 9IMs representing the relationships between their component regions.

We begin by making two observations about the expression of local interactions between the component regions of map geometries in the 9IM that represents the geometries' relationship. The first observation is that if the interiors of two regions intersect, then the interiors of the map geometries intersect. This is because the presence of additional regions in a map geometry cannot cause two intersecting regions to no longer intersect. This observation also holds for interior/boundary intersections, and boundary/boundary intersections between regions composing map geometries. Therefore, if two map geometries contain respective regions such that those regions' interiors or boundaries intersect each other, then the 9IM for those regions will have the corresponding entry set to *true*.

The first observation does not hold for local interactions involving the exteriors of regions composing two map geometries. Consider Figure 4. This figure depicts two map geometries, one with a solid boundary (which we name A), and a second one with a dashed boundary (which we name B) that is completely contained in the closure of the first. Note that if we examine the topological relationship between the leftmost regions in each geometry, it is clear that interior of the region from B intersects the exterior of the region from A. However, this does not occur globally due to the other regions contained in the geometries. However, we make the observation that the union of all regions that compose a map geometry is a complex region whose exterior is equivalent to the exterior of the map geometry. Therefore, a map geometry's interior or boundary intersects the exterior of a second map geometry if the interior or boundary of a single region in the first map geometry intersects the exterior of the union of all regions in the second. Therefore, we can discover the values in the 9IM representing the topological relationship between two map geometries by examining the interactions of those geometries' component regions and their unions. Given a map geometry π a we define the set of all complex regions in π as $R(\pi) = \{\overline{r} | r \in \rho(\pi)\}$

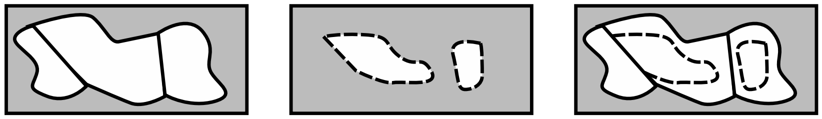


Fig. 4. Two example map geometries shown individually and overlaid.

(recall that the regions in the set $\rho(\pi)$ are open point sets, and complex regions are defined as closed point sets):

Definition 3. *Let A and B be map geometries and $U_r = \bigcup_{r \in R(A)} r$ and $U_s = \bigcup_{s \in R(B)} s$. Let $M_{(A,B)}{}^{\circ\circ}$ be the entry in the matrix M representing a topological relationship between A and B corresponding to the intersection of the interiors of A and B, etc. The entries in the 9IM representing the topological relationship between two map geometries can be defined as follows:*

$M_{(A,B)}{}^{\circ\circ} = \exists r \in R(A), s \in R(B) | r^\circ \cap s^\circ \neq \varnothing$
$M_{(A,B)}{}^{\circ}\partial = \exists r \in R(A), s \in R(B) | r^\circ \cap \partial s \neq \varnothing$
$M_{(A,B)}\partial^{\circ} = \exists r \in R(A), s \in R(B) | \partial r \cap s^\circ \neq \varnothing$
$M_{(A,B)}\partial\partial = \exists r \in R(A), s \in R(B) | \partial r \cap \partial s \neq \varnothing$
$M_{(A,B)}{}^{-\circ} = \exists s \in R(B) | U_r^- \cap s^\circ \neq \varnothing$
$M_{(A,B)}{}^{-}\partial = \exists s \in R(B) | U_r^- \cap \partial s \neq \varnothing$
$M_{(A,B)}{}^{\circ-} = \exists r \in R(A) | r^\circ \cap U_s^- \neq \varnothing$
$M_{(A,B)}\partial^{-} = \exists r \in R(A) | \partial r \cap U_s^- \neq \varnothing$

At this point we are able to characterize a 9IM representing the topological relationship between two map geometries based on intersections of the components of the regions that make up the map geometries. However, our goal is to go a step further and characterize such a relationship based on the topological predicates between complex regions that are currently implemented in spatial systems. In order to achieve this, we represent a topological relationship between map geometries as an ordered triple of sets of topological predicates between complex regions, which we denote a *component based topological relationship* (CBTR). This triple consists of a set of topological predicates between the component regions in two map geometries (from which we can directly identify whether the interiors, boundaries, or interiors and boundaries of the map geometries intersect), a set of topological predicates between the regions of the first map geometry and union of the regions in the second (to determine whether the interior and boundary of the first intersect the exterior of the second), and a set of topological predicates between the regions of the second map geometry and the union of the regions in the first.

Definition 4. *Let P_{CR} be the set of topological predicates between complex regions. A component based topological relationship between map geometries A and B is an ordered triple $CBTR(A,B) = (P, N_1, N_2)$ of sets of topological predicates between complex regions such that:*

$P = \{p \in P_{CR} | r \in R(A) \;\wedge\; s \in R(B) \;\wedge\; p(r,s)\}$
$N_1 = \{p \in P_{CR} | r \in R(A) \;\wedge\; U_s = \bigcup_{s \in R(B)} s \;\wedge\; p(r, U_s)\}$
$N_2 = \{p \in P_{CR} | U_r = \bigcup_{r \in R(A)} r \;\wedge\; s \in R(B) \;\wedge\; p(U_r, s)\}$

Definition 4 allows us to identify a CBTR given two map geometries. However, we still cannot determine the topological relationship between two map geometries given their CBTR. In order to do this, we must define a correspondence between 9IMs representing topological relationships between map geometries and CBTRs. Given such a correspondence, we can identify which CBTRs correspond to a particular 9IM, and vice versa. Furthermore, if we can show that each CBTR corresponds to a single 9IM for map geometries, then we will be able to uniquely identify a topological relationship between two map geometries by determining their CBTR.

In order to find the possible CBTRs that correspond to a particular 9IM between map geometries R, we find all possible values for each set in the triple (P, N_1, N_2). We then find all combinations of these values that form a valid CBTR. To find all possible values of the set P, we take the powerset of the set of 9IMs representing topological relationships between complex regions. We then keep only the sets in the power set such that (i) for each interaction between interiors, boundaries, or interior and boundary in R with a value of *true*, at least one 9IM exists in the set that has a corresponding entry set to *true*, and (ii), for each interaction between interiors, boundaries, or interior and boundary in R with a value of *false*, all 9IMs in the set have a corresponding entry of *false*. This follows directly from the observations made about the intersections of interiors and boundaries among the regions that make up a map geometry. The set of possible values for N_1 and N_2 are computed identically, except entries corresponding to interactions involving the exterior of a map geometry are used. A CBTR that corresponds to the topological relationship R is then a triple (P, N_1, N_2) consisting of any combination of the computed values for each set. Because Definition 3 defines each entry in a 9IM based on an equivalence to information found in the CBTR, it follows that each CBTR corresponds to a single topological relationship between map geometries. Therefore, we are able to uniquely represent a topological relationship between map geometries as a CBTR, which consists of topological relationships between complex regions. To use this in a spatial database, one must compute the CBTR for two given map geometries, and then use the rules in Definition 3 to construct the 9IM that represents their topological relationship.

6 Conclusions and Future Work

In this paper, we have defined and provided examples of the 49 topological relationships between map geometries modeled as spatial partitions. These relationships can be used to pose topological queries over map geometries in spatial systems. Furthermore, we have shown how these topological relationships can be computed using the existing topological relationships between complex regions that are currently implemented in many spatial systems. For future work, we plan to investigate a map geometry querying mechanism. Because map geometries are more complex than other spatial types, it is not yet clear if the traditional spatial querying mechanisms are adequate for handling general queries over map geometries.

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