Circle Packing and its Applications

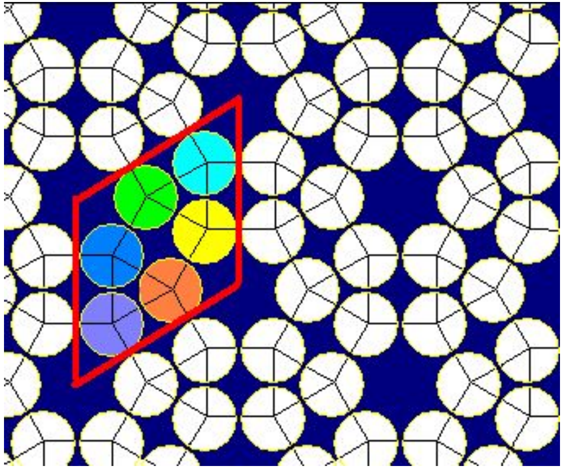
Jeremy Youngquist

Problem Statement

Many variants on circle packing exist, we will look at two of them:

1. Filling a domain with *n* circles of maximum radius
2. Find radii of circles which satisfy a tangency condition

NP-Hard optimization problem

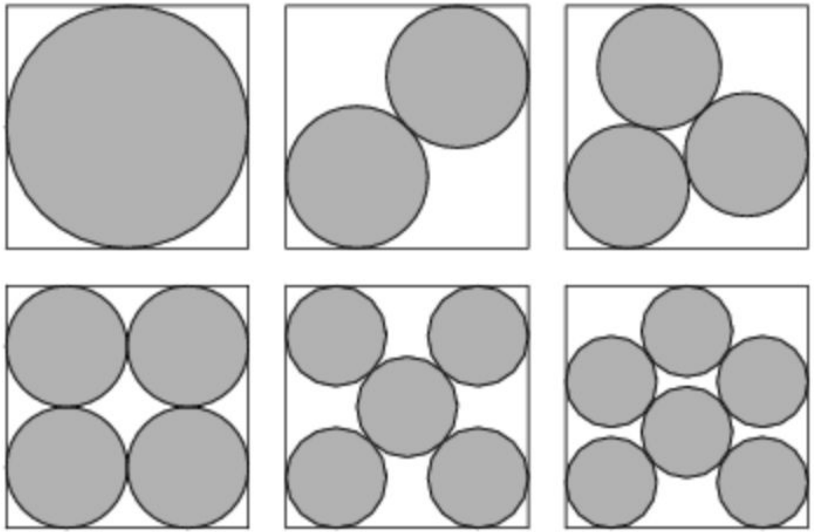


In report: Applications to mesh generation and modeling granular materials/isostatic circle packings

Filling a Square with Circles

Given a unit square, how large of a radius can *n* circles have and still fit?

Reformulation: find the locations of n points in the unit square such that the minimum distance between any two points is maximized

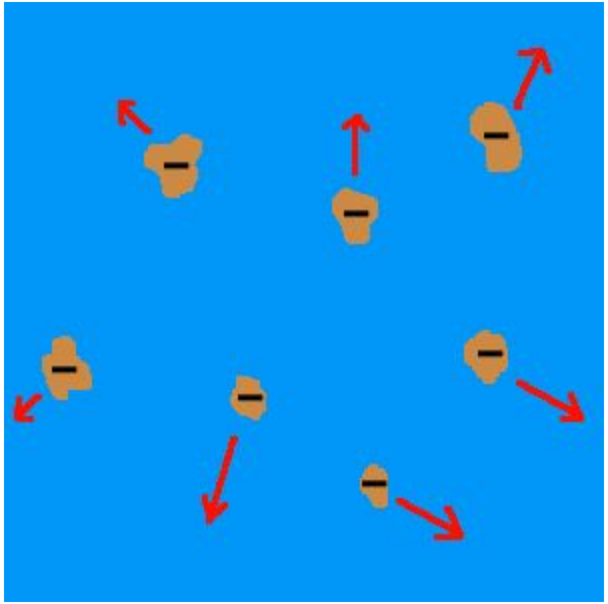


Approach 1: Energy Minimization

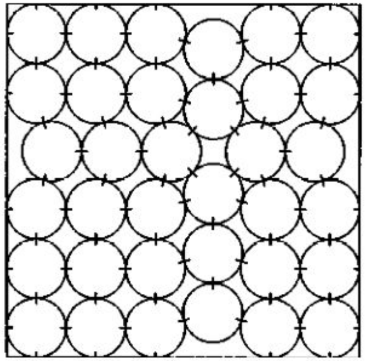
Observe that the problem restatement behaves similarly to electrons in a potential well

Minimize total energy to find packing

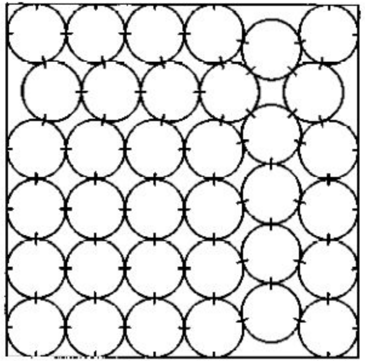
$$E = \sum_{1 \le i < j \le n} \left(\frac{\lambda}{d_{ij}^2} \right)^m$$



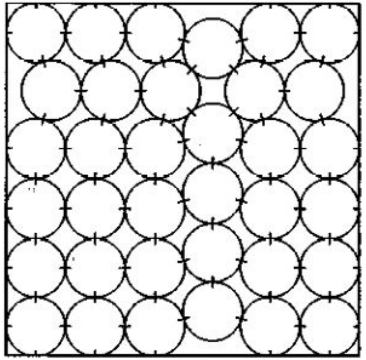
Energy-Minimal Packing Examples



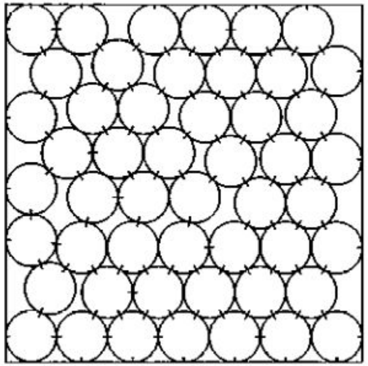
$n - 35(a)$



$n = 35(b)$



$n = 35(c)$

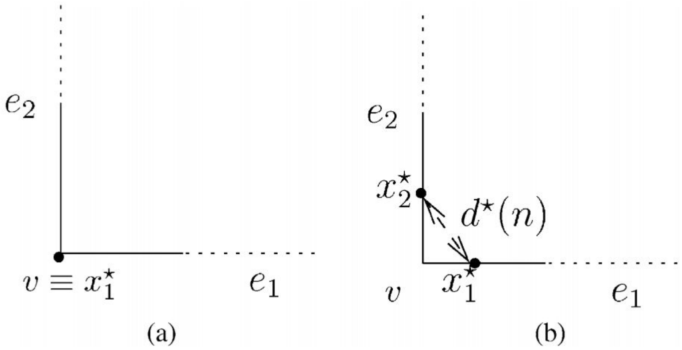


$n = 50$

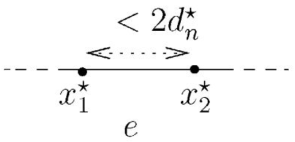
Quadratic Programming

Note the following reduction to a mathematical programming problem:

$$
\begin{aligned}
&\max t \\
&t - ||x_i - x_j||_2^2 \leqslant 0, \quad 1 \leqslant i < j \leqslant n, \\
&x_i \in U, \qquad\qquad\qquad i = 1, \ldots, n,
\end{aligned}
$$



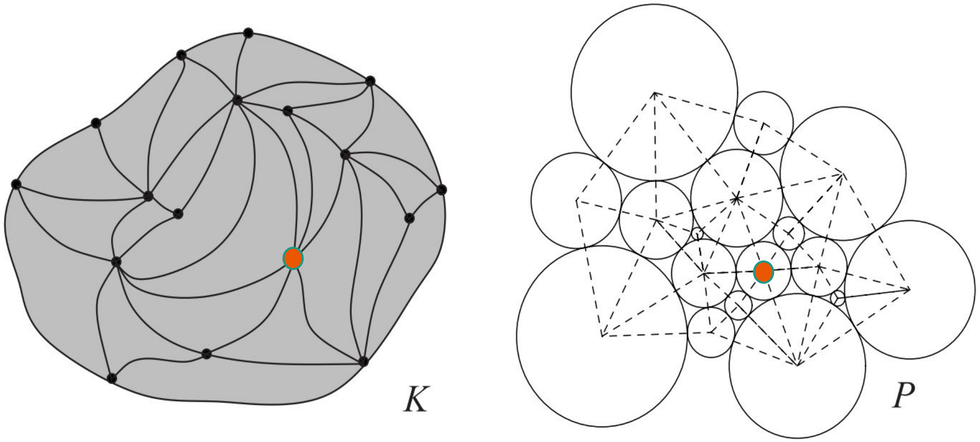
Use specialized solvers which take advantage of geometry



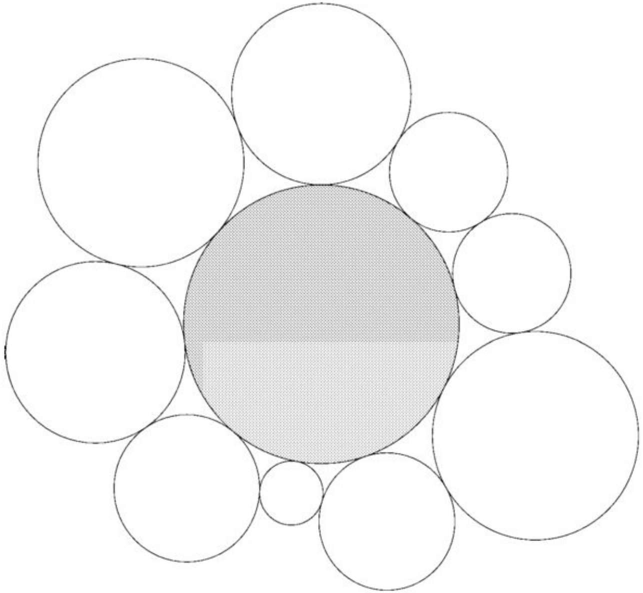
Satisfying Tangency Conditions

Given a graph *K*, find a circle packing *P* whose tangency graph is *K*

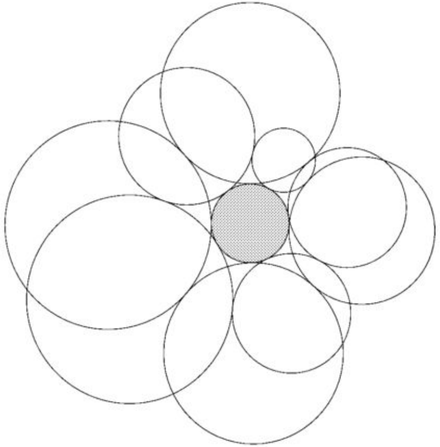
Generally motivated by applications to conformal mappings

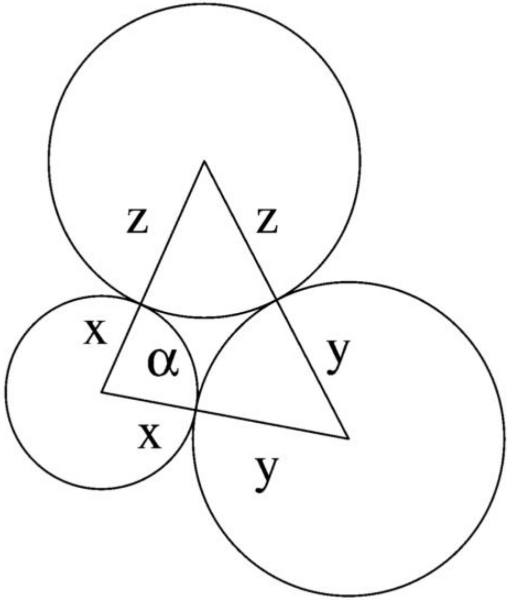


Angle Sums and Flowers



$$\theta(v) = \sum \alpha$$





The Algorithm

Algorithm Pack

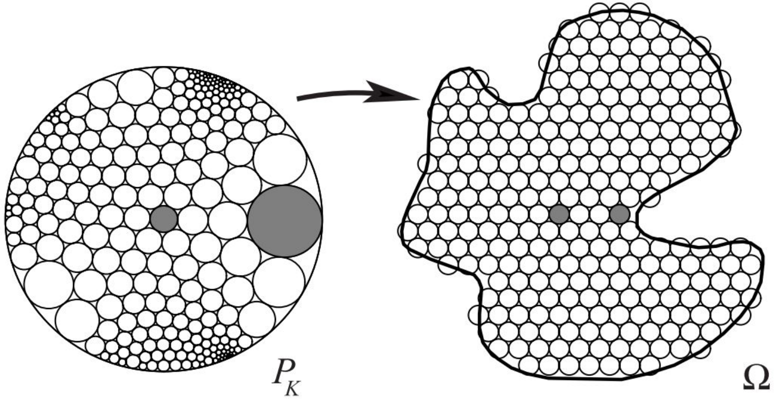
procedure PACK(K)
    while $\sum_v |\theta(v, R) - 2\pi n| > \epsilon$ **do**
        Choose an internal vertex v of K.
        Calculate $\theta(v)$
        Find r s.t. if all circles in flower had radius r, $\theta(v)$ would not change.
        Set radius of v to be the radius for which k circles of radius r would have $\theta(v) = 2\pi$
    end while
end procedure

Conformal Mappings

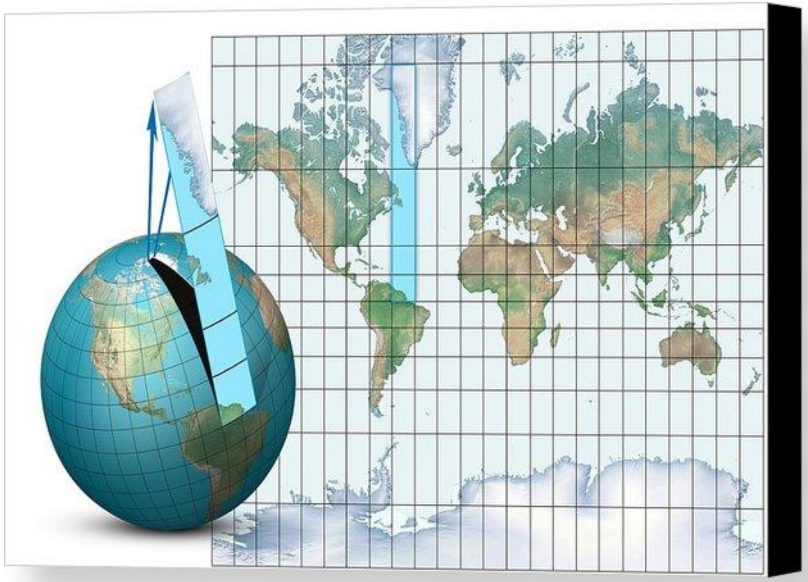
A conformal map is a map which is locally angle-preserving.

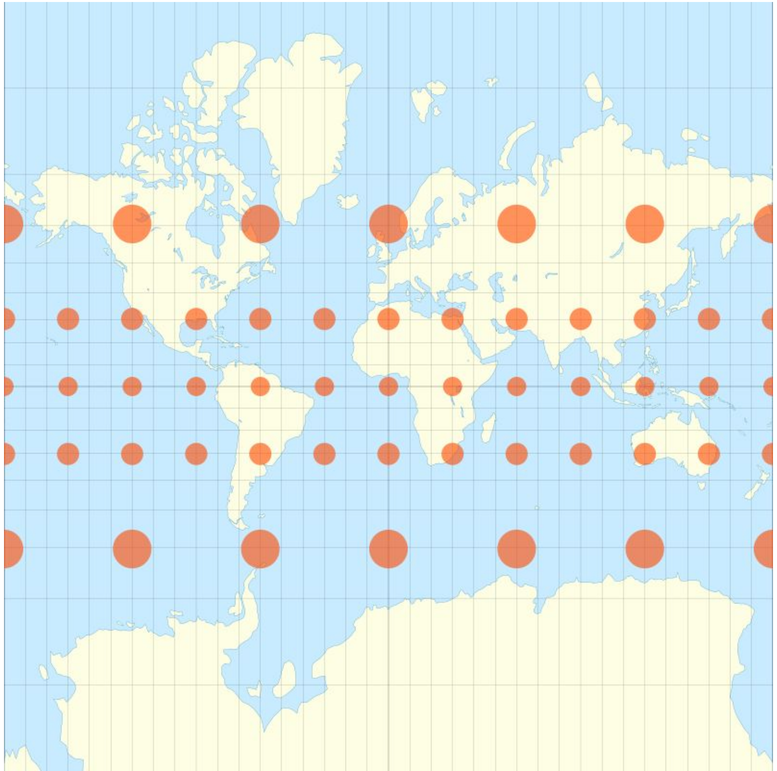
Intuitively it maps infinitesimal circles to infinitesimal circles.

What if we map larger circles to larger circles?



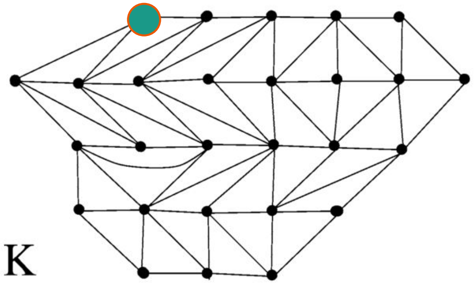
Mercator Projection



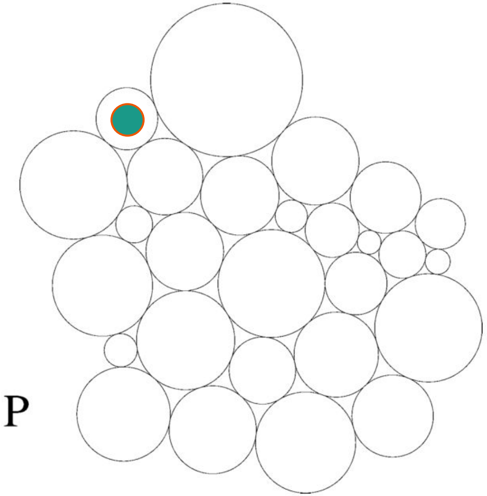


A Key Theorem

Let K be a triangulation of an oriented topological surface. Then there exists an essentially unique circle packing P_k for K in the unit disk.



K



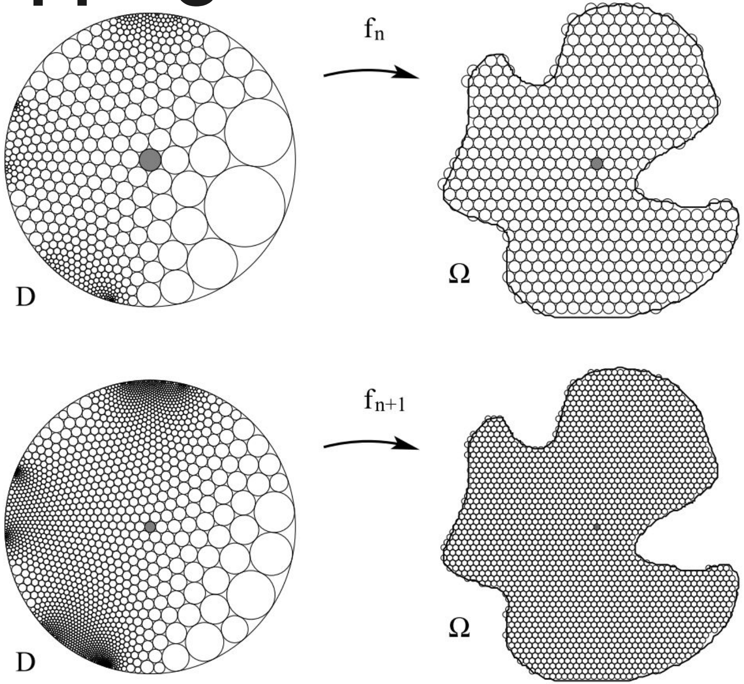
P

Converging to a Riemann Mapping

Tile the entire plane with a hexagonal circle packing of circles with radius ε.

Carve out portion which fills domain Ω and let K be its tangency graph.

Find packing for K in unit disk.



References (1/2)

[1] Marshall Bern and David Eppstein. Quadrilateral meshing by circle packing. 10, 09 1999.

[2] Marshall W. Bern, Scott A. Mitchell, and Jim Ruppert. Linear-size nonobtuse triangulation of polygons. In *Symposium on Computational Geometry*, 1994.

[3] Charles R. Collins and Kenneth Stephenson. A circle packing algorithm. *Computational Geometry*, 25(3):233 – 256, 2003.

[4] Robert Connelly. Rigidity of packings. *European Journal of Combinatorics*, 29(8):1862 – 1871, 2008. In honor of Ludwig Danzer's 80th birthday.

[5] Robert Connelly, Evan Solomonides, and Maria Yampolskaya. The isostatic conjecture, 2017.

[6] Aleksandar Donev, Salvatore Torquato, Frank H. Stillinger, and Robert Connelly. Jamming in hard sphere and disk packings. *Journal of Applied Physics*, 95(3):989–999, 2004.

[7] Mhand Hifi. A literature review on circle and sphere packing problems: Models and methodologies. *Advances in Operations Research*, (22), 2009.

[8] Marco Locatelli and Ulrich Raber. Packing equal circles in a square: a deterministic global optimization approach. *Discrete Applied Mathematics*, 122(1):139 – 166, 2002.

[9] Gary Miller, Dafna Talmor, Shang-Hua Teng, and Noel Walkington. On the radius-edge condition in the control volume method. *SIAM J. Numer. Anal.*, 36, 1999.

[10] John W. Morgan. Chapter v on thurston's uniformization theorem for three-dimensional manifolds. volume 112 of *Pure and Applied Mathematics*, pages 37 – 125. Elsevier, 1984.

References (2/2)

[11] K. J. Nurmela and P. R. J. Östergård. Packing up to 50 equal circles in a square. *Discrete & Computational Geometry*, 18(1):111–120, Jul 1997.

[12] Burt Rodin and Dennis Sullivan. The convergence of circle packings to the riemann mapping. *J. Differential Geom.*, 26(2):349–360, 1987.

[13] Steffen Rohde. Oded schramm: From circle packing to sle, 2010.

[14] Kenneth Stephenson. The approximation of conformal structures via circle packing. In *In Computational Methods and Function Theory 1997, Proceedings of the Third CMFT Conferent*, pages 551–582. World Scientific.

[15] Kenneth Stephenson. Circle packing: a mathematical tale. *Notices of the AMS*, 50(11):1376–1388, 2003.

[16] Terrence Tao. 246c notes 2: Circle packings, conformal maps, and quasiconformal maps, 2018.

[17] S. Torquato and F. H. Stillinger. Multiplicity of generation, selection, and classification procedures for jammed hard-particle packings. *The Journal of Physical Chemistry B*, 105(47):11849–11853, 2001.

[18] Eric W. Weisstein. Circle packing. From MathWorld—A Wolfram Web Resource.

[19] Eric W. Weisstein. Conformal mapping. From MathWorld—A Wolfram Web Resource.

Mesh Generation

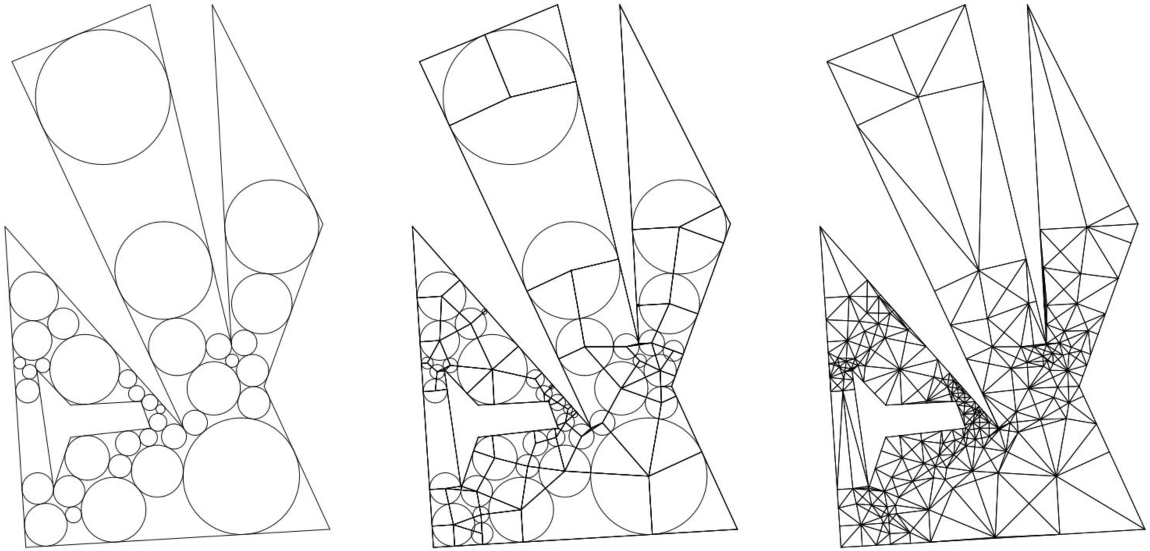


Figure 1. (a) Disk packing. (b) Induced small polygons. (c) Final triangulation.