

# Jörg Peters' Box Spline Synopsis

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Box splines represent a generalization of univariate spline theory to several variables. Box splines were introduced by de Boor and DeVore [3]. The rich theory is collected in the “box spline book” [1] which serves as reference for the following exposition of these piecewise polynomial functions.

The box spline  $M_{\Xi}$  in  $s$  variables is defined by the  $s \times n$  matrix  $\Xi$  (pronounced Xi) with columns in  $R^s \setminus 0$ . For the purposes of this paper we may assume that the first  $s$  columns of  $\Xi$  form the identity matrix  $I$ . This yields the following inductive definition of the box spline. If  $\Xi = I$ , then  $M_{\Xi}$  is the function that is 1 on the unit cube and 0 elsewhere:

$$M_I(x) := \begin{cases} 1, & \text{if } x \in [0..1]^s, \\ 0, & \text{else.} \end{cases}$$

This box spline is piecewise constant, has degree zero and is discontinuous. If  $\Xi \cup \xi$  is any matrix formed from  $\Xi$  by the addition of the column  $\xi \in R^s$ , then the box spline  $M_{\Xi \cup \xi}$  is given by the convolution equation

$$M_{\Xi \cup \xi}(u) = \int_0^1 M_{\Xi}(u - t\xi).$$

For  $s = 1$  this is exactly the B-spline construction by convolution (c.f. Figure 1).

## 1.1 Box spline properties

The box spline has the following properties.

- (i)  $M_{\Xi}$  is positive and its shifts sum to one:  
 $\sum_{\alpha \in Z^s} M_{\Xi}(\cdot - \alpha) = 1$ .
- (ii) The support of  $M_{\Xi}$  is  $\Xi[0..1]^s$ , i.e. the set sum of the columns contained in  $\Xi$ .

(iii)  $M_{\Xi}$  is piecewise polynomial of degree  $n - s$ . That is, each convolution in another direction  $\xi$  increases the degree by one.

(iv)  $M_{\Xi}$  is  $\rho - 2$  times continuously differentiable, where  $\rho$  is the minimal number of columns that need to be removed from  $\Xi$  to obtain a matrix whose columns do not span  $R^s$ .

(v)  $M_{\Xi}$  reproduces all polynomials of degree  $m := \rho - 1$  and none of degree higher than  $n - s$ .

(vi) The  $L^p$  approximation-order of the spline space  $S := \text{span}(M(\cdot - \alpha))$  is  $\rho$ . That is with the refinement of the lattice  $x \rightarrow hx, h < 1$ ,  
 $\text{dist}(f, \sum a(\alpha)M_{\Xi}((\cdot - \alpha)/h)) = O(h^{\rho})$  for all sufficiently smooth  $f$ .

Thus the  $n$  columns of  $\Xi$ , which may be interpreted as directions in  $R^s$ , determine the support of the piecewise polynomial and its continuity properties. Understanding the number  $\rho$  requires an analysis of the independent submatrices of  $\Xi$ .

## 1.2 Box spline examples

We develop three examples relevant to this paper.

- 1. The well-known univariate uniform cubic B-spline has the matrix (direction set)

$$\Xi := [1 \quad 1 \quad 1 \quad 1 \quad ]$$

Figure 1 shows in order the characteristic function of the 1-dimensional cube and its repeated convolution in the direction 1 yielding the linear ‘hat’ function, the quadratic and finally the cubic B-spline. We determine the characteristic numbers as

$$s = 1, n = 4, \text{ and } \rho = 4$$

Figure 1: Uniform univariate splines

since all elements of the set have to be removed to make it nonspanning in  $R^s$ . The degree of the B-spline pieces is  $n - s = 3$  and the continuity is of order  $\rho - 2 = 2$  as expected. The cubic spline formed as a linear combination of B-splines is guaranteed to at least reproduce polynomials of degree 3 and none of degree higher than 3. The approximation order is 4.

- 2. The bivariate box spline  $z$  based on the matrix

$$\Xi := \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

is called Zwart-Powell element [6], [4], [5]. It is displayed in Figure 4 (lower right). The characteristic numbers are

$$s = 2, n = 4, \text{ and } \rho = 3$$

and hence the element is of degree 2 and its polynomial pieces are connected  $C^1$ . Since  $n - s = 2 = \rho - 1$  linear combinations of the ZP-element reproduce exactly all quadratic polynomials; that is any quadratic  $q(x, y)$  can be written as

$$q(x, y) = \sum_{\alpha \in Z^2} a(\alpha) z((x, y) - \alpha).$$

The Zwart-Powell element stands out among the low-degree box-splines defined over the plane, in that it has maximal smoothness equal to the degree minus one and is piecewise polynomial over a regular triangulation.

- 3. The 7-direction box spline is a similar serendipity element among the trivariate box splines. It is based on the direction matrix

$$\Xi := \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The seven directions defined by the columns of the matrix cut  $R^3$  into a symmetric regular arrangement of tetrahedra. The characteristic numbers of the 7-direction box spline are

$$s = 3, n = 7, \text{ and } \rho = 4.$$

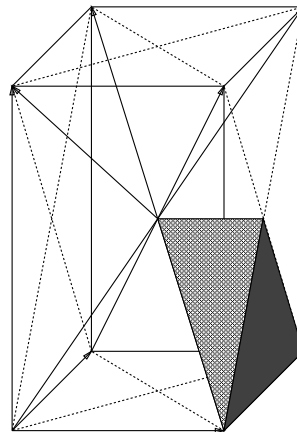


Figure 2: The 7 directions of the box spline and its domain tetrahedra

Thus the polynomial piece defined over each tetrahedron is of degree  $n - s = 4$  and splines formed as a linear combination of shifts of the box spline are  $C^{\rho-2} = C^2$ . Elements of the spline space reproduce all cubics in three variables (and some additional polynomials of degree four) and the approximation order is 4.

### 1.3 Box spline subdivision

To quickly approximate any box spline we may use subdivision. Since the shifts of the box spline  $M_\Xi$  form a nonnegative, local partition of unity, a spline formed as a linear combination of shifts of the box spline is a finite convex combination of its coefficients  $a(\alpha)$ . To the extent that the local variation of the coefficients is small, the coefficients  $a(\alpha)$  approximate the spline well. This is the basis for fast algorithms for graphic display and rendering. The key observation is that the variation of the coefficients is reduced when the spline is expressed in terms of box splines corresponding to the refined lattice  $\frac{1}{2}Z^s$ :

$$\sum_{j \in Z^s} a(j) M(x - j) = \sum_{k \in \frac{1}{2}Z^s} a_{1/2}(k) M(2(x - k)).$$

$$\begin{array}{cccc}
& & a & a & b & b \\
4a & 4b & \rightarrow & a & a & b & b \\
& & & c & c & d & d \\
4c & 4d & & c & c & d & d \\
\end{array}$$

$$\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\rightarrow^{1/2} & aa & ab & bb & \cdot & \rightarrow^{1/4} & \cdot & \cdot & \cdot & \cdot \\
& ac & bc & bd & \cdot & & aabc & abbd & \cdot & \cdot \\
& cc & cd & dd & \cdot & & \cdot & accd & bcdd & \cdot \\
\end{array}$$

Figure 3: Four-direction box spline subdivision.

The successive computation of a sequence of refined coefficients  $a_1, a_{1/2}, \dots$  is called a **subdivision algorithm**: We compute  $a_{h/2}$  from  $a_h$  for  $\alpha$  on the finer mesh  $\frac{h}{2}Z^s$ . First set

$$a_{h/2}(\alpha) := \begin{cases} 2^s a_h(\alpha), & \text{if } \alpha \in hZ^s \\ 0, & \text{else} \end{cases}$$

Then average in each of the directions in  $\Xi$ . That is for each  $\xi \in \Xi$  compute, careful not to overwrite still needed values,

$$a_{h/2}(\alpha) \leftarrow (a_{h/2}(\alpha) + a_{h/2}(\alpha - \xi/2))/2.$$

Under mild assumptions on the matrix  $\Xi$  that are satisfied by all three box splines defined above, the sequence of control points converges quadratically to the spline [1], (30)Theorem, page 169. The sequence of array entries for the subdivision of a spline with coefficients  $a, b, c, d$  and the ZP-element as  $M$  is displayed in Figure 3.

To illustrate the effect of subdivision as approximate evaluation, we choose one box-spline coefficient (at the origin) non-zero, and all other coefficients equal zero, i.e.

$$a(\alpha) = \begin{cases} 1, & \text{if } \alpha_1 = \alpha_2 = 0 \\ 0, & \text{else} \end{cases} \quad \alpha := (\alpha_1, \alpha_2).$$

Then

$$\sum_{\alpha} a(\alpha) M_{\Xi}((x, y) - \alpha) = M_{\Xi}(x, y),$$

and the spline represents just a single basis function. Figure 4 below shows four steps of subdivision on the spline coefficients. The central spike is of height 1.

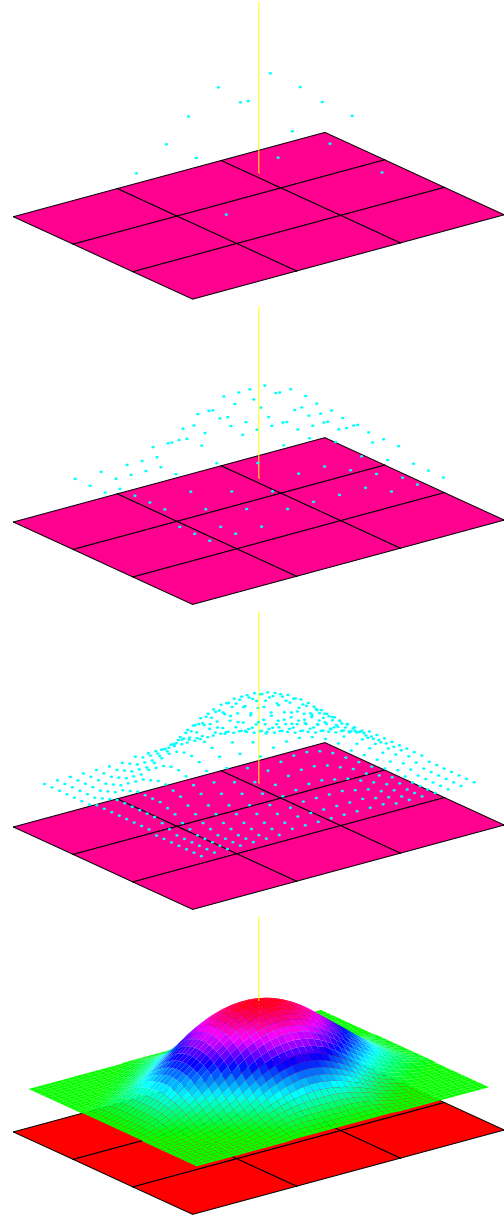


Figure 4: The Zwart-Powell element  $M_{\Xi}$  approximated using 4 steps of subdivision. The point cloud are the coefficients generated by the refinement.

In principle, one can convert any piecewise polynomial in box-spline form into any other piecewise polynomial representation such as the power form or the Bernstein form. For example, in the Bernstein-Bézier form, the Zwart element is represented by 28 quadratic pieces with coefficients  $1/2$ ,  $1/4$ ,  $1/8$  and 0.

## References

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