ABSTRACT

Limited-data Computed Tomography (CT) presents challenges for image reconstruction algorithms and has been an active topic of research aiming at reducing the exposure to X-ray radiation. We present a novel formulation for tomographic reconstruction based on sparse approximation of the image gradients from projection data. Our approach leverages the interdependence of the partial derivatives to impose an additional curl-free constraint on the optimization problem. The image is then reconstructed using a Poisson solver. The experimental results show that, compared to total variation methods, our new formulation improves the accuracy of reconstruction significantly in few-view settings.

Index Terms— Gradient-Domain Sparsity, Compressed Sensing, Computed Tomography, Total Variation

1. INTRODUCTION

Tomographic reconstruction from limited projection data (e.g., few-view and limited-angle [1]) has been the subject of active research with the goal of reducing radiation exposure. Accurate image reconstruction from very low dose data is key to making mass screening (e.g., for pre-metastasis breast cancer) a viable option. In the limited data setting, classical approaches, such as filtered back projection (FBP), produce visible artifacts and result in unacceptable images. Statistical image reconstruction algorithms offer significant improvements over analytical methods (e.g., FBP) by minimizing a penalized-likelihood cost function, modeling the statistics of X-ray imaging, with the caveat of expensive computations [2, 3, 4]. On the other hand, principles of compressed sensing (CS) and sparse approximation offer attractive prospects for imaging from limited data [5] and there is a growing body of evidence that CT can significantly benefit from the integration of sparsity into the reconstruction algorithms [1].

The sparsity of the representation for the signal (being reconstructed) is crucial to the success of optimization-based reconstruction methods from limited data. While transform-domain sparsity (e.g., wavelets, ridgelets) is often used for CS reconstruction, sparsity in the gradient domain is a suitable choice for the class of piecewise constant signals. Many biomedical images can be closely modeled as homogeneous regions with variations only occurring at boundaries between the neighboring regions [2, 3]. Motivated by sparsity in the gradient domain, many approaches formulate the reconstruction as an optimization problem where the total variation (TV) norm is minimized in the solution [6, 3, 7]. Sidky et al. [3] proposed an iterative algorithm for CT reconstruction that minimizes TV norm, subject to non-negativity of the image and consistency of the estimated projections with the available data. Choi et al. [6] presented an efficient first-order method for solving TV minimization constrained with statistically weighted least-squares of the projection data in CT. Mueller and Siltanen [7] formulated the limited-angle CT as a Bayesian inversion problem, where the prior (e.g., space of bounded variation functions) is integrated into the reconstruction algorithm by seeking the maximum a posteriori (MAP) estimate of the image unknowns.

TV minimization approaches reconstruct an image whose projections are consistent with the given measurements while its gradient magnitude is sparse. We propose to reconstruct the gradient (vector) field of the image in a sparse approximation framework for computed tomography. The image is then recovered from the gradient field as the second step. The advantage of reconstructing the gradient field (instead of directly recovering the image) is that the sparsity exhibited in the individual partial derivatives (i.e., components of the gradient vector) is often greater than the sparsity observed in the field of gradient magnitudes (i.e., TV).

Patel et al. introduced a gradient-based image recovery from partial Fourier samples in [8], where the horizontal and vertical partial derivatives of the image are recovered separately. While we show that this approach can be adapted for tomographic reconstruction, we demonstrate that the separate recovery of partial derivatives can lead to less competitive results. In contrast, we propose a novel formulation for limited-data CT imaging, that exploits the interdependence of partial derivatives of an image and enforces the integrability constraint at the sparse approximation stage.

Leveraging the relation between the gradient components, has been shown to be effective for partial Fourier sensing [9], where deriving partial data corresponding to derivative images is straightforward. In this paper, we provide a formulation for deriving X-ray measurements corresponding to partial derivatives, for CT applications. The experimental re-
results show that, in a few-view setting, the proposed approach, which we call Sparse Gradient Field (SGF), outperforms recovery based on TV minimization as well as the approach based on separate recovery of partial derivatives.

2. SPARSE APPROXIMATION OF GRADIENTS

TV minimization methods [5, 1] seek for an image, represented by a vector \( \mathbf{f} \in \mathbb{R}^n \), whose measurements (partial Fourier or Radon transforms) are consistent with the given data \( \mathbf{p} \), while penalizing solutions with large total variations:

\[
\hat{\mathbf{f}} = \arg\min_{\mathbf{u} \in \mathbb{R}^n} \|A\mathbf{u} - \mathbf{p}\|_2^2 + \lambda(\|D_x\mathbf{u}\|_1 + \|D_y\mathbf{u}\|_1). \quad (1)
\]

Here \( A \) is the sensing operator (tomographic system matrix or partial Fourier) and \( D_x \) and \( D_y \) are the horizontal and vertical differential (or difference) operators.

Let \( f_x = D_x f \) and \( f_y = D_y f \) denote the images corresponding to horizontal and vertical components of the gradient of \( f \). Moreover, let \( p_x \) and \( p_y \) be the measurements corresponding to the images \( f_x \) and \( f_y \), respectively. The fact that individual gradient components, \( f_x \) and \( f_y \), are each sparser than the gradient magnitude image \( |f_x| + |f_y| \), motivates separate recovery of partial derivatives through independent convex optimizations:

\[
\hat{f}_x = \arg\min_{u_x \in \mathbb{R}^n} \|A_{u_x} - p_x\|_2^2 + \lambda_1\|u_x\|_1 \quad (2a)
\]

\[
\hat{f}_y = \arg\min_{u_y \in \mathbb{R}^n} \|A_{u_y} - p_y\|_2^2 + \lambda_2\|u_y\|_1 \quad (2b)
\]

In the context of MR imaging (i.e., partial Fourier sensing), this formulation was shown to be competitive to TV when the gradient field is very sparse [8]. In this approach each component is approximated separately, and therefore the recovered vector field \( [\hat{f}_x, \hat{f}_y] \) may not be integrable (i.e., exhibits a non-zero curl). However, a vector field corresponding to the gradient of an image will necessarily have zero curl:

\[
\text{curl}(\nabla f) = D_x f_y - D_y f_x = 0. \quad (3)
\]

The solution proposed in [8], is a post-processing approach where an integrable gradient field is estimated (e.g., via least-squares) from the sparse approximation problems in (2).

In our formulation, \( A \) is the tomographic system matrix and \( p \) is the projection data (i.e., Radon transform). We will describe how to derive \( p_x \) and \( p_y \) from the given projections \( p \), in Section 3. In contrast to [8], we exploit the interdependence of partial derivatives to impose \( n \) additional curl-free constraints at the sparse approximation stage, which allows for recovery from fewer number of samples. This is achieved by tying the two optimizations in (2a) and (2b) with the additional curl-free constraint that the recovered vector field needs to be curl free. Therefore, the sparse gradient recovery is formulated as:

\[
\hat{f}_x, \hat{f}_y = \arg\min_{u_x, u_y \in \mathbb{R}^n} \|A_{u_x} - p_x\|_2^2 + \|A_{u_y} - p_y\|_2^2 + \\
\lambda(\|u_x\|_1 + \|u_y\|_1) + \gamma(\|D_y u_x - D_x u_y\|_2^2) \quad (4)
\]

To solve (4), we define the gradient field \( g = [f_x, f_y]^T \in \mathbb{R}^{2n} \) and a system matrix \( H \) that integrates the curl-free constraint together with the derivatives projection data \( p' \):

\[
H = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad p' = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \quad (5)
\]

The proposed SGF method recovers the gradient field as a solution to the sparse approximation problem defined by:

\[
\hat{g} = \arg\min_{u' \in \mathbb{R}^{2n}} \|H u' - p'\|_2^2 + \lambda\|u'\|_1. \quad (6)
\]

This problem can be solved using any \( \ell_1 \) solver (e.g., L1-LS [10], or ADMM [4]). We note that \( H \) is implemented as an operator and does not need to be an explicit matrix.

Once the partial derivatives are recovered from the sparse approximation problem (6), the image can be reconstructed using the Poisson equation (with the Dirichlet boundary conditions) [8]:

\[
\Delta \hat{f} = \text{div}(\hat{g}) \quad (7)
\]

where \( \Delta \) is the Laplacian operator and \( \text{div}(\hat{g}) = D_x f_x + D_y f_y \) is the divergence of the recovered gradient field.

3. PROJECTION OF PARTIAL DERIVATIVES

In order to solve (6), the projections of the partial derivative images (i.e., \( p_x \) and \( p_y \)) need to be computed from of the data observed in the sinogram domain \( p \). In the parallel projection geometry, as described below, the projection-slice theorem [11] provides a simple recipe for deriving \( p_x \) and \( p_y \), while this process is more involved for non-parallel geometries.

Let \( S_\theta \) denote the slice operator along a direction \( \theta = [\cos(\theta), \sin(\theta)]^T \) and \( P_{\theta^\perp} \) be the operator projecting the image along the direction perpendicular to \( \theta \). From the projection-slice theorem we have that the Fourier transform of a projection \( P_{\theta^\perp}(f) \) is the restriction of the Fourier transform of the image \( F\{f\} \) to the slice:

\[
s = S_\theta(F\{f\}) = F\{P_{\theta^\perp}(f)\}. \quad (8)
\]

Let \( (\omega_x, \omega_y) \) denote the horizontal and vertical frequency variables corresponding to the \((x, y)\) space-domain variables. We have \( F\{f_x\} = j\omega_x F\{f\} \) with \( j = \sqrt{-1} \). When the slice \( s(\omega) \) is parameterized by the variable \( \omega \) in 1-D, with \( \omega_x = \cos(\theta)\omega \), we have:

\[
S_\theta(F\{f_x\})(\omega) = S_\theta(j\omega_x F\{f\}) = \cos(\theta)j\omega s(\omega). \quad (9)
\]
Since $j\omega s(\omega)$ corresponds to the Fourier transform of the derivative of the (1-D) projection data, $P_\theta(\mathbf{f})$, this implies:

$$P_\theta(\mathbf{f}) = \cos(\theta)D P_\theta(\mathbf{f})$$

where $D$ denotes the 1-D differential (difference) operator acting in the sinogram domain. This means that $p_x$, for a given angle $\theta$, is the derivative of the sinogram data $P_\theta(\mathbf{f})$, (with respect to its parameterization in the sinogram domain) scaled by $\cos(\theta)$. Similarly for $p_y$, the derivative of sinogram data is scaled by $\sin(\theta)$. In a discrete setting, derivative of given measurements is replaced by finite differencing.

4. RESULTS AND DISCUSSION

In order to study the effectiveness of the sparse gradient field approach, we compare our method against the traditional FBP, TV-based CS reconstruction [7] and reconstruction from separate recovery of gradient components.

Our ground truth images consist of a real CT dataset, obtained from a high-resolution scan of Catphan phantom, a realistic brain phantom [12] and the Shepp-Logan dataset, all of size $256 \times 256$ and shown in Fig. 1. The equation in (6) is solved using an $\ell_1$ solver as in [10], and the numerical accuracy in each reconstruction is measured in terms of Signal to Noise Ratio (SNR), reported in logarithmic (dB) scale. The regularization parameters were found empirically and $\lambda$ was set to 1, while $\gamma$ was set to 100 and 0, for SGF and separate recovery, respectively. For a fair comparison the parameters for TV minimization were also tuned to achieve the best results.

Fig. 2 shows the reconstruction results for FBP, TV minimization and the proposed approach with 15 projection angles. For such a small number of projections the FBP reconstruction results in significant noise and streak artifacts. The TV-based CS reconstruction improves the reconstruction accuracy, however, the noise is still present and visible in homogeneous regions. As a result many of the small features are not recovered clearly. Reconstruction based on the proposed SGF method is free from stair-casing artifacts and outperforms TV and FBP approaches. The SGF approach provides a clear reconstruction of the small features.

The next experiment was performed with a very small number of projection angles. The reconstruction results, shown in Fig. 3, demonstrate that when the number of projections is lowered to 10 angles TV minimization and separate recovery result in poor reconstructions, which shows the significance of the additional $256 \times 256$ (dependent) curl-free constraints in a few-view setting.

Fig. 4 shows reconstruction accuracy versus the number of projection angles for the Catphan dataset. As the graph suggests, in a few-view setting, SGF outperforms TV-based reconstruction as well as separate recovery of gradient components. Another observation is that the SGF method is more resilient to the reduction of projection angles. The accuracy of TV minimization and separate recovery drastically goes from 34 dB down to 23.5 dB when the projections are reduced from 30 to 10 angles. However, SGF is more resilient
and its reconstruction accuracy drops from 34.8 dB to 27.03 dB for the same reduction in angles.

Our next experiment examines the effect of background noise introduced by physical factors. As discussed in [1] the errors remaining from x-ray flux determination, scatter and beam-hardening can be idealistically modeled as independent, slowly varying background noise on each sinogram. Similar to [1], we model this type of noise by adding a random offset, following a Gaussian distribution with standard deviation of 10% of the maximum intensity in the sinogram data, to each projection view. This type of noise is naturally eliminated by our derivative-based SGF approach. Fig. 5 shows the resultant images from the three methods for Shepp-Logan and brain datasets reconstructed from 15 and 27 projection angles, respectively. The figure demonstrates that while TV minimization is significantly affected by the background noise, SGF and separate recovery exhibit robustness with no or minimal degradation in visual or numerical accuracy, which is due to the differentiation in the projection domain. This agrees with the observation in [1] where minimizing $\ell_2$ norm of the data derivatives results in favorable reconstructions compared to minimizing the standard $\ell_2$ data-norm.

5. CONCLUSION

We proposed a novel cost function for tomographic reconstruction that recovers the image’s gradient vector field in a sparse approximation framework. Our formulation benefits from the interdependence of partial derivatives that improves the accuracy of reconstruction by imposing additional curl-free constraints on the optimization problem. The results, in a parallel geometry, with few-view setting show that SGF method outperforms TV minimization as well as FBP. While the results are presented for 2D images, the method can be readily extended to 3D where the advantages of exploiting interdependence of partial derivatives are greater. Since SGF is based on differentiation in the sinogram domain, it eliminates the slowly varying background noise as discussed in [1].

6. REFERENCES