B-splines and the Riemann’s Zeta Function on Integers

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1 Background and definitions

The Poisson’s summation formula is:
\[ \sum_{k \in \mathbb{Z}} f(kT) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \hat{f}(n \frac{2\pi}{T}), \]  
(1)
where \( \hat{f} \) is the Fourier transform of \( f \):
\[ \hat{f}(\omega) = \int_{\mathbb{R}} f(x) \exp(-i\omega x) \, dx. \]

The Poisson’s summation formula holds for all integrable functions, given the series on the right hand side is absolutely convergent.

The simplest (centered) B-spline, \( \beta_0 \), is the characteristic function of \([-1/2, 1/2)\). Higher order B-splines \( \beta_m \) are defined recursively by \( \beta_m = \beta_{m-1} * \beta_0 \). Fourier transform of a B-spline is:
\[ \hat{\beta}_m(\omega) = \text{sinc}^{m+1}(\omega), \]  
(2)
where \( \text{sinc}(t) := \sin(t)/(t) \).

Lemma 1 (Partition of Unity).
\[ \sum_{k \in \mathbb{Z}} \beta_m(k) = 1 \]

Proof. Using Poisson’s summation formula, setting \( T = 1 \), we have:
\[ \sum_{k \in \mathbb{Z}} \beta_m(k) = \sum_{n \in \mathbb{Z}} \text{sinc}^{m+1}(2\pi n) = 1, \]
since \( \sin(\pi n) = 0 \), for all \( n \in \mathbb{Z} \) and the only non-zero term in the right hand side comes from \( n = 0 \): \( \text{sinc}(0) := 1 \) that agrees with the actual limit of sinc at 0. \( \square \)
1.1 Zeta function

The Riemann’s Zeta function is defined as:

\[ \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}. \tag{3} \]

2 Relationship between \( \beta_m \) and \( \zeta(m + 1) \)

By playing around with the parameter \( T \), one can recover parts of the series. The linear B-spline can be related to the \( \zeta(2) \) and generally, \( \beta_m \) can be related to \( \zeta(m + 1) \). For the case of linear B-spline, by setting \( T = 2 \) in (1), we have:

\[
\sum_{k \in \mathbb{Z}} \beta_1(2k) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \text{sinc}^2(n\pi)
\]

\[
= \frac{1}{2} + \sum_{n=1}^{\infty} \text{sinc}^2(n\pi)
\]

\[
= \frac{1}{2} + \frac{1}{(\pi/2)^2} \sum_{n=1}^{\infty} \frac{\text{sinc}^2(n\pi/n^2)}{n^2}
\]

\[
= \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\text{sinc}^2(n\pi/n^2)}{n^2}.
\]

The last series is nearly \( \zeta(2) \) except that \( \text{sinc}^2(n\pi/n^2) \) zeroes out the even terms. But it is, surely, convergent, since it is a sub-series of \( \zeta(2) \), so as the series with remaining terms of \( \zeta(2) \). The last series can be resolved, however, using \( \zeta(2) \):

\[
\sum_{n=1}^{\infty} \frac{\text{sinc}^2(n\pi/n^2)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}
\]

\[
= \zeta(2) - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
= \zeta(2) - \frac{1}{4} \zeta(2)
\]

\[
= \frac{3}{4} \zeta(2).
\]

Therefore, we can relate \( \beta_1 \) to \( \zeta(2) \) by:

\[
\sum_{k \in \mathbb{Z}} \beta_1(2k) = \frac{1}{2} + \frac{3}{\pi^2} \zeta(2). \tag{4}
\]

Since the support of \( \beta_1 \) is \([-1, 1)\), the left hand side is 1 and we have:

\[
\zeta(2) = \frac{\pi^2}{6}.
\]

\( \zeta(2) \) has been used in the so called **Basel problem**. Its inverse is the probability that two randomly selected integers are relatively prime.
2.1 Result

Using the above approach, one can derive the exact values for all even integers of \( \zeta \) using evaluation of B-splines on even integers. Wikipedia lists \( \zeta(2), \zeta(4) \) and \( \zeta(6) \), but, apparently, evaluation of \( \zeta \), in general is difficult and there are papers for evaluation such as [1].

Is there a closed-form solution for all B-splines of odd order:

\[
\sum_{k \in \mathbb{Z}} \beta_{2m+1}(2k) = ?
\]

(5)

3 The difficult, but interesting case

The odd values of \( \zeta \), are interesting; for instance, \( \zeta(3) \), known as Apéry’s constant [2] is a curious number that occurs in various physical problems. The exact value of this constant is not known and it is an open problem whether this number is transcendental.

To derive Apéry’s constant using the B-spline approach, we shall focus on the quadratic B-spline; employing the Poisson’s sum (1), and \( T = 2 \), we have:

\[
\sum_{k \in \mathbb{Z}} \beta_2(2k) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \text{sinc}^3(\pi n)
\]

\[
= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin^3 \left( n \frac{\pi}{2} \right)}{(n \frac{\pi}{2})^3}
\]

\[
= \frac{1}{2} + \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin^3 \left( n \frac{\pi}{2} \right)}{n^3}
\]

even terms are zero.

The last series in the above derivation is more difficult to tackle since \( \sin^3 \left( n \frac{\pi}{2} \right) \) is alternating its sign for the non-zero terms.

\[
\sum_{n=1}^{\infty} \frac{\sin^3 \left( n \frac{\pi}{2} \right)}{n^3} = \sum_{n=0}^{\infty} \frac{\sin^3 \left( (2n+1) \frac{\pi}{2} \right)}{(2n+1)^3}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}
\]

(6)

\[
= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3}
\]
On the other hand:

\[
\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \\
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{(2n)^3} \\
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n^3} \\
= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \frac{1}{8} \zeta(3).
\]

Hence, we have:

\[
\frac{7}{8} \zeta(3) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \\
= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} + \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3} \\
\tag{7}
\]

(7) and (6) are different and hence we can not resolve \( \zeta(3) \) using this approach.

4 S.O.S

The question is: is there a more appropriate choice than \( T = 2 \)? By choosing different values like \( T = 3/2 \), we get different sub-series of \( \zeta(3) \). Can we build \( \zeta(3) \), perhaps, from multiple choices for \( T \)?

References
