Optimal 3D Lattices in Scientific Visualization and Computer Graphics

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Outline

1. Visualization and Graphics
2. Interpolation/Approximation
3. Further Research Orientation
Visualization - Rendering

Data Sources

- Numerical Simulations
- CT, PET Scanner
- MRI machine
- Material Science, Microstructures
- Geological or Seismic

Cartesian

⇒

+ Interpolation

By Stefan Bruckner
Visualization - Rendering

Data Sources
Numerical Simulations
CT, PET Scanner
MRI machine
Material Science, Microstructures
Geological or Seismic

⇒

Optimal

⇒

Interpolation

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Optimal 3D Lattices in Visualization and Graphics

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Optimal Sampling Lattice
Sphere Packing - History
The Optimal Lattice

- 30% more information with the same number of samples.

Comparison:
- Cartesian Lattice: 763K data points, 326 seconds
- Optimal Lattice: 741K data points, 167 seconds

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The Optimal Lattice

- 30% more information with the **same** number of samples.

**Cartesian Lattice**
- 763K data points
- 326 seconds

**Optimal Lattice**
- 741K data points
- 167 seconds
The Uncertainty Principle

- The farther the samples, the closer the replicas
- The closer the samples, the farther replicas

Space Domain:

Fourier Domain:

- Coarsely Sampled
- Densely Sampled
- Optimally Sampled
Optimal 2D lattice

- Uniform resolution on *all* orientations: Isotropic spectrum
- Pack the replicas as tight as possible

- Cartesian Sampling
- Hexagonal Sampling

- Hex: Fewer Samples
- Hex: More Info

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How to pack 3D spheres as tight as possible?
Packing Cannon-balls

- Thomas Harriot (17th): How to pack cannon-balls on a ship as many as possible?

![Image of cannon-balls]

Courtesy of Mathew Brady/Library of Congress
Hexagonal Packing: Optimal packing in 2D
Hexagonal Packing: Optimal packing in 2D
3D - Kepler’s Conjecture

- Kepler suggested the grocer’s method is the best.
- Face Centered Cubic (FCC) Packing
Gauß proved it for a *regular* packing.
Abstracted in the Hilbert’s 18th problem.
Hales announced a computer aided proof in 1998.
The Face Centered Cubic packing:
Optimal 3D lattice

- Densest Sphere Packing in 3D: Face Centered Cubic (FCC)
- Dense packing in Fourier domain!
- Dual Lattice: Body Centered Cubic (BCC)
Interpolation or Reconstruction

- Given function values at lattice sites, interpolate at an arbitrary point.
- Tensor product? Not applicable to BCC.
- Radial basis functions? Too generic, not satisfying results
- Ad-hoc methods - splitting to Cartesian
Nearest Neighbor Interpolant

The Voronoi Cell: *Truncated Octahedron*
Linear Order Interpolant

The Nearest Neighbors Cell: *Rhombic Dodecahedron*
Linear Interpolator?

Linear drop off

Linear B-spline
Rhombic Dodecahedron

- *Projection* of four dimensional hypercube (tesseract) [Entezari, Dyer, Möller, IEEE VIS 2004]
- Maximum support projection along *antipodal* axis
Lower dimensional accidents

- Linear B-spline
- Linear hexagonal spline
Properties

- Projection Matrix:

\[ \Xi = [\xi_1 \xi_2 \xi_3 \xi_4] \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \]

- Linear box spline: \( C^0 \) interpolant

- Fourier Transform:

\[ \hat{M}_\Xi(\omega) = \prod_{k=1}^{4} \text{sinc} (\xi_k \cdot \omega) \]
Higher order splines

- Successive convolutions of linear box spline with itself
- $C^2$ box spline $= \text{Linear} \ast \text{Linear}$
- Similar to tri-cubic B-spline on the Cartesian lattice

![Linear B-spline](image1)

* ![Linear B-spline](image2)

= ![Cubic B-spline](image3)
Box spline shifts on the BCC lattice!

The approximate $\tilde{f}_h$ of $f$ by a BCC lattice scaled by $h$:
[Entezari, Dyer, Möller, IEEE VIS 2004]
- $C^0$ box spline: $\|f - \tilde{f}_h\| = O(h^2)$, hence 2\textsuperscript{nd} order
- $C^2$ box spline: $\|f - \tilde{f}_h\| = O(h^4)$, hence 4\textsuperscript{nd} order
Performance, key in applications

The commonly used tri-cubic B-spline: neighborhood of $4 \times 4 \times 4 = 64$ points

The $C^2$ box spline on BCC: 32 points

Twice faster!

BCC shifts of box splines: an efficient shift-invariant space for approximation
Computational Cost

Non-separable, **twice** more efficient!
Results - Carp Dataset

2,744K Cart samples 658 sec  2,735K BCC samples 335 sec
Results - Explicit Test Function

68K Cart samples 67 sec

65K BCC samples 35 sec
Results - Errors

68K Cart samples 67 sec
65K BCC samples 35 sec
Multiresolution

- Visualization and processing of *large* datasets
- Compression, Denoising, . . .
- Tensor-product solution: change of resolution by 8
Granular Multiresolution

- Subgroups of Cartesian: $\|\text{FCC}\| = \frac{1}{2}\|\mathbb{Z}^3\|$ and $\|\text{BCC}\| = \frac{1}{4}\|\mathbb{Z}^3\|$

Original Cart  
FCC:1/2  
BCC:1/4

[Entezari, Meng, Bergner, Möller, EuroVIS 2006]
Reconstruction on FCC

- Visualization of QuantumDot project at Purdue University [Qiao, Ebert, Entezari, Korkusinski, Klimeck, IEEE VIS 2005]
- FCC sampling optimally minimizes aliasing
- FCC sampling has been proposed for Video processing
Visualization of QuantumDot Project
Seven-directional box spline [Peters 1996].
Similar to tri-cubic B-spline, 20% faster! [Entezari, Möller, IEEE TVCG 2006]
Stair-casing in Reconstruction

Voxelized surface  tri-cubic B-spline  box spline
Cartesian Box Spline

tri-cubic B-spline

box spline
Summary

- The Optimal BCC Lattice
  - More accurate
  - More computationally efficient

- Accuracy: A key challenge in NIH/NSF 2006 report on Visualization
Reconstruction

- Accuracy is important!
- Medical acquisition devices, algorithms
- CT reconstruction algorithms, adopt to optimal lattices
- MRI efficient sampling
- Under the same exposure time, more quality

By NASA
Unlike Cartesian, hexagonal images have only one type of neighborhood: face-connected

Face + Vertex Connected

Face-Connected

3D: FCC has only one type of neighborhood: face-connected

Mathematical Morphology
Fluid, Fire and Smoke Visualization

- A demanding field in graphics and entertainment industry
- Develop Level-set/fast marching methods on BCC/FCC
- Iso-surface extraction for higher order interpolants on Cartesian, BCC and FCC

By Ron Fedkiw
Lattice-Boltzmann Model

- A popular model in computer graphics
- Relies on discretization of space and particle velocities
- FCC/BCC offers better discretization of space and more isotropic discretization of velocities

By Ron Fedkiw
Visualization of high-dimensional datasets
Typically around 100 variables: temperature, pressure, volume geometry, . . .
Sampling is very costly (around 1 min per sample)
Efficient high dimensional lattices, sphere packings
Numerical PDE: Collocation Method

- Spline space for approximation
- Banded matrices are desired
- Efficient BCC shift-invariant space $\Rightarrow$ smaller bandwidths for the same smoothness and approximation power

\[
\begin{bmatrix}
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\_ & \_ & \_ & \_ \\
\end{bmatrix}
\]
Efficient Box Spline Convolution

- Develop and generalize efficient piecewise polynomial evaluation
- Exploit the power of Graphics Processing Units (GPU) for interactive visualizations
Acknowledgments

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Thank You