Vector-Valued Image Regularization


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Motivation

- Vector data is a rich source of information in computer vision.

- Examples: color images (RGB), optical flow vectors, etc.

- Need for smoothing vector-valued images.
Overview

- Classification of diffusion schemes for SCALAR images.
  1. Energy Minimization
  2. PDEs expressed using diffusion tensor
  3. Use of oriented Laplacians
- Extension to Vector-Valued Images
- Proposed “unified” approach for regularization
- Applications
  1. Denoising
  2. Image Inpainting
  3. Flow-field visualization
  4. Image Magnification
Diffusion of Scalar Images: Energy Minimization

- Image smoothing as minimization of an energy functional as follows:

\[
\min_{I: \Omega \rightarrow \mathbb{R}} \int_{\Omega} \phi(\| \nabla I \|) \, d\Omega
\]

- Energy dependent upon global image properties (some function of the image gradients).
Area Minimization
Diffusion of Scalar Images: Diffusion Tensors

- Diffusion schemes are also represented by the following PDE:
  \[
  \frac{\partial I}{\partial t} = \text{div} \left( D \nabla I \right)
  \]

- \( D = \) diffusion tensor, designed using local image properties.
- \( D = d \times d \) matrix (\( d = \) dimensionality of image, 2 for images, 3 for volumes).
- Eigenvectors = direction of maximal and minimal change of gradient (locally).
- Eigenvalues = amount of change of gradient in the two directions.
A note about diffusion tensors

- The spectral properties of diffusion tensor $D$ do not always describe the anisotropicity.
- Consider the following two tensors

$$D_1 = \frac{\text{Id}}{\|\nabla I\|} \quad \text{and} \quad D_2 = \frac{1}{\|\nabla I\|^3} (\nabla I \nabla I^T)$$

- Observe that $D_1$ is isotropic, $D_2$ is anisotropic.
- But they lead to the same kind of diffusion, i.e.

$$\text{div} \left( D_1 \nabla I \right) = \text{div} \left( D_2 \nabla I \right) = \text{div} \left( \frac{\nabla I}{\|\nabla I\|} \right)$$
Diffusion: Oriented 1D Laplacians

- 2D Diffusion = Cascade of two 1D diffusions
- This means

\[
\frac{\partial I}{\partial t} = c_1 \frac{\partial^2 I}{\partial \xi^2} + c_2 \frac{\partial I}{\partial \eta^2}
\]

- \(\xi\) and \(\eta\) are orthogonal directions.
- Relative weighting of the coefficients controls the “anisotropicity” of the diffusion.
Fig. 2. Principle of regularization techniques based on oriented Laplacians: Two 1D smoothing are done along orthogonal axes $\xi$ and $\eta$ that are different for each image points.
For scalar images, the aforementioned three forms are known to be equivalent. This means that ENERGY MINIMIZATION = DIVERGENCE EXPRESSION = LAPLACIAN SMOOTHING.

\[
(A) \min_{I: \Omega \to \mathbb{R}} \int_{\Omega} \phi(\|\nabla I\|) \, d\Omega \\
\Rightarrow (B) \frac{\partial I}{\partial t} = \text{div} \left( \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|} \nabla I \right) \\
\Rightarrow (C') \frac{\partial I}{\partial t} = \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|} I_{\xi\xi} + \phi''(\|\nabla I\|) I_{\eta\eta}
\]
Extension to vector-valued images: Preliminaries

- Vector-valued image given as $\mathbf{I} : \Omega \rightarrow \mathbb{R}^n$
- The "$i$"th channel is denoted as $I_i$.
- Consider structure tensor $G$

$$G = \sum_{j=1}^{n} \nabla I_j \nabla I_j^T.$$ 

- $G$ can be written in terms of its eigenvectors and eigenvalues:

$$G = \lambda_+ \theta_+ \theta_+^T + \lambda_- \theta_- \theta_-^T$$
Energy Minimization

- The energy functional for vector-valued images:

\[
\min_{\mathbf{I}: \Omega \rightarrow \mathbb{R}^n} E(\mathbf{I}) = \int_{\Omega} \psi(\lambda_+, \lambda_-) \, d\Omega
\]

- The function \( \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) describes local image variations.
- Example (Blomgren and Chan, 1998), the function is given as:

\[
\sqrt{\lambda_+ - \lambda_-}
\]
Connections to the Diffusion Tensor

- Given earlier energy functional, the Euler-Lagrange equation is
  \[
  \frac{\partial I_i}{\partial t} = \text{div} (D \nabla I_i) \quad (i = 1..n)
  \]

- The diffusion tensor \( D \) can be proved to be as follows:
  \[
  D = \frac{\partial \psi}{\partial \lambda_+} \theta_+ \theta_+^T + \frac{\partial \psi}{\partial \lambda_-} \theta_- \theta_-^T
  \]

- \( D \) has the same eigenvectors as the structure tensor \( G \), but different eigenvalues, dictated by the function \( \psi \).
Oriented Laplacians

- The PDE can be written as:

\[ \frac{\partial I_i}{\partial t} = c_1 I_{i\xi\xi} + c_2 I_{i\eta\eta} = \text{trace} \left( T H_i \right) \quad (i = 1..n) \]

- Here the Hessian of the ‘i’ th image channel is \( H_i \).

- \( T = 2\times2 \) tensor defined as \( T = c_1 \xi\xi^T + c_2 \eta\eta^T \)

- It can be proved that the formal solution to the above PDE is equivalent to

\[ I_i(t) = I_{i(0)} \ast G^{(T,t)} \quad (i = 1..n) \]

where \( G^{(T,t)}(x) = \frac{1}{4\pi t} \exp \left( -\frac{x^T T^{-1} x}{4t} \right) \quad \text{with} \quad x = (x \ y)^T \)
Oriented Laplacians

- Koenderink had proposed that solution to the heat equation is equivalent to convolution of an image with a (circular) Gaussian.

- The previous equation is an extension to this idea.

- The case of simple isotropic diffusion is equivalent to having a tensor $T$ equal to the identity matrix.

- $T$ may not be constant over the entire image (non-linear PDE).
Examples of convolution with oriented Gaussians
Examples of convolution with oriented Gaussians
Relation between D and T

- The divergence equation can be expressed as:

\[
\text{div} (\mathbf{D} \nabla I_i) = \text{trace} (\mathbf{D} \mathbf{H}_i) + \nabla I_i^T \, \text{div} (\mathbf{D})
\]

- Here \( \text{div} () \) is a divergence operator defined as follows:

\[
\mathbf{D} = (d_{ij}), \text{div} (\mathbf{D}) = \begin{pmatrix}
\text{div}(d_{11}) & d_{12}^T \\
\text{div}(d_{21}) & d_{22}^T
\end{pmatrix}
\]

- The additional term \( \nabla I_i^T \, \text{div} (\mathbf{D}) \) contributes to the diffusion, and smoothes the image in directions different from the eigenvectors of \( \mathbf{D} \).

- This term represents a spatial variation of \( \mathbf{D} \).
Relation between $D$ and $T$.

- Reconsider

$$\text{div} \ (D \nabla I_i) = \text{trace} \ (DH_i) + \nabla I_i^T \cdot \text{div} \ (D)$$

- We can express it as follows:

$$\text{div} \ (D \nabla I_i) = \sum_{j=1}^{n} \text{trace} \ (\delta_{ij}D + Q^{ij}H_j)$$

- The additional diffusion tensors that appear ($Q^{ij}$) contribute to the diffusion.

- Instead of $D$, it is more accurate if we specify $T$, while designing a filter.
New Multi-valued Regularization PDE

- We wish to specify a $T$ such that:
  1. The diffusion depends on local properties of the image (which is given by the structure tensor)
    \[ G = \sum_{j=1}^{n} \nabla I_j \nabla I_j^T \]
  2. Smooth isotropically in homogenous regions, that is no preferred direction,
    \[ \frac{\partial I_i}{\partial t} \approx \Delta I_i = \text{trace}(\mathbf{H}_i) \]
  3. Smooth image along vector edges, given by
    \[ \frac{\partial I_i}{\partial t} = \text{trace} (\theta \theta^T \mathbf{H}_i) \]
New Multi-valued Regularization PDE

- Such as $T$ is given as follows:

$$T = f_-(\sqrt{\lambda_+^* + \lambda_-^*}) \theta_-^* \theta_-^T + f_+(\sqrt{\lambda_+^* + \lambda_-^*}) \theta_+^* \theta_+^T$$

$$f_+(s) = \frac{1}{1 + s^2} \quad \text{and} \quad f_-(s) = \frac{1}{\sqrt{1 + s^2}}$$

$$\frac{\partial I_i}{\partial t} = \text{trace} \left( \left[ f_+ \left( \sqrt{\lambda_+^* + \lambda_-^*} \right) \theta_-^* \theta_-^T + f_- \left( \sqrt{\lambda_+^* + \lambda_-^*} \right) \theta_+^* \theta_+^T \right] H_i \right)$$

- Observe: $f_+(s)$ and $f_-(s)$ tend to 1, as $s$ tends to 0 (isotropic diffusion in homogenous regions).
- The ratio of $f_+(s)$ to $f_-(s)$ tends to 0, as $s$ tends to infinity (diffuse along salient edges and not across them).
Multi-valued Regularization

PDE: Choice of Norm

- The norm $\mathcal{N} = \sqrt{\lambda_+}$ has high value at edges and corners.

- The norm $\mathcal{N}_- = \sqrt{\lambda_+ - \lambda_-}$ has high value at edges, but is zero-valued at corners.

- The norm $\mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-}$ has high value at edges, and even higher value at corners (also does not require explicit eigendecomposition of $G$).
Choice of Norm

(a) Color checkerboard (real size 40 × 40)
(b) $\mathcal{N} = \sqrt{\lambda_+}$
(c) $\mathcal{N}_- = \sqrt{\lambda_+ - \lambda_-}$
(d) $\mathcal{N}_+ = \sqrt{\lambda_+ + \lambda_-}$

Figure 2.11: A list of possible vector variation norms.
Results for image denoising
Results for image denoising
Numerical Implementation

- The PDE is given as:
  \[
  \frac{\partial I_i}{\partial t} = \text{trace} \ (TH_i) \quad (i = 1..n)
  \]

- Conventional Implementation: Use a discretization approach using central differences.

- Alternative: Compute (3 by 3) oriented Gaussian \(G(T,t)\) kernel for each pixel and convolve it with a (3 by 3) neighborhood around the pixel.

- Advantage: avoids computation of second derivatives!
Numerical Implementation

\[
[\text{Trace}(TH)](x,y) = \begin{cases} 
  i(x-1,y-1) & i(x,y) & i(x+1,y) \\
  i(x-1,y) & i(x,y) & i(x+1,y) \\
  i(x-1,y+1) & i(x,y+1) & i(x+1,y+1) 
\end{cases} \ast \begin{cases} 
  G(-1,-1) & G(0,-1) & G(1,-1) \\
  G(-1,0) & G(0,0) & G(1,0) \\
  G(-1,1) & G(0,1) & G(1,1) 
\end{cases}
\]
Numerical Implementation

(a) Noisy color image  (b) Restored with the Hessian spatial discretization scheme  (c) Restored with the local filtering scheme, with $3 \times 3$ masks.

Figure 3.7: Comparison of the two proposed numerical schemes.
Application to Inpainting

- Inpainting = filling holes in an image by interpolating neighborhood data.

- Apply proposed PDE only in the holes (specified manually).

- Disallow isotropic diffusion.

- Boundary pixels will be diffused until they fill up the hole.
Inpainting Results

This horrible green text hides the beautiful market place of Bar sur Loup, a typical village in the Alpes-Maritimes. Would Inpainting algorithms succeed to remove this text?

(a) 

(b)
Inpainting Results
Inpainting Results
Fig. 12. Using vector-valued regularization PDE’ for image magnification (×4). (a) Original color image (64 × 64). (b) Bloc interpolation. (c) Linear interpolation. (d) Interpolation with PDEs.
Flow-Field Visualization

- Consider a 2D flow field on an image, given by
  \[ \mathcal{F} : \Omega \rightarrow \mathbb{R}^2 \]

- Visualization using vector graphics:

- Demerit: cannot be used to visualize dense flow-fields
Flow-Field Visualization

- Solution: Take any colored image $I$.
- Diffuse it using the proposed PDE.
- But to compute $T$, do not use image geometry, but the directions of the flow field $F$.
- Modified PDE is (unidirectional):

$$\frac{\partial I_i}{\partial t} = \text{trace} \left( \left[ \frac{1}{\|F\|} F F^T \right] H_i \right) \quad (i = 1..n)$$

$$\frac{\partial I}{\partial t} = \|F\| \frac{F}{\|F\|} \frac{F}{\|F\|} = \|F\| \frac{\partial^2 I}{\partial (\frac{F}{\|F\|})^2}$$
Fig. 13. Using our vector-valued regularization PDE (15), for flow visualization (1). (a) Flow visualization with arrows. (b) Flow visualization with diffusion PDEs (5 iter.). (c) Flow visualization with diffusion PDEs (15 iter.).
Flow-Field Visualization
Results

Fig. 14. Using our vector-valued regularization PDE (15), for flow visualization (2).
Proof of Relation between $D$ and $\psi(\lambda_+, \lambda_-)$

- Consider energy functional

\[
\min_{I: \Omega \to \mathbb{R}^n} E(I) = \int_{\Omega} \psi(\lambda_+, \lambda_-) \, d\Omega
\]

- Consider corresponding PDE

\[
\frac{\partial I_i}{\partial t} = \text{div} \left( D \nabla I_i \right) \quad (i = 1..n)
\]

- To prove that

\[
D = \frac{\partial \psi}{\partial \lambda_+}(\lambda_+, \lambda_-) \theta_+ \theta^T_+ + \frac{\partial \psi}{\partial \lambda_-}(\lambda_+, \lambda_-) \theta_- \theta^T_-
\]
Proof of Relation between $D$ and $\psi(\lambda_+, \lambda_-)$

- Euler Lagrange of the energy functional

\[
\frac{\partial I_i}{\partial t} = \text{div} \left( \begin{pmatrix} \frac{\partial \psi}{\partial I_{ix}} \\ \frac{\partial \psi}{\partial I_{iy}} \end{pmatrix} \right) \quad (i = 1..n)
\]

- By chain rule

\[
\begin{pmatrix} \frac{\partial \psi}{\partial I_{ix}} \\ \frac{\partial \psi}{\partial I_{iy}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \lambda_+}{\partial I_{ix}} & \frac{\partial \lambda_-}{\partial I_{ix}} \\ \frac{\partial \lambda_+}{\partial I_{iy}} & \frac{\partial \lambda_-}{\partial I_{iy}} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi}{\partial \lambda_+} \\ \frac{\partial \psi}{\partial \lambda_-} \end{pmatrix}
\]
Proof of Relation between $D$ and $\psi(\lambda_+, \lambda_-)$

Knowing that $\lambda_{\pm}$ are eigenvalues of $G$, and applying chain rule, we get

$$\frac{\partial \lambda_{\pm}}{\partial I_{ix}} = \sum_{k,l} \frac{\partial \lambda_{\pm}}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial I_{ix}}$$

and

$$\frac{\partial \lambda_{\pm}}{\partial I_{iy}} = \sum_{k,l} \frac{\partial \lambda_{\pm}}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial I_{iy}}$$

Further, we can show that:

$$\begin{pmatrix}
\frac{\partial \lambda_{\pm}}{\partial I_{ix}} \\
\frac{\partial \lambda_{\pm}}{\partial I_{iy}}
\end{pmatrix}
= \begin{pmatrix}
2 \frac{\partial \lambda_{\pm}}{\partial g_{11}} & \frac{\partial \lambda_{\pm}}{\partial g_{12}} \\
\frac{\partial \lambda_{\pm}}{\partial g_{12}} & 2 \frac{\partial \lambda_{\pm}}{\partial g_{22}}
\end{pmatrix} \nabla I_i$$
Proof of Relation between $D$ and $\psi(\lambda_+, \lambda_-)$

- Using $\mathbf{G} = \lambda_+ \theta_+\theta_+^T + \lambda_- \theta_-\theta_-^T$ and the fact that the eigenvectors are orthonormal, we get:

$$\frac{\partial \lambda_+}{\partial g_{kl}} = \theta_+^T \frac{\partial \mathbf{G}}{\partial g_{kl}} \theta_+ \quad \text{and} \quad \frac{\partial \lambda_-}{\partial g_{kl}} = \theta_-^T \frac{\partial \mathbf{G}}{\partial g_{kl}} \theta_-$$

- This can further be simplified to the following:

$$\left( \begin{array}{c} \frac{\partial \lambda_+}{\partial I_{ix}} \\ \frac{\partial \lambda_+}{\partial I_{iy}} \end{array} \right) = 2 \theta_+ \theta_+^T \nabla I_i \quad \text{and} \quad \left( \begin{array}{c} \frac{\partial \lambda_-}{\partial I_{ix}} \\ \frac{\partial \lambda_-}{\partial I_{iy}} \end{array} \right) = 2 \theta_- \theta_-^T \nabla I_i$$
Proof of Relation between $D$ and $\psi(\lambda_+, \lambda_-)$

Hence, we have

\[ \frac{\partial I_i}{\partial t} = \text{div} \left( D \nabla I_i \right) \quad \text{where} \quad D = 2 \frac{\partial \psi}{\partial \lambda_+} \theta_+ \theta_+^T + 2 \frac{\partial \psi}{\partial \lambda_-} \theta_- \theta_-^T \]
The structure tensor $G$ was given by:

$$ G = \sum_{i=1}^{n} \nabla I_i \nabla I_i^T $$

The following formula presents a justification:

$$ G = \begin{pmatrix} R_x^2 + G_x^2 + B_x^2 & R_x R_y + G_x G_y + B_x B_y \\ R_x R_y + G_x G_y + B_x B_y & R_y^2 + G_y^2 + B_y^2 \end{pmatrix} $$
To prove \[ \text{div} (\mathbf{D} \nabla I_i) = \text{trace} (\mathbf{D} \mathbf{H}_i) + \nabla I_i^T \mathbf{d} \text{iv} (\mathbf{D}) \]

- Let the diffusion tensor be

\[ \mathbf{D} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]

- Then we have

\[
\begin{align*}
\text{div} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix}
\text{div} \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) \\
\text{div} \left( \begin{pmatrix} c \\ d \end{pmatrix} \right)
\end{pmatrix} \\
&= \begin{pmatrix}
a_x + b_y \\
c_x + d_y
\end{pmatrix}
\end{align*}
\]
To prove \( \nabla \cdot (\mathbf{D} \nabla I_i) = \text{trace} (\mathbf{D} \mathbf{H}_i) + \nabla I_i^T \cdot \mathbf{D} \)
Proof that \( \frac{\partial I_i}{\partial t} = \text{trace} \left( \mathbf{TH}_i \right) \iff I_{i(t)} = I_{i(t=0)} \ast G^{(T,t)} \)

\[ G^{(T,t)}(\mathbf{x}) = \frac{1}{4\pi t} \exp \left( -\frac{\mathbf{x}^T \mathbf{T}^{-1} \mathbf{x}}{4t} \right) \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \]

- We have:

\[ \frac{\partial G^{(T,t)}}{\partial t} = -\frac{1}{4\pi t^2} \exp \left( -\frac{\mathbf{x}^T \mathbf{T}^{-1} \mathbf{x}}{4t} \right) \left( 1 - \frac{\mathbf{x}^T \mathbf{T}^{-1} \mathbf{x}}{4t} \right) \]

- The spatial and temporal derivatives of G are:

\[ \nabla G^{(T,t)} = -\frac{1}{8\pi t^2} \exp \left( -\frac{\mathbf{x}^T \mathbf{T}^{-1} \mathbf{x}}{4t} \right) \mathbf{T}^{-1} \mathbf{x} \]

\[ \mathbf{H}_{G^{(T,t)}} = -\frac{1}{8\pi t^2} \exp \left( -\frac{\mathbf{x}^T \mathbf{T}^{-1} \mathbf{x}}{4t} \right) \mathbf{T}^{-1} \left( \mathbf{Id} - \frac{\mathbf{x} \mathbf{x}^T \mathbf{T}^{-1}}{2t} \right) \]
Proof that \[ \frac{\partial I_i}{\partial t} = \text{trace} \left( \mathbf{T} \mathbf{H}_i \right) \quad \iff \quad I_{i(t)} = I_{i(t=0)} * G^{(\mathbf{T},t)} \]

- It follows that:
  \[
  \text{trace} \left( \mathbf{T} \mathbf{H}_{G^{(\mathbf{T},t)}} \right) = - \frac{1}{8\pi t^2} \exp \left( -\frac{x^T \mathbf{T}^{-1} x}{4t} \right) \text{trace} \left( \mathbf{I}d - \frac{xx^T \mathbf{T}^{-1}}{2t} \right) \\
  = - \frac{1}{8\pi t^2} \exp \left( -\frac{x^T \mathbf{T}^{-1} x}{4t} \right) \left( 2 - \frac{x^T \mathbf{T}^{-1} x}{2t} \right) \\
  = \frac{\partial G^{(\mathbf{T},t)}}{\partial t}
  \]

- By linearity of convolution, we have:
  \[
  \frac{\partial (I_{i_0} * G^{(\mathbf{T},t)})}{\partial t} = I_{i_0} * \frac{\partial G^{(\mathbf{T},t)}}{\partial t} \\
  = I_{i_0} * \text{trace} \left( \mathbf{T} \mathbf{H}_{G^{(\mathbf{T},t)}} \right) \\
  = \text{trace} \left( \mathbf{T} \mathbf{H}_{I_{i_0} * G^{(\mathbf{T},t)}} \right)
  \]
To prove \[ \text{div} \left( \mathbf{D} \nabla I_i \right) = \sum_{j=1}^{n} \text{trace} \left( (\delta_{ij} \mathbf{D} + Q^{ij}) \mathbf{H}_j \right) \]

Let us decompose \( \mathbf{D} \) into anisotropic and isotropic parts:

\[
\mathbf{D} = \alpha(\lambda_+, \lambda_-) \mathbf{G} + \beta(\lambda_+, \lambda_-) \mathbf{I}d
\]

\[
\alpha = \frac{f_1(\lambda_+, \lambda_-) - f_2(\lambda_+, \lambda_-)}{\lambda_+ - \lambda_-}
\]

\[
\beta = \frac{\lambda_+ f_2(\lambda_+, \lambda_-) - \lambda_- f_1(\lambda_+, \lambda_-)}{\lambda_+ - \lambda_-}
\]

Using \( \mathbf{D} \) in above form, we have:

\[
\text{div}(\mathbf{D}) = \alpha \text{div}(\mathbf{G}) + \mathbf{G} \nabla \alpha + \nabla \beta
\]
To prove \( \text{div} (D \nabla I_i) = \sum_{j=1}^{n} \text{trace} \left( (\delta_{ij}D + Q^{ij})H_j \right) \)

Expanding the three terms on the RHS of the previous equation gives:

\[
\nabla I_i^T \nabla (D) = \sum_{j=1}^{n} \text{trace} \left( H_j P^{ij} \right)
\]

\[
P^{ij} = \alpha \nabla I_i^T \nabla I_j \text{Id}
\]

\[
+ 2 \left( \frac{\partial \alpha}{\partial \lambda_+} \theta_+ \theta_+^T + \frac{\partial \alpha}{\partial \lambda_-} \theta_- \theta_-^T \right) \nabla I_j \nabla I_i^T \text{G}
\]

\[
+ 2 \left( (\alpha + \frac{\partial \beta}{\partial \lambda_+}) \theta_+ \theta_+^T + (\alpha + \frac{\partial \beta}{\partial \lambda_-}) \theta_- \theta_-^T \right) \nabla I_j \nabla I_i^T
\]
Example of difference between D and T

Consider:
\[
\frac{\partial I}{\partial t} = \text{div} (D \nabla I) \quad \text{and} \quad \frac{\partial I}{\partial t} = \text{trace} (TH)
\]

And see the following:

\[
\frac{\partial I}{\partial t} = \text{div} \left( \frac{\phi' (\| \nabla I \|)}{\| \nabla I \|} \nabla I \right) \\
= \frac{\phi'' (\| \nabla I \|)}{\| \nabla I \|} I_{\xi \xi} + \phi'' (\| \nabla I \|) I_{\eta \eta} \\
= \text{trace} \left( \left[ \frac{\phi' (\| \nabla I \|)}{\| \nabla I \|} \xi \xi^T + \phi'' (\| \nabla I \|) \eta \eta^T \right] H \right)
\]
Example of difference between D and T

- We have \( \text{div} \left( \mathbf{D} \nabla I \right) = \text{trace} \left( \mathbf{T} \mathbf{H} \right) \)

- Where

\[
\mathbf{D} = \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|} \mathbf{I} \]

\[
\mathbf{T} = \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|} \xi \xi^T + \phi''(\|\nabla I\|) \eta \eta^T
\]

- Consider \( \frac{\partial I_i}{\partial t} = \text{trace} \left( \mathbf{D} \mathbf{H}_i \right) \) and \( \frac{\partial I_i}{\partial t} = \text{div} \left( \mathbf{D} \nabla I_i \right) \)

where \( \mathbf{D} = \frac{1}{\|\nabla I\|} \mathbf{I} \)
Example of difference between D and T

(a) Noisy scalar image

(b) $\frac{\partial I}{\partial t} = \text{trace} \left( \frac{1}{\|
abla I\|} H \right)$

(c) $\frac{\partial I}{\partial t} = \text{div} \left( \frac{1}{\|
abla I\|} \nabla I \right)$
Summary

- PDE-based formalisms from the literature on scalar-valued diffusion.
- Extensions of the above formalisms to vector-valued images.
- Proposed framework unifies several previously existing methods of diffusion.
- Accurate design of PDE for any type of diffusion by specifying tensor $T$.
- Applications in image denoising, inpainting, magnification and flow field visualization.
- Possible extensions: constrained vector-valued diffusion (e.g. direction diffusions where vectors must have unit norm).
References


- D. Tschumperle, “PDEs Based Regularization of Multi-valued Images”, *PhD Thesis*, INRIA.