1. Introduction

The standard model of the bucket brigade assembly line [1] for the n workers m stations case (n ≪ m) considers an assembly line that is partitioned into m stations, each station corresponding to a subtask of the total work content. A job has to be processed at all m stations, in sequence, to be completed. The workers are ordered 1 to n, upstream to downstream, and this order is maintained across stations at all times. Each worker picks a job and processes it on a station with a velocity commensurate with his skill at that station. The worker then takes the job to the next station to continue processing it. In the blocking model, two workers are not allowed to occupy the same station simultaneously. The downstream worker has precedence over the upstream worker in the sense that the upstream worker has to wait until the downstream worker has released the station. When a worker arrives at a station that is busy, he is considered blocked on that station and he does not seek any work until his successor leaves that station. When the last worker completes processing his job, all workers simultaneously hand off their jobs in their current states to their respective successors, picking up the job of their respective predecessors; the first worker starts processing a new job. In the instantaneous walk-back model, the entire set of hand-offs happens instantaneously.

In this note we consider the 2-worker m-stations bucket brigade assembly line with blocking and instantaneous walk back. We provide a complete characterization of the dynamics under the model where the workers can have different velocities on different stations, albeit constant velocities over each station. The assembly line is represented by the interval I = [0, 1]. The processing of a job begins at 0 and ends at 1. The station S_i is represented by the interval [P(i), P(i+1)), with P(1) = 0, P(m+1) = 1. The upstream worker is denoted by W_1 and the downstream worker by W_2.

2. Prior work and contributions made

Bartholdi et al. [1] analyzed the n workers m stations case with blocking and instantaneous walk-back, for the special case where the workers are sequenced slowest to fastest. They showed that if the workers can be sequenced such that each is faster than their predecessor at all stations, then there is a unique stable fixed point to which the system converges independent of the starting positions of the workers. They then studied the 2 and 3 workers case [2] under the assumption that each worker has a constant velocity over the entire assembly line with the work content distributed uniformly over the entire assembly line (hence, the concept of stations does not exist). In this framework, if workers can be sequenced from slowest to fastest, they can never be blocked. Furthermore, the production rate under such conditions is the sum of the velocities of the workers and is the maximum achievable across all sequencing of the workers [1].

Armbruster et al. [3] studied the dynamics of the 2 workers case where W_1 is faster than W_2 in the interval [0, X) and slower in the interval [X, 1]. They considered both the cases where W_2 is allowed to pass over W_1 and where W_1 can be blocked by W_2. Although not as restrictive as the assumption that one worker’s speed dominates the other uniformly, this framework cannot model the general case
where \( W_1 \) is faster/slower than \( W_2 \) on an arbitrary, not necessarily contiguous set of stations.

In this note we generalize the results reported in [3]. We fully characterize \( f \) that specifies the successive hand-off locations between \( W_1 \) and \( W_2 \) in Section 3. After demonstrating that \( f \) has a unique fixed point and can have no periodic cycles of period >2, we show how to algorithmically compute the fixed point and the critical point (defined in Section 3), in Section 4. In Section 5, we determine the necessary and sufficient conditions for the global stability of the fixed point, and show how to algorithmically ascertain this in Section 6. In Section 7 we analyze throughput, and in Section 8, we present concluding remarks.

3. Characterization of the mapping function

We begin by characterizing the mapping \( f \) that specifies the successive hand-off locations between \( W_1 \) and \( W_2 \). Specifically, if \( W_i \) begins at the start of the assembly line (i.e., at \( 0 \)) and \( W_j \) begins at \( x \in [0, 1] \), then \( f(x) \) denotes the position of \( W_i \) at the time when \( W_j \) reaches the end of the assembly line (i.e., 1). Naturally, after hand-off, \( W_i \) begins at 0 and \( W_j \) at \( f(x) \). We characterize \( f \) through the following set of theorems.

**Theorem 3.1.** \( f \) is continuous and piece-wise linear.

**Proof.** Let \( V_{\text{max}} \) and \( V_{\text{min}} \) be, respectively, the maximum and minimum velocities of \( W_1 \) and \( W_2 \) over the entire assembly line. Let \( x_0 \in (0, 1) \) be given. Consider \( x \in [0, 1) \) such that \( |x_0 - x| < \delta \) for small \( \delta \). When \( W_2 \) starts at \( x \) instead of \( x_0 \), the amount of time gained or lost by \( W_1 \) when \( W_2 \) reaches 1 is \( \Delta t < \frac{\delta}{V_{\text{min}}} \). Therefore \( |f(x_0) - f(x)| \leq \Delta t \). For a given \( \epsilon > 0 \) one can choose a \( \delta \) such that \( \Delta t < \frac{\delta}{V_{\text{min}}} < \epsilon \). Hence if \( |x_0 - x| < \delta \) then \( |f(x_0) - f(x)| < \epsilon \) which proves the continuity of \( f \) at \( x_0 \). To prove that \( f \) is piece-wise linear, let \( x_0 \) and \( f(x_0) \) lie in the interior of their respective stations, i.e., not at their station’s boundaries. Let \( V_1 \) and \( V_2 \) be, respectively, the velocities of \( W_1 \) at \( x_0 \) and \( W_2 \) at \( x_0 \). Let \( x = x_0 \pm \Delta x \). Then \( f(x) = (f(x_0) + \frac{\Delta x V_1}{V_2}) \), i.e., \( f \) is linear in the neighborhood of \( x_0 \). \( \square \)

**Definition.** \( W_1 \) is said to be blocked on \( x \) if, for \( W_2 \) beginning at \( x \) and \( W_1 \) at \( 0 \) there exists a station \( S_j \) such that \( W_1 \) is blocked at \( S_j \), i.e., \( W_1 \) reaches \( S_j \) before \( W_2 \) leaves \( S_j \).

**Theorem 3.2.** \( f \) is non-increasing. It is constant up to a point \( \tilde{x} \) beyond which it is strictly decreasing.

**Proof.** If \( W_1 \) is not blocked on \( x \), then \( \forall y > x, W_1 \) is not blocked on \( y \). If \( W_1 \) is blocked on \( x \) at a station \( S_i \), then \( \forall y < x, W_1 \) will be blocked at \( S_i \) and hence \( f(y) = f(x) \) in this range. Therefore if \( W_1 \) is blocked \( \forall y < \tilde{x} \), then \( \forall y < \tilde{x}, f(x) = f(\tilde{x}) \). Finally, let \( t_0 \) be the total time for which \( W_1 \) processes a job when \( W_2 \) begins at \( x \). Then for \( \tilde{x} < x < y, t_0 > t_i \), and therefore, \( f(\tilde{x}) > f(x) > f(y) \). In other words, \( f \) is strictly decreasing after \( \tilde{x} \). \( \square \)

We label \( \tilde{x} \) the critical point. Fig. 1 presents an example of \( f \).

**Theorem 3.3.** \( f \) has a unique fixed point and has no periodic cycles of period >2.

**Proof.** It follows from the Brouwer’s fixed point theorem that \( f \) has a fixed point. Since \( f \) is non-increasing this fixed point is unique. Moreover, it is well known from dynamical systems theory that a monotonically non-increasing function (in our case, the mapping \( f \)) cannot have periodic cycles of period >2. We provide a proof here for completeness.

Let \( x_1, x_2, \ldots, x_n \) form a cycle of period \( n > 2 \) with \( f(x_j) = x_{j+1} \). Without loss of generality let \( x_1 < x_i, \forall i \neq 1 \). Since \( x_2 > x_1 \), it follows from **Theorem 3.2** that \( f(x_2) = x_3 < f(x_1) = x_2 \), i.e., \( x_1 < x_3 < x_2 \), and hence \( x_1 < x_3 < x_4 < x_2 \), and hence \( x_1 < x_3 < x_4 < x_3 < x_2 \). It follows that the \( x_i \)’s are ordered as \( x_1 < x_3 < x_5 < \cdots < x_6 < x_4 < x_2 \). Importantly \( x_1 < x_3 < x_2 \). Therefore \( f(x_3) = x_1 > f(x_2) = x_3 \), leading to a contradiction. \( \square \)

4. Algorithmic computation of the critical point and the fixed point

4.1. Critical point \( (\tilde{x}) \)

Let \( x = \tilde{x} - \Delta x \), where \( \Delta x \) is an arbitrarily small positive number. Let \( S_p \) be the last station at which \( W_1 \) is blocked when \( W_2 \) begins at \( x \). Hence when \( W_2 \) begins at \( \tilde{x}, W_1 \) will enter \( S_p \) and \( W_2 \) will leave \( S_p \) simultaneously. This property can be used to compute the value of \( \tilde{x} \).

Let the final station be \( S_l = \{ P(m), P(m+1) = 1 \} \). Allow \( W_2 \) to begin at positions \( P(m), P(m+1), \ldots \), till the first position \( P(k) \) is found on which \( W_1 \) is blocked (i.e., \( W_1 \) beginning at \( P(1) = 0 \) is blocked when \( W_2 \) begins at \( P(k) \)). Hence \( P(k) < \tilde{x} < P(k+1) \). Let \( S_p \) be the corresponding last station at which \( W_1 \) is blocked. Then \( \tilde{x} \) is the position for which the time taken by \( W_2 \) to reach \( P(p(m+1) = 1 \) beginning at \( x \) equals the total time taken by \( W_1 \) to reach \( P(p(m+1) = 1 \) at 0. Let \( V_i(j) \) denote the velocity of worker \( i \) at station \( j \). Then,

\[
\tilde{x} = (P(k+1) - V_2(k) \sum_{j=1}^{p-1} \frac{P(j+1) - P(j)}{V_1(j)}) - \sum_{j=k+1}^{m} \frac{P(j+1) - P(j)}{V_2(j)}
\]

4.2. Fixed point \( (x_0) \)

Since \( x_0 \) is a fixed point, \( f(x_0) = x_0 \). We consider all three possibilities: \( x_0 \leq x_0 \), and \( x_0 > x_0 \), and show how \( x_0 \) can be computed in each case. Let \( S_p \) denote the station in which \( \tilde{x} \) occurs, i.e., \( P(k) \leq \tilde{x} < P(k+1) \).

4.2.1. Case 1: \( x_0 = \tilde{x} \)

As defined above let \( V_1(j) \) denote the velocity of worker \( i \) at station \( j \). Then if \( \sum_{j=1}^{k} \frac{P(j+1) - P(j)}{V_1(j)} + \frac{x - P(k)}{V_1(k)} = \frac{P(k+1) - x}{V_2(k)} + \sum_{j=k+1}^{m} \frac{P(j+1) - P(j)}{V_2(j)} \) then \( x_0 = \tilde{x} \).

4.2.2. Case 2: \( x_0 < \tilde{x} \)

**Lemma 4.1.** If \( x_0 < \tilde{x} \) and \( x_0 \) occurs in station \( S_p \), then \( S_p \) is the only station at which \( W_1 \) is blocked when \( W_2 \) begins at \( x_0 \).
Since $x_0 < \tilde{x}$, $W_1$ is blocked on $x_0$. Furthermore, since $W_2$ begins at $S_0$, $S_1$ is the first station at which $W_1$ can be blocked. Since $x_0$ is a fixed point and lies in $S_1$, $W_1$ does not reach $S_{L+1}$ when $W_2$ begins at $x_0$. The claim then follows. □

Based on the definition of $\tilde{x}$, we see that $S_1 = S_k$ and $P(k) \leq x_0 < \tilde{x} < P(k+1)$, i.e., $x_0$ and $\tilde{x}$ occur in the same station and $x_0 = P(k)$ if and only if $k = m$. If $k \neq m$, it follows from Lemma 4.1 that the time taken by $W_1$ to reach $x_0$ beginning at $P(k)$ equals the time taken by $W_2$ to reach the end of the assembly line beginning at $P(k+1)$. Therefore, $x_0 = P(k) + V_1(k) = \sum_{j=k+1}^m \frac{P(j)-P(0)}{V_2(j)}$.

4.2.3. Case 3: $x_0 > \tilde{x}$

Let $x_0$ occur in station $S_i$. Since $x_0 > \tilde{x}$, $W_1$ is not blocked. Therefore, the time taken by $W_1$ to reach $x_0$ beginning at $P(1) = 0$ equals the time taken by $W_2$ to reach $P(m+1) = 1$ beginning at $x_0$, which can be computed as,

$$x_0 = \frac{V_1(0) + V_2(0)}{V_1(0) + V_2(0)} * \sum_{j=1}^{m} \frac{P(j+1) - P(j)}{V_2(j)} - \sum_{j=1}^{m-1} \frac{P(j+1) - P(j)}{V_1(j)} + \frac{P(i+1)}{V_1(i)} + \frac{P(i)}{V_2(i)}.$$ 

5. Necessary and sufficient conditions for a globally stable fixed point

**Theorem 5.1.** If $x_0 \leq \tilde{x}$ then $x_0$ is globally stable.

**Proof.** From Theorem 3.2, $\forall x \leq \tilde{x}$, $f(x) = f(\tilde{x}) = f(x_0) = x_0$. Moreover, $\forall x > \tilde{x}$, $f(x) < f(\tilde{x}) = x_0$ and hence $f^2(x) = x_0$. Thus $x_0$ is a globally stable fixed point. □

Hereafter we consider the more interesting case: $x_0 > \tilde{x}$. We first provide the necessary conditions for the stability of the fixed point and the necessary conditions to avoid stable period 2 cycles. From these we derive the necessary and sufficient conditions for the global stability of the fixed point.

**Theorem 5.2.** $x_0$ is a stable fixed point if and only if

1. If $f$ is differentiable at $x_0$ then $f'(x_0) < 1$.
2. If $f$ is not differentiable at $x_0$ then for $\Delta x \to 0^+$, $x_1 = x_0 + \Delta x, x_2 = x_0 - \Delta x, f'(x_1) * f'(x_2) < 1$.

**Proof.** Case (i) is a well known condition from dynamical systems theory. For case (ii) let $x_1 = x_0 + \Delta x, x_2 = x_0 - \Delta x$. It follows from Theorem 3.2 that $f^2(x_1) > x_0$. Let $\Delta x = f^2(x_0) - x_0$ and let $t = \frac{\Delta x}{x_1 - x_0} = \frac{\Delta x}{x_2 - x_0}$, where $\Delta y = f_1(x) - x_0$. As $\Delta x \to 0^+$, $t = f'(x_1) * f'(x_2)$. Since $f$ is piece-wise linear, $f'(x_1) * f'(x_2) < 1$. Therefore $f'(x_1) * f'(x_2) < 1$ then $\Delta x < 0$ and hence $t = f'(x_1) * f'(x_2) < 1$. □

**Theorem 5.3.** If $f^2(\tilde{x}) < \tilde{x}$ then $f$ has a stable period 2 cycle.

**Proof.** Let $f(\tilde{x}) = \tilde{y}$, $f(\tilde{y}) = \tilde{w}$, and $\tilde{w} < \tilde{x}$. From Theorem 3.2 $f(\tilde{w}) = \tilde{y}$, and therefore $f$ has a period 2 cycle: $(\tilde{w}, \tilde{y}, \tilde{w})$. Also, the intervals $[0, \tilde{x}] \ni \tilde{w}$ and $f^{-1}(\tilde{x}), 1 \ni \tilde{y}$ converge to the cycle in a single period. Hence, the cycle is stable. □

We also notice that if $f^2(\tilde{x}) = \tilde{x}$ then $\forall x \leq \tilde{x}, f^2(x) = \tilde{x}$, and therefore the interval $[0, \tilde{x}]$ converges to the period 2 cycle.

Moreover, in this case it is possible that for $x > \tilde{x}, f^2(x)$ may not converge to $\tilde{x}$, i.e., $f$ might be structurally unstable. Hence the necessary conditions for the global stability of the fixed point is $f^2(\tilde{x}) > \tilde{x}$.

We now provide necessary and sufficient conditions for the global stability of the fixed point.

**Theorem 5.4.** $x_0$ is globally stable if and only if $x_0 \to x_0$ under the iterates of $f$.

**Proof.** If $x_0$ is globally stable, then by definition $f^n(\tilde{x}) \to x_0$. To prove the converse, let $\tilde{x} \to x_0$ and define $y = f(\tilde{x})$. It follows from Theorem 3.2 that $\forall x \leq \tilde{x}, f(x) = y$ and hence $\forall x \leq \tilde{x}, x \to x_0$. Moreover, since $\tilde{x} \to x_0$ it follows from Theorem 5.3 that $f'(y) > 1$. Let $\tilde{x} > y$ be such that $f(\tilde{x}) = \tilde{x}$. We see that $\forall z \geq \tilde{x}, f(z) = \tilde{x}$ and hence $\forall z \geq \tilde{x}, z \to x_0$. What remains to be shown is the convergence of the points in the interval $I = (\tilde{x}, \tilde{z})$ to $x_0$. Define $l_0 = (x_0, \tilde{z}),$ and $l_n = (f^{n-1}(\tilde{x}), x_0)$ for $n \geq 1$, where $f^{n}(\tilde{x}) = \tilde{x}$. We note that $l_n \to l_0 \cup l_1 \cup x_0$ and $h(l_n) = l_{n+1}, \forall n \geq 0$. Since the endpoint $f^{n}(\tilde{x})$ of $l_n$ converges to $x_0$, it follows that all points in $I$ converge to $x_0$. Hence $\forall x \in [0, 1], x \to x_0$, making $x_0$ a globally stable fixed point. □

As noted earlier some period 2 cycles can assume a dual role both as an attractor and a repellor, i.e., their domain of attraction is one-sided with the other side being a repellor, for example, when $f$ is structurally unstable. If these period 2 cycles are counted twice for their dual role, then we have the following.

**Theorem 5.5.** If $x_0$ is a stable fixed point, then the number of period 2 cycles is even.

**Proof.** Since $x_0$ is sandwiched between the points of any period 2 cycle, it suffices to consider the region $(x_0, 1]$. Consider the map $f^2$ and the diagonal line $L$ defined by $f^2(x) = x$. Label any intersection of $f^2$ with $L$ proper, if $f^2$ completely intersects $L$ and does not merely touch $L$. Since $x_0$ is stable, for $x = x_0 + \Delta x$, infinitesimal $\Delta x < 0$, we have $f^2(x) < x$. Moreover, since $W_1$ never enters the last station, $\forall x, f'(x) < 1$ and hence $f^2(x) < 1$. Therefore the number of proper intersections of $f^2$ with $L$ in the range $(x_0, 1]$ is even. If $f^2$ merely touches $L$ at $y$, then it is easy to see that the period 2 cycle involving $y$ is structurally unstable and hence is counted twice for its dual role. Hence the number of period 2 cycles is even. □

**Corollary 5.6.** If $x_0$ is stable and a period 2 cycle exists, then $x_0$ is not globally stable.

**Proof.** If the period 2 cycle is a repellor, then from Theorem 5.5 it follows that an attracting period 2 cycle exists and hence there exists an interval converging to it. Hence the fixed point $x_0$ is not globally stable. □

6. Algorithmic determination of the global stability of the fixed point

We saw in the previous section that whether or not the fixed point $x_0$ is globally stable can be determined by considering the following exhaustive list of scenarios: (i) if $x_0 \leq \tilde{x}$ then $x_0$ is globally stable, (ii) if $x_0 > \tilde{x}$ and $f^2(\tilde{x}) \leq \tilde{x}$ then $x_0$ is not globally stable (period 2 cycle exists), and (iii) if $x_0 > \tilde{x}$ and $f(\tilde{x}) < f^{-1}(\tilde{x})$ then $x_0$ is globally stable iff $\tilde{x} \to x_0$ under the iterates of $f$.

Scenarios (i) and (ii) can be easily verified. Scenario (iii) concerns the dynamics of the points in the interval $I = (\tilde{x}, \tilde{z})$ where $\tilde{z} = f^{-1}(\tilde{x})$, since in this case $f(I) \subset I$ and furthermore $f^2([0, 1]) \subset I$. Based on the observation that worker $W_i$ is not
Case 1: $x > x_0$. Let $t$ be the time taken by $W_2$ to reach $x$ beginning at $x_0$, $W_1$ would then have traveled for an additional time $t$ before hand-off, had $W_2$ begun at $x_0$. Therefore, if $W_1$ travels for time $t$ from $f(x)$ she would reach $f(x_0) = x_0$, or equivalently, if $W_1$ begins at $x_0$ and travels backward for time $t$ she would reach $f(x)$. Let the fixed point be verified by plotting $f$ the inverse of the velocities of the workers on the corresponding check if $\forall \in (0,2)$, $[0.2,0.25],[0.25,0.3],[0.3,0.45],[0.45,0.5],[0.5,0.7],[0.7,0.9]$ and $(0.9,1)$ with the velocity of $W_i$ being 1 on all stations and the velocity of $W_2$ being $[1,0.9,1.4,1.1,0.9,1.1,0.5,3]$ on the respective stations. Simulation shows that this assembly line has a stable fixed point at 0.5603, an unstable period 2 cycle (0.3987, 0.7173), followed by a stable period 2 cycle (0.3687, 0.7333). The production rate for the stable fixed point and the stable period 2 cycle are 1.7848 and 1.7966, respectively (\(T_{diff} = 0.0074\)) indicating that the stable period 2 cycle has a higher production rate than the stable fixed point.

7.2. Dependence of throughputs on the velocities of the workers

At first sight one would expect the throughput to increase with an increase in the velocity of either of the workers. This however is not true as the following simple example demonstrates.

Consider a two station assembly line where $W_i$ is faster than $W_j$ in station 1. Assume, in addition, that the fixed point $x_0$ lies in station 1. Clearly, $W_i$ remains blocked until $W_j$ reaches station 2. When the velocity of $W_j$ is increased in station 2, the fixed point shifts to the left. Let the new fixed point be denoted by $x_0$. Then, $x_0 < x_0$. The difference in time to produce one item between the former and the latter scenario can be shown to be equal to the difference in the times taken by $W_i$ and $W_j$ to reach $x_0$ from $x_0$. Since $W_i$ is faster than $W_j$ in station 1, this quantity is negative implying that the former scenario has a higher throughput than the latter. In essence, the production rate drops when the velocity of $W_i$ is increased.

8. Conclusion

This note generalizes the results in [3]. When applied to the 2 worker case, [1] holds that $x_0$ is globally stable when $W_i$ dominates $W_j$ over the entire assembly line. We show in this case the imposed condition on $g$ defined in Section 6 is satisfied.

Let \(x = x_0 + \Delta x\), \(\Delta x > 0\). Let \(g(x) = f^2(x) = x_0 + \Delta z\), \(\Delta z > 0\). We demonstrate that if $W_j$ dominates $W_i$ then \(\Delta z < \Delta x\), and hence $x_0$ is globally stable. Let $t_i, i \in \{1, 2\}$ be the time taken by $W_i$ to reach $x$ beginning at $x_0$. Since $W_j$ dominates $W_i$, $t_2 < t_1$. Let $W_i$ reach position $y$ traveling backward from $x_0$ for time $t_2$. Then, \(f(x) = y\). Since $W_j$ dominates $W_i$, the time taken by $W_j$ to reach $y$, say $t_1$, is less than $t_2$. Hence $t_2 < t_3 < t_1$. Since $\Delta z$ (respectively, \(\Delta x\)) is the distance covered by $W_i$ traveling forward from $x_0$ for time $t_3$ (respectively, $t_1$), \(\Delta z < \Delta x\) implying that $x_0$ is globally stable. However, it is a simple exercise to construct examples that show that the criterion is a sufficient and not necessary condition for the global stability of the fixed point.

References