

## Supplementary Material to the manuscript “Connectomic Constraints on Computation in Feedforward Networks of Spiking Neurons”

Venkatakrishnan Ramaswamy · Arunava Banerjee

### Online Resource A: Relationship of the abstract neuron model to some widely-used neuron models

Here, we demonstrate that the properties that our abstract model of the neuron is contingent on are satisfied, up to arbitrary accuracy, by several widely-used neuron models such as the Leaky Integrate-and-Fire Model and Spike Response Model.

#### Leaky Integrate-and-Fire Model

Consider the standard form of the Leaky Integrate-and-Fire Model:

$$\tau_m \frac{du}{dt} = -u(t) + RI(t) \quad (1)$$

where  $\tau_m = RC$ . When  $u(t^{(f)}) = v$ , the neuron fires a spike and the reset is given by  $u(t^{(f)} + \Delta) = u_r$ , where  $v$  is the threshold and  $\Delta$  is the absolute refractory period. Suppose an output spike has occurred at time  $\hat{t} - \Delta$ , the above differential equation has the following solution:

$$u(t) = u_r \exp\left(-\frac{t - \hat{t}}{\tau_m}\right) + \frac{1}{C} \int_0^{t - \hat{t}} \exp\left(-\frac{s}{\tau_m}\right) I(t - s) ds \quad (2)$$

Suppose  $I(t) = \sum_j w_j \sum_i \alpha(t - t_j^{(i)})$  and  $\alpha(\cdot)$  had a finite support. Then, it is clear from the above expression that the contribution of the previous output spike fired by the present neuron as well as the contribution of input spikes from presynaptic neurons decays exponentially with time. Therefore, one can compute the membrane potential to arbitrary accuracy

---

Computer and Information Science and Engineering,  
University of Florida, Gainesville, FL 32611, USA.  
E-mail: {vr1, arunava}@cise.ufl.edu

*Present address* of V. Ramaswamy:  
Interdisciplinary Center for Neural Computation,  
The Edmond and Lily Safra Center for Brain Sciences,  
The Hebrew University of Jerusalem, Jerusalem 91904, Israel.  
E-mail: venkat.ramaswamy@mail.huji.ac.il

by choosing input and output “windows” of appropriate size so that  $u(\cdot)$  is a function only of input spikes and output spikes in those windows. It is easy to verify that the all the axioms of our model are satisfied: Clearly, the model above has an absolute refractory period, a past output spike has an inhibitory effect on membrane potential, and upon receiving no input and output spikes in the said windows, it settles to resting potential. Thus, an instantiation of our abstract model can simulate a Leaky Integrate-and-Fire Model to arbitrary accuracy.

### Spike Response Model

Consider now the standard form of the Spike Response Model(Gerstner and Kistler, 2002).

In the absence of spikes, the membrane potential  $u(\cdot)$  is set to the value  $u_r = 0$ . Otherwise, the membrane potential is given by

$$u(t) = \eta(t - \hat{t}_i) + \sum_j w_j \sum_i \epsilon_{ij}(t - \hat{t}_i, t - t_j^{(i)}) \quad (3)$$

where  $\eta(\cdot)$  describes the after-hyperpolarization after an output spike at  $\hat{t}_i$  and  $\epsilon_{ij}(\cdot)$  describes the response to incoming spikes  $t_j^{(i)}$ , which are the spikes fired by presynaptic neuron  $j$  with  $w_j$  being synaptic weights.  $\eta(\cdot)$ , is set to a sufficiently low value for  $\Delta$  milliseconds after an output spike so as not to cause another spike, where  $\Delta$  is the absolute refractory period. The functions  $\eta(\cdot)$  and  $\epsilon_{ij}(\cdot)$  typically decay exponentially with time and therefore, as before, one can compute the membrane potential to arbitrary accuracy by choosing input and output “windows” of appropriate size so that the  $u(\cdot)$  is a function only of input spikes and output spikes in those windows. Likewise, it is easy to verify that the all the axioms of our model are satisfied: Clearly, the model above has an absolute refractory period, a past output spike has an inhibitory effect on membrane potential, and upon receiving no input and output spikes in the said windows, it settles to resting potential. Thus, it is straightforward to verify that an instantiation of our abstract model can simulate a Spike Response Model to arbitrary accuracy.

## Online Resource B: Proofs and Technical Remarks

### Technical Remarks from Section 4

It might be argued that the input spike-train to a neuron cannot possibly be infinitely long, since every neuron begins existence at a certain point in time. However, this begs the question whether the neuron was at the resting potential when the first input spikes arrived<sup>1</sup>. An assumption to this effect would be significant, particularly if the current membrane potential depended on it. It is easy to construct an example along the lines of the example described in Figure 1, where the current membrane potential is different depending on whether this assumption is made or not. Assuming infinitely long input spike-train ensembles, on the other hand, obviates the need to make any such assumption. We retain this viewpoint for the rest of the paper with the understanding that the alternative viewpoint discussed at the beginning of this paragraph can also be expounded along similar lines.

### Proofs from Section 5

*Proof of Gap Lemma.* Since, in each  $\mathbf{x}_0$  consistent with  $\chi$ , with respect to  $\mathbb{N}$ , the interval  $(t + 2\rho, t + 3\rho)$  of  $\mathbf{x}_0$  and the  $(t + \mathcal{T} + \rho, t + \mathcal{T} + 2\rho)$  of  $\chi$  are arbitrary, the sequence of spikes present in the interval  $(t + \rho, t + 2\rho)$  of  $\mathbf{x}_0$  could be arbitrary. However,  $\chi^*$  and  $\chi$  are identical in  $(t, t + \rho + \mathcal{T})$ . Thus, it follows from Axiom 2 in the formal definition of a neuron that for every  $t' \in (t, t + \rho)$ ,  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t'}(\chi)), \Xi_{(0, \rho)}(\sigma_{t'}(\mathbf{x}_0)))$  is at most the value of  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t'}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t'}(\mathbf{x}_0^*)))$ , because  $\Xi_{(0, \rho)}(\sigma_{t'}(\mathbf{x}_0^*))$  is  $\phi$ , i.e. empty. Since  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t'}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t'}(\mathbf{x}_0^*)))$  is less than  $\tau$  for every  $t' \in (t, t + \rho)$ ,  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t'}(\chi)), \Xi_{(0, \rho)}(\sigma_{t'}(\mathbf{x}_0)))$  is less than  $\tau$  in the same interval, as well. Therefore,  $\mathbf{x}_0$  has no spikes in  $(t, t + \rho)$ .

That  $2\rho$  is the smallest possible gap length in  $\mathbf{x}_0^*$  for this to hold, follows from the counterexample in Figure 1, where the present conclusion did not hold, when  $\mathbf{x}_0^*$  had gaps of length  $2\rho - \delta$ , for arbitrarily small  $\delta > 0$ . □

*Proof of Corollary 1.* (1) is immediate from the Gap Lemma, when we set  $\chi = \chi^*$ .

For (2), the proof is by strong induction on the number of spikes since  $t$ . Let  $\mathbf{x}_0$  be an arbitrary spike-train that is consistent with  $\chi^*$ , with respect to  $\mathbb{N}$ . Notice that from (1) we have that  $\mathbf{x}_0$  is identical to  $\mathbf{x}_0^*$  in  $(t, t + \rho)$ . The base case is to show that both  $\mathbf{x}_0^*$  and  $\mathbf{x}_0$  have their first spike since  $t$  at the same time. Assume, without loss of generality, that the first spike of  $\mathbf{x}_0$  at  $t_1 \leq t$ , is no later than the first spike of  $\mathbf{x}_0^*$ . We have  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t_1}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t_1}(\mathbf{x}_0^*))) = P(\Xi_{(0, \mathcal{T})}(\sigma_{t_1}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t_1}(\mathbf{x}_0)))$  since  $\Xi_{(0, \rho)}(\sigma_{t_1}(\mathbf{x}_0^*)) = \Xi_{(0, \rho)}(\sigma_{t_1}(\mathbf{x}_0)) = \phi$ . Therefore  $\mathbf{x}_0^*$  also has its first spike since  $t$  at  $t_1$ . Let the induction hypothesis be that both  $\mathbf{x}_0^*$  and  $\mathbf{x}_0$  have their first  $k$  spikes since  $t$  at the same times. We show that this implies that the  $(k + 1)^{th}$  spike in each spike-train is also at the same time instant. Assume, without loss of generality, that the  $(k + 1)^{th}$  spike since  $t$  of  $\mathbf{x}_0$  at  $t_{k+1}$ , is no later than the  $(k + 1)^{th}$  spike since  $t$  of  $\mathbf{x}_0^*$ . Now,  $\Xi_{(0, \rho)}(\sigma_{t_{k+1}}(\mathbf{x}_0^*))$  is identical to  $\Xi_{(0, \rho)}(\sigma_{t_{k+1}}(\mathbf{x}_0))$  from the induction hypothesis since  $(t + \rho) - t_{k+1} \geq \rho$ . Thus,  $P(\Xi_{(0, \mathcal{T})}(\sigma_{t_{k+1}}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t_{k+1}}(\mathbf{x}_0^*))) = P(\Xi_{(0, \mathcal{T})}(\sigma_{t_{k+1}}(\chi^*)), \Xi_{(0, \rho)}(\sigma_{t_{k+1}}(\mathbf{x}_0)))$  and therefore  $\mathbf{x}_0^*$  also has its  $(k + 1)^{th}$  spike at  $t_{k+1}$ . This completes the proof of (2).

<sup>1</sup> Note that our axiomatic definition of a neuron does not address this question.

(3) follows from the Gap Lemma and (2). □

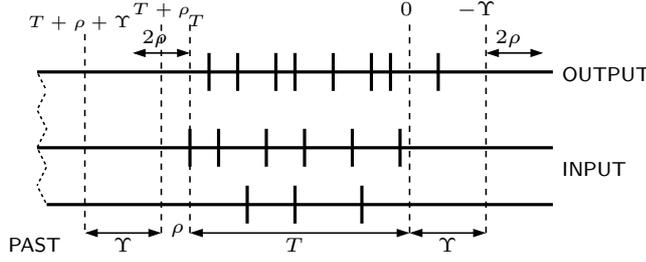
**Proposition 1.** *Let  $\chi$  be a spike-train ensemble that satisfies a  $T$ -Gap criterion for a neuron  $N(\alpha, \mathcal{T}, \rho, \tau, \lambda, m, P : \bar{S}_{(0, \mathcal{T})}^m \times \bar{S}_{(0, \rho)} \rightarrow [\lambda, \tau])$ , where  $T \in \mathbb{R}^+$ . Then, there is exactly one spike-train  $\mathbf{x}_0$ , such that  $\mathbf{x}_0$  is consistent with  $\chi$ , with respect to  $N$ .*

*Proof of Proposition 1.* Since  $\chi$  satisfies a  $T$ -Gap criterion, there exists a spike-train  $\mathbf{x}_0$  with at least one gap of length  $2\rho$  in every interval of time of length  $T - \mathcal{T} + 2\rho$ , so that  $\mathbf{x}_0$  is consistent with  $\chi$  with respect to  $N$ . For the sake of contradiction, assume that there exists another spike-train  $\mathbf{x}_0'$ , not identical to  $\mathbf{x}_0$ , which is consistent with  $\chi$ , with respect to  $N$ . Let  $t'$  be the time at which one spike-train has a spike but another doesn't. Let  $t > t'$  be such that  $\mathbf{x}_0$  has a gap in the interval  $(t, t + 2\rho)$ . By Corollary 1 to the Gap Lemma, it follows that  $\mathbf{x}_0'$  is identical to  $\mathbf{x}_0$  after time instant  $t + \rho$ . This contradicts the hypothesis that  $\mathbf{x}_0'$  is different from  $\mathbf{x}_0$  at  $t'$ . □

**Lemma 2.** *Consider a feedforward network  $\mathcal{N}$ . Let  $\chi$  satisfy a  $T$ -Gap criterion for  $\mathcal{N}$ , where  $T \in \mathbb{R}^+$ . Then the output neuron of  $\mathcal{N}$  produces a unique output spike-train when  $\mathcal{N}$  receives  $\chi$  as input. Furthermore, the membrane potential of the output neuron at any time instant depends on at most the past  $T$  milliseconds of input in  $\chi$ .*

*Proof of Lemma 2.* We prove that the output of the network is unique by strong induction on depth. Let  $N_i$ , for  $1 \leq i \leq d$ , be the set of neurons in  $\mathcal{N}$  of depth  $i$ . Each neuron  $N \in N_1$  receives all inputs from spike-trains in  $\chi$ . Since,  $N$  satisfies a Gap criterion with those input spike-trains, its output is unique. The induction hypothesis then is that for all  $i \leq k < d$ , each neuron  $N \in N_i$  produces a unique output spike-train when  $\mathcal{N}$  is driven by  $\chi$ . Consider arbitrary  $N' \in N_{k+1}$ . It is clear that all inputs to  $N'$  are from spike-trains from  $\chi$  or neurons in  $\bigcup_{i=1}^k N_i$ , for otherwise the depth of  $N'$  would be greater than  $k + 1$ . Since, all its inputs are unique by the induction hypothesis and they satisfy a Gap criterion for  $N'$ , its output is also unique.

Next, we show that the membrane potential of the output neuron at any time instant depends on at most the past  $T$  milliseconds of input in  $\chi$ . Since the output neuron satisfies a  $(\frac{T}{d})$ -Gap Criterion, its membrane potential at any point depends on at most the past  $(\frac{T}{d})$  milliseconds of the inputs it receives (some of which may be output spike-trains of other neurons). Consider one such ‘‘penultimate layer’’ neuron. Again, its output membrane potential at any time instant, likewise, depends on its inputs in the past  $(\frac{T}{d})$  milliseconds. Therefore, the current potential of the output neuron is dependent on the input received by the penultimate layer neuron in at most the past  $(\frac{2T}{d})$  milliseconds. Similar arguments can be put forth until, for each path, one reaches a neuron, all of whose inputs do not come from other neurons. Since the longest such path is of length  $d$ , it is straightforward to verify that the membrane potential of the output neuron depends on at most  $T$  milliseconds of past input in  $\chi$ . □



**Fig. 1** Illustration showing that an input spike-train ensemble satisfying a Flush Criterion also satisfies a Gap Criterion.

### Proofs from Section 6

**Lemma 3.** *An input spike-train ensemble  $\chi$  for a neuron  $N(\alpha, \Upsilon, \rho, \tau, \lambda, m, P : \bar{\mathcal{S}}_{(0, \Upsilon)}^m \times \bar{\mathcal{S}}_{(0, \rho)} \rightarrow [\lambda, \tau])$  that satisfies a  $T$ -Flush Criterion also satisfies a  $(T + 2\Upsilon + 2\rho)$ -Gap Criterion for that neuron.*

*Proof of Lemma 3.* Figure 1 accompanies this proof. The neuron on being driven by  $\chi$  cannot have output spikes outside the interval  $(-\Upsilon, T)$ . This easily follows from Axiom 2 and 3 of the neuron because the neuron does not have input spikes before time instant  $T$  and in the interval  $(-\Upsilon, 0)$  and onwards. Now, to see that  $\chi$  satisfies a  $(T + 2\Upsilon + 2\rho)$ -Gap Criterion, recall that with a  $T'$ -Gap Criterion, distance between any two gaps of length  $2\rho$  on the output spike-train is at most  $T' - \Upsilon - 2\rho$ . With  $\chi$ , we observe that the distance between any two  $2\rho$  gaps on the output spike-train is at most  $T + \Upsilon$ . Thus,  $T' - \Upsilon - 2\rho = T + \Upsilon$ , which gives us  $T' = T + 2\Upsilon + 2\rho$ . The result follows.  $\square$

**Lemma 4.** *An input spike-train ensemble  $\chi$  for a feedforward network that satisfies a  $T$ -Flush Criterion also satisfies a  $(dT + d(d + 1)\Upsilon + 2d\rho)$ -Gap Criterion for that network, where  $\Upsilon, \rho$  are upper bounds on the same parameters taken over all the neurons in the network and  $d$  is the depth of the network.*

*Proof of Lemma 4.* Following the proof of the previous lemma, we know that neurons that receive all their inputs from  $\chi$  have no output spikes outside the interval  $(-\Upsilon, T)$ . Similarly, neurons that have depth 2 with respect to the input vertices of the network have no output spikes outside  $(-2\Upsilon, T)$ . Likewise, the output neuron, which has depth  $d$ , has no output spikes outside  $(-d\Upsilon, T)$ . It follows that the output neuron obeys a  $(T + (d + 1)\Upsilon + 2\rho)$ -Gap Criterion. Also, every other neuron obeys this criterion because the distance between the  $2\rho$  output gaps for every neuron is at most that of the output neuron, since their depth is bounded from above by the depth of the output neuron. Thus, from the definition of the Gap criterion for feedforward networks, we have that  $\chi$  satisfies a  $(dT + d(d + 1)\Upsilon + 2d\rho)$ -Gap Criterion for the current network.  $\square$

## Proofs from Section 7

*Proof of Lemma 5.* We prove the easy direction first. If  $\exists \mathcal{N}' \in \Sigma_2$  such that  $\forall \mathcal{N} \in \Sigma_1, \mathcal{T}_{\mathcal{N}'}|_{\mathcal{F}_m} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{F}_m}$ , then it follows that  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{G}_{12}} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{G}_{12}}$  because  $\mathcal{F}_m \subseteq \mathcal{G}_{\mathcal{N}}$ .

For the other direction, let  $\exists \mathcal{N}' \in \Sigma_2$  such that  $\forall \mathcal{N} \in \Sigma_1, \mathcal{T}_{\mathcal{N}'}|_{\mathcal{G}_{12}} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{G}_{12}}$ . We construct  $\mathcal{F}' \subseteq \mathcal{F}_m$ , so that  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{F}'} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{F}'}$ . This immediately implies  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{F}_m} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{F}_m}$ . Consider arbitrary  $\mathcal{N} \in \Sigma_1$ . From the hypothesis, we have  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{G}_{12}} \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{G}_{12}}$ . Therefore  $\exists \chi \in \mathcal{G}_{12}$  such that  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{G}_{12}}(\chi) \neq \mathcal{T}_{\mathcal{N}}|_{\mathcal{G}_{12}}(\chi)$ . Additionally, there exist  $T_1, T_2 \in \mathbb{R}^+$ , so that  $\chi$  satisfies a  $T_1$ -Gap Criterion for  $\mathcal{N}$  and a  $T_2$ -Gap Criterion for  $\mathcal{N}'$ . Let  $T = \max(T_1, T_2)$ . Let  $\mathcal{T}_{\mathcal{N}'}|_{\mathcal{G}_{12}}(\chi) = \mathbf{x}_0'$  and  $\mathcal{T}_{\mathcal{N}}|_{\mathcal{G}_{12}}(\chi) = \mathbf{x}_0$ . Let  $\tilde{\mathcal{F}} = \bigcup_{t \in \mathbb{R}} \Xi_{(0,2T)}(\sigma_t(\chi))$ . Note that each element of  $\tilde{\mathcal{F}}$  satisfies a  $2T$ -Flush Criterion. The claim, then, is that  $\mathcal{T}_{\mathcal{N}'}|_{\tilde{\mathcal{F}}} \neq \mathcal{T}_{\mathcal{N}}|_{\tilde{\mathcal{F}}}$ . We have  $\Xi_{(0,T)}(\mathcal{T}_{\mathcal{N}'}(\Xi_{(0,2T)}(\sigma_t(\chi)))) = \Xi_{(0,T)}(\sigma_t(\mathbf{x}_0'))$  and  $\Xi_{(0,T)}(\mathcal{T}_{\mathcal{N}}(\Xi_{(0,2T)}(\sigma_t(\chi)))) = \Xi_{(0,T)}(\sigma_t(\mathbf{x}_0))$ . This follows from the fact that  $\chi$  satisfies the  $T$ -Gap Criterion with both  $\mathcal{N}$  and  $\mathcal{N}'$  and therefore when  $\mathcal{N}$  and  $\mathcal{N}'$  are driven by any segment of  $\chi$  of length  $2T$ , the output produced in the latter  $T$  milliseconds of that interval agrees with  $\mathbf{x}_0$  and  $\mathbf{x}_0'$  respectively. Therefore, if  $\mathbf{x}_0 \neq \mathbf{x}_0'$ , it is clear that there exists a  $t$ , so that  $\mathcal{T}_{\mathcal{N}'}(\Xi_{[0,2T]}(\sigma_t(\chi))) \neq \mathcal{T}_{\mathcal{N}}(\Xi_{[0,2T]}(\sigma_t(\chi)))$ .  $\mathcal{F}'$  is obtained by taking the union of such  $\tilde{\mathcal{F}}$  for every  $\mathcal{N} \in \Sigma_1$ . The result follows.  $\square$

## Technical Remarks from Section 8

Some technical remarks concerning the mechanics of proving complexity results are stated below.

For two sets of feedforward networks,  $\Sigma_1$  and  $\Sigma_2$  with  $\Sigma_1 \subseteq \Sigma_2$ , in order to prove that  $\Sigma_2$  is more complex than  $\Sigma_1$ , it is sufficient to show a transformation  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  that no network present in  $\Sigma_1$  can perform, while demonstrating a network in  $\Sigma_2$  that can effect it. This involves constructing such a transformation, i.e. prescribing an output spike train for every element in  $\mathcal{F}_m$ . Recall that  $\mathcal{F}_m$  consists of spike-train ensembles of order  $m$ , with the property that for each such ensemble there exists a positive real number  $T$ , so that the ensemble satisfies a  $T$ -Flush criterion. In practice, however, it usually suffices to prescribe output spike trains for a small subset<sup>2</sup> of elements of  $\mathcal{F}_m$ , and prove that no network in  $\Sigma_1$  can map the input spike trains in that subset to their prescribed outputs. The second step would involve demonstrating a network in  $\Sigma_2$  that maps this subset of  $\mathcal{F}_m$  to the prescribed output, while mapping the rest of  $\mathcal{F}_m$  to arbitrary output spike trains. Strictly speaking then, the transformation  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  we prescribe comprises the mapping from  $\mathcal{F}_m$  to output spike trains, as effected by *this* network in  $\Sigma_2$ . For convenience however, we shall refer to the mapping prescribed for some small subset of  $\mathcal{F}_m$  as the prescribed transformation.

The next remark concerns timescales of the parameters  $\gamma$  and  $\rho$  of each neuron in the network and the timescale at which the transformation operates. Recall that the parameters  $\gamma$  and  $\rho$  correspond to the timescale at which the neuron integrates inputs it receives and the relative refractory period respectively. It would be reasonable to expect that the values of these parameters lie within a certain range as constrained by physiology, although this range might be different for different local neuronal networks in the brain. Suppose we have an upper bound on the value of each such parameter. Then, when we prove a complexity

<sup>2</sup> albeit typically one that contains, for each positive real number  $T$ , at least one spike-train ensemble satisfying a  $T$ -Flush Criterion.

result, there would exist a timescale  $T$ , which is a function of these upper bounds, such that there exists a transformation on this timescale that cannot be performed by any network with the said architecture, whose parameters are governed by these upper bounds. More precisely, there would exist a transformation that maps a set of inputs satisfying a  $T$ -Flush criterion to an output spike train that (provably) cannot be performed by any network with the architecture in question. When stating and proving a complexity result, however, for the sake of succinctness, we do not explicitly state the relation between these bounds and the corresponding  $T$ . We simply let  $\mathcal{X}$ ,  $\rho$  and  $T$  remain unbounded. It is straightforward for the reader to derive a bound on  $T$  as a function of bounds on  $\mathcal{X}$  and  $\rho$ , as discussed.

The final remark is about our neuron model and the issue of what we can assume about the neurons when demonstrating that a certain network *can* effect a given transformation. Recall that our neuron model assumes that our neurons satisfy a small number of elementary properties but are otherwise unconstrained. This allowed our model to accomodate a large variety of neuronal responses. This was convenient when faced with the task of showing that no network of a certain architecture could perform a given transformation, no matter what response properties its neurons have. However, when we wish to show that a certain transformation can be done by a specific network, some caution is in order. In this case, it is prudent to restrict ourselves to as simple a neuron model as possible, so that whether the neuronal responses employed are achievable by a real biological neuron, is not in question. In practice, we describe the neurons in the construction, so that they can certainly be effected by a highly-reduced neuron model such as the Spike Response Model SRM<sub>0</sub> (Gerstner and Kistler, 2002).

#### Proofs from Section 9

*Proof of Lemma 6.* Let  $\mathcal{N}$  be a network that effects  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$ .

**$\mathcal{T}(\cdot)$  is causal.** Consider arbitrary  $\chi_1, \chi_2 \in \mathcal{F}_m$  with  $\Xi_{(t, \infty)} \chi_1 = \Xi_{(t, \infty)} \chi_2$ , for some  $t \in \mathbb{R}$ . We wish to show that  $\Xi_{[t, \infty)} \mathcal{T}(\chi_1) = \Xi_{[t, \infty)} \mathcal{T}(\chi_2)$ . Let  $N_i$ , for  $1 \leq i \leq d$ , be the set of neurons in  $\mathcal{N}$  of depth  $i$ , where  $d$  is the depth of  $\mathcal{N}$ . Each neuron  $N \in N_1$  receives all its inputs from spike-trains in  $\chi$ . When the network receives  $\chi_1$  and  $\chi_2$  as input, suppose  $N$  receives  $\chi'_1$  and  $\chi'_2$  respectively as input. Also, clearly,  $\Xi_{(t, \infty)} \chi'_1 = \Xi_{(t, \infty)} \chi'_2$ . Let  $\mathbf{x}'_1$  and  $\mathbf{x}'_2$  be the output produced by  $N$  on receiving  $\chi'_1$  and  $\chi'_2$  respectively. Since  $\chi'_1, \chi'_2 \in \mathcal{F}_m$ , there exists a  $T \in \mathbb{R}^+$ , so that  $\Xi_{[T, \infty)} \chi'_1 = \Xi_{[T, \infty)} \chi'_2 = \phi^{m'}$ , where  $m'$  is the number of inputs to  $N$ . Therefore, by Axiom (3) of the neuron, we have  $\Xi_{[T, \infty)} \mathbf{x}'_1 = \Xi_{[T, \infty)} \mathbf{x}'_2 = \phi$ . Now, for all  $t' \in \mathbb{R}$ ,  $\Xi_{t'} \mathbf{x}'_j = \langle t' \rangle$  if and only if  $P_N(\Xi_{(0, t'_N)}(\sigma_{t'}(\chi'_j))), \Xi_{(0, \rho_N)}(\sigma_{t'}(\mathbf{x}'_j)) = \tau_N$ , for  $j = 1, 2$ . It is immediate that for  $t' > t$ , we have  $\Xi_{(0, t'_N)}(\sigma_{t'}(\chi'_1)) = \Xi_{(0, t'_N)}(\sigma_{t'}(\chi'_2))$ . Now, by an induction argument on the spike number since  $T$ , it is straightforward to show that for all  $t' > t$ ,  $\Xi_{(0, \rho_N)}(\sigma_{t'}(\mathbf{x}'_1)) = \Xi_{(0, \rho_N)}(\sigma_{t'}(\mathbf{x}'_2))$ . Thus, we have  $\Xi_{[t, \infty)} \mathbf{x}'_1 = \Xi_{[t, \infty)} \mathbf{x}'_2$ . Similarly, using a straightforward induction argument on depth, one can show that for every neuron in the network, its output until time instant  $t$  is identical in either case. We therefore have  $\Xi_{[t, \infty)} \mathcal{T}(\chi_1) = \Xi_{[t, \infty)} \mathcal{T}(\chi_2)$ .

**$\mathcal{T}(\cdot)$  is time-invariant.** Consider arbitrary  $\chi \in \mathcal{F}_m$  and  $t \in \mathbb{R}$  with  $\sigma_t(\chi) \in \mathcal{F}_m$ . We wish to show that  $\mathcal{T}(\sigma_t(\chi)) = \sigma_t(\mathcal{T}(\chi))$ . As before, let  $N_i$ , for  $1 \leq i \leq d$ , be the set of neurons in  $\mathcal{N}$  of depth  $i$ , where  $d$  is the depth of  $\mathcal{N}$ . Each neuron  $N \in N_1$  receives all its inputs from spike-trains in  $\chi$ . When the network receives  $\chi$  and  $\sigma_t(\chi)$  as input, suppose  $N$  receives  $\chi'$  and  $\sigma_t(\chi')$  respectively as input. Let  $\mathbf{x}'_1$  and  $\mathbf{x}'_2$  be the output produced by  $N$  on receiving  $\chi'$  and  $\sigma_t(\chi')$  as input respectively. We wish to show that  $\mathbf{x}'_2 = \sigma_t(\mathbf{x}'_1)$ . Since  $\chi' \in \mathcal{F}_m$ , there exists a  $T \in \mathbb{R}^+$ , so that  $\Xi_{[T, \infty)} \chi' = \Xi_{[T-t, \infty)} \sigma_t(\chi') = \phi^{m'}$ ,

where  $m'$  is the number of inputs to  $N$ . Therefore, by Axiom (3) of the neuron, we have  $\Xi_{[T, \infty)} \mathbf{x}_1' = \Xi_{[T-t, \infty)} \mathbf{x}_2' = \phi$ . Now, for all  $t' \in \mathbb{R}$ ,  $\Xi_{t'} \mathbf{x}_1' = \langle t' \rangle$  if and only if  $P_N(\Xi_{(0, \tau_N)}(\sigma_{t'}(\chi')), \Xi_{(0, \rho_N)}(\sigma_{t'}(\mathbf{x}_1'))) = \tau_N$ . It is therefore straightforward to make an induction argument on the spike number, starting from the oldest spike in  $\mathbf{x}_1'$  to show that  $\mathbf{x}_1'$  has a spike at some  $t'$  iff  $\mathbf{x}_2'$  has a spike at  $t' - t$  and therefore we have  $\mathbf{x}_2' = \sigma_t(\mathbf{x}_1')$ . Similarly, using a straightforward induction argument on depth, one can show that for every neuron in the network, its output in the second case is a time-shifted version of the one in the first case. We therefore have  $\mathcal{T}(\sigma_t(\chi)) = \sigma_t(\mathcal{T}(\chi))$ .

**$\mathcal{T}(\cdot)$  is resettable.** Let  $\mathcal{Y}$  and  $\rho$  be upper bounds on those parameters over all the neurons in  $\mathcal{N}$ . If  $\mathcal{Y} < \rho$ , then set the value of  $\mathcal{Y} = \rho$ . The claim is that for  $W = d(\mathcal{Y} + \rho) + \rho$ ,  $\mathcal{T}(\cdot)$  is  $W$ -resettable, where  $d$  is the depth of  $\mathcal{N}$ . Consider arbitrary  $\chi \in \mathcal{F}_m$  so that  $\chi$  has a gap in the interval  $(t, t + d(\mathcal{Y} + \rho) + \rho)$ , for some  $t \in \mathbb{R}$ . As before, let  $N_i$ , for  $1 \leq i \leq d$ , be the set of neurons in  $\mathcal{N}$  of depth  $i$ . Each neuron  $N \in N_1$  receives all its inputs from spike-trains in  $\chi$ . Therefore by Axiom (3) of the neuron, it is straightforward to see that the output of  $N$  has a gap in the interval  $(t, t + (d-1)(\mathcal{Y} + \rho) + 2\rho)$ . By similar arguments, we have that output of each neuron  $N \in N_i$ , for  $1 \leq i \leq d$  has a gap in the interval  $(t, t + (d-i)(\mathcal{Y} + \rho) + (i+1)\rho)$ . Thus, in particular, the output neuron has a gap in the interval  $(t, t + (d+1)\rho)$ . Since  $d \geq 1$ , the Gap Lemma applies, and at time instant  $t$  the output of the output neuron depends on spikes in the interval  $(t, t + (\mathcal{Y} + \rho))$  of its inputs. All inputs to the output neuron have a gap in the interval  $(t, t + (\mathcal{Y} + \rho) + d\rho)$ , since they have depth at most  $(d-1)$ . Since those inputs have a gap in the interval  $(t + (\mathcal{Y} + \rho), t + (\mathcal{Y} + \rho) + d\rho)$ , for  $d \geq 2$ , the Gap Lemma applies and the output neuron's output at time instant  $t$  depends on outputs of the "penultimate layer" in the interval  $(t, t + 2(\mathcal{Y} + \rho))$ . Therefore by similar arguments, the output of the output neuron at time instant  $t$  at most depends on inputs from  $\chi$  in the interval  $(t, t + d(\mathcal{Y} + \rho))$ . That is to say that  $\mathcal{T}(\chi')$ , for every  $\chi'$  identical to  $\chi$  in the interval  $(-\infty, t + d(\mathcal{Y} + \rho))$ , has the same output as  $\mathcal{T}(\chi)$  in the interval  $[t, -\infty)$ , following the corollary to the Gap Lemma. In particular,  $\Xi_{(-\infty, t]} \chi$  is one such  $\chi'$ . We therefore have  $\Xi_{(-\infty, t]} \mathcal{T}(\chi) = \mathcal{T}(\Xi_{(-\infty, t]} \chi)$  upon noting that  $\Xi_{(-\infty, t]} \mathcal{T}(\Xi_{(-\infty, t]} \chi) = \mathcal{T}(\Xi_{(-\infty, t]} \chi)$ , since  $\mathcal{T}(\cdot)$  has no spikes in  $(t, \infty)$ . Thus,  $\mathcal{T}(\cdot)$  is resettable.  $\square$

*Proof of Proposition 2.* Assume that the hypothesis in the proposition is true. Let  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  be  $W$ -Resettable for some  $W \in \mathbb{R}^+$ .

We first show a construction for the neuron  $O$ , prove that it obeys all the axioms of the abstract model and then show that it has the property that for every  $\chi \in \mathcal{F}_m$ ,  $\mathcal{T}(\chi)$  is consistent with  $\mathcal{T}_J(\chi) \sqcup \chi$  with respect to  $O$ .

We first construct the neuron  $O(\alpha_O, \mathcal{Y}_O, \rho_O, \tau_O, \lambda_O, m_O, P_O : \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}^{m_O} \times \bar{\mathcal{S}}_{(0, \rho_O)} \rightarrow [\lambda_O, \tau_O])$ . Set  $\alpha_O = \alpha$  and  $\rho_O, \tau_O \in \mathbb{R}^+$ ,  $\lambda_O \in \mathbb{R}^-$  arbitrarily with  $\rho_O \geq \alpha_O$ . Set  $\mathcal{Y}_O = \max\{U, W\}$  and  $m_O = m + 1$ . The function  $P_O : \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}^{m_O} \times \bar{\mathcal{S}}_{(0, \rho_O)} \rightarrow [\lambda_O, \tau_O]$  is constructed as follows.

For  $\chi' \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}^{m_O}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$ , set  $P_O(\chi', \mathbf{x}_0') = \tau_O$  and  $P_O(\chi', \phi) = \tau_O$  if and only if there exists  $\chi \in \mathcal{F}_m$  and  $t \in \mathbb{R}$  so that  $\Xi_t \mathcal{T}(\chi) = \langle t \rangle$  and  $\chi' = \Xi_{(0, \mathcal{Y}_O)}(\sigma_t(\mathcal{T}_J(\chi) \sqcup \chi))$  and  $\mathbf{x}_0' = \Xi_{(0, \rho_O)}(\sigma_t(\mathcal{T}(\chi)))$ . Everywhere else, the value of this function is set to zero.

Next, we show it obeys all of the axioms of the single neuron.

We prove that  $O$  satisfies Axiom (1) by showing that its contrapositive is true. Let  $\chi' \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}^{m_O}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$  be arbitrary so that  $P_O(\chi', \mathbf{x}_0') = \tau_O$ . If  $\mathbf{x}_0' = \phi$ , Axiom (1) is immediately satisfied. Thus, consider the case when  $\mathbf{x}_0' = \langle x_0^{1'}, x_0^{2'}, \dots, x_0^{k'} \rangle$ . Then  $x_0^{1'} \geq \alpha$ ,

otherwise, from the construction of  $P_O(\cdot)$ , it is immediate that there exists a  $\chi \in \mathcal{F}_m$  with  $\mathcal{T}(\chi) \notin \mathcal{S}$ .

Next, we prove that  $O$  satisfies Axiom (2). Let  $\chi' \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}^{m_O}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$  be arbitrary. If  $P_O(\chi', \mathbf{x}_0') = \tau_O$ , then it is immediate from the construction that  $P_O(\chi', \phi) = \tau_O$ . On the contrary, if  $P_O(\chi', \mathbf{x}_0') \neq \tau_O$ , from the construction of  $O$ , we have  $P_O(\chi', \mathbf{x}_0') = 0$ . Then the ‘‘tie-breaker’’ condition in the hypothesis implies that  $P_O(\chi', \phi) \neq \tau_O$ . Therefore,  $P_O(\chi', \phi) = 0$ . Thus, Axiom (2) is satisfied either way.

With Axiom (3), we wish to show  $P_O(\phi^{m+1}, \phi) = 0$ . Here, we will show that  $P_O(\mathbf{x}_J \sqcup \phi^m, \mathbf{x}_0') = 0$ , for all  $\mathbf{x}_J \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$  which implies the required result. Assume, for the sake of contradiction, that there exists a  $\mathbf{x}_J \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$ , so that  $P_O(\mathbf{x}_J \sqcup \phi^m, \mathbf{x}_0') = \tau_O$ . From the construction of  $O$ , this implies that there exists  $\chi \in \mathcal{F}_m$  and  $t \in \mathbb{R}$  so that  $\Xi_t \mathcal{T}(\chi) = \langle t \rangle$  and  $\Xi_{(0, \mathcal{Y}_O)}(\sigma_t(\chi)) = \phi^m$ . That is,  $\chi$  has a gap in the interval  $(t, t + W)$ , since  $\mathcal{Y}_O \geq W$ . Since  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is causal, time-invariant and  $W$ -resettable, by Corollary 3 (stated and proved later in the present write-up), we have  $\Xi_t \mathcal{T}(\chi) = \phi$ , which is a contradiction. Therefore, we have  $P_O(\mathbf{x}_J \sqcup \phi^m, \mathbf{x}_0') \neq \tau_O$  and by construction of  $O$ ,  $P_O(\mathbf{x}_J \sqcup \phi^m, \mathbf{x}_0') = 0$ , for all  $\mathbf{x}_J \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_O)}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0, \rho_O)}$ . This implies  $P_O(\phi^{m+1}, \phi) = 0$ , satisfying Axiom (3).

Finally, we wish to show that for every  $\chi \in \mathcal{F}_m$ ,  $\mathcal{T}(\chi)$  is consistent with  $\mathcal{T}_J(\chi) \sqcup \chi$  with respect to  $O$ . That is, we wish to show that for every  $\chi \in \mathcal{F}_m$  and for every  $t \in \mathbb{R}$ ,  $\Xi_0 \sigma_t(\mathcal{T}(\chi)) = \langle 0 \rangle$  if and only if  $P_O(\Xi_{(0, \mathcal{Y}_O)}(\sigma_t(\mathcal{T}_J(\chi) \sqcup \chi)), \Xi_{(0, \rho_O)}(\sigma_t(\mathcal{T}(\chi)))) = \tau_O$ . Consider arbitrary  $\chi \in \mathcal{F}_m$  and  $t \in \mathbb{R}$ . If  $\Xi_0 \sigma_t(\mathcal{T}(\chi)) = \langle 0 \rangle$ , then it is immediate from the construction of  $O$  that  $P_O(\Xi_{(0, \mathcal{Y}_O)}(\sigma_t(\mathcal{T}_J(\chi) \sqcup \chi)), \Xi_{(0, \rho_O)}(\sigma_t(\mathcal{T}(\chi)))) = \tau_O$ . To prove the converse, suppose  $\Xi_0 \sigma_t(\mathcal{T}(\chi)) \neq \langle 0 \rangle$ . Then, from the contrapositive of the ‘‘tie-breaker’’ condition, it follows that for all  $\tilde{\chi} \in \mathcal{F}_m$  and for all  $\tilde{t} \in \mathbb{R}$  with  $\Xi_{(0, \mathcal{Y}_O)}(\sigma_{\tilde{t}}(\mathcal{T}_J(\tilde{\chi}) \sqcup \tilde{\chi})) = \Xi_{(0, \mathcal{Y}_O)}(\sigma_{\tilde{t}}(\mathcal{T}_J(\chi) \sqcup \chi))$ , we have  $\Xi_0 \sigma_{\tilde{t}}(\mathcal{T}(\tilde{\chi})) = \Xi_0 \sigma_{\tilde{t}}(\mathcal{T}(\chi)) \neq \langle 0 \rangle$ . Therefore, from the construction, we have  $P_O(\Xi_{(0, \mathcal{Y}_O)}(\sigma_t(\mathcal{T}_J(\chi) \sqcup \chi)), \Xi_{(0, \rho_O)}(\sigma_t(\mathcal{T}(\chi)))) \neq \tau_O$ .  $\square$

*Proof of Proposition 3.* Assume that the hypothesis in the proposition is true. Let  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  be  $W'$ -Resettable for some  $W' \in \mathbb{R}^+$ . Set  $W = \max\{W', 12\alpha\}$ . One readily verifies that  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is also  $W$ -resettable.

We first show a construction for the neuron  $J$ , prove that it obeys all the axioms and then show that it has the property that there exists a  $U \in \mathbb{R}^+$  so that for all  $t_1, t_2 \in \mathbb{R}$  and  $\chi_1, \chi_2 \in \mathcal{F}_m$  with  $\Xi_0 \sigma_{t_1}(\mathcal{T}(\chi_1)) \neq \Xi_0 \sigma_{t_2}(\mathcal{T}(\chi_2))$ , we have  $\Xi_{(0, U)}(\sigma_{t_1}(\mathcal{T}_J(\chi_1) \sqcup \chi_1)) \neq \Xi_{(0, U)}(\sigma_{t_2}(\mathcal{T}_J(\chi_2) \sqcup \chi_2))$ , where  $\mathcal{T}_J : \mathcal{F}_m \rightarrow \mathcal{S}$  is such that for each  $\chi \in \mathcal{F}_m$ ,  $\mathcal{T}_J(\chi)$  is consistent with  $\chi$  with respect to  $J$ .

We first construct the neuron  $J\langle \alpha_J, \mathcal{Y}_J, \rho_J, \tau_J, \lambda_J, m_J, P_J : \bar{\mathcal{S}}_{(0, \mathcal{Y}_J)}^{m_J} \times \bar{\mathcal{S}}_{(0, \rho_J)} \rightarrow [\lambda_J, \tau_J] \rangle$ . Set  $\alpha_J = \alpha$ . Let  $p, q, r \in \mathbb{R}^+$ , with<sup>3</sup>  $p = 8\alpha, q = 2\alpha$  and  $r = \alpha$ . Set  $\mathcal{Y}_J = p + q + r + W$ ,  $\rho_J = 2p - r$  and  $m_J = m$ . Let  $\tau_J \in \mathbb{R}^+, \lambda_J \in \mathbb{R}^-$  be chosen arbitrarily. The function  $P_J : \bar{\mathcal{S}}_{(0, \mathcal{Y}_J)}^{m_J} \times \bar{\mathcal{S}}_{(0, \rho_J)} \rightarrow [\lambda_J, \tau_J]$  is constructed as follows.

For  $\chi \in \bar{\mathcal{S}}_{(0, \mathcal{Y}_J)}^{m_J}$  and  $\mathbf{x}_0 \in \bar{\mathcal{S}}_{(0, \rho_J)}$ , set  $P_J(\chi, \mathbf{x}_0) = \tau_J$  and  $P_J(\chi, \phi) = \tau_J$  if and only if one of the following is true; everywhere else, the function is set to zero.

1.  $\Xi_{(p, p+W)}\chi = \phi^{m_J}$ ,  $\Xi_p\chi \neq \phi^{m_J}$  and  $\Xi_{(0, p]}\mathbf{x}_0 = \phi$ .
2.  $\Xi_{(0, p+q]}\mathbf{x}_0 = \langle t \rangle$ , where  $q \leq t < (q + r)$  and  $(t - q) = \varepsilon(0, \Xi_{(0, p]}\sigma_t(\chi))$ . Moreover,  $\Xi_{(t+p, t+p+W)}\chi = \phi^{m_J}$  and  $\Xi_{(p+t)}\chi \neq \phi^{m_J}$ .

<sup>3</sup> The choice of values for  $p, q, r$  and  $W$  was made so as to satisfy the following inequalities, which we will need in the proof:  $p < W, p > 2(q + r)$  and  $q > \alpha$ .

3.  $\Xi_{(0,2p-(q+r))}\mathbf{x}_0 = \langle t_x, t_y \rangle$  with  $(p - (q + r)) < t_x \leq (p - q) \leq t_y = p$ . Also, for all  $t' \in [0, p]$ ,  $\Xi_{(t',t'+W)}\chi \neq \phi^{m_j}$ .
4.  $\Xi_{(0,2p-r)}\mathbf{x}_0 = \langle t, t_x, t_y \rangle$  with  $q \leq t < (q + r) < (p - r) \leq t_x \leq p \leq t_y = p + t$  and  $(t - q) = \varepsilon((t_y - t_x - q), \Xi_{(0,p)}\sigma_t(\chi))$ . Furthermore, for all  $t' \in [0, p + t]$ ,  $\Xi_{(t',t'+W)}\chi \neq \phi^{m_j}$ .

where  $\varepsilon : [0, r) \times \bar{\mathcal{S}}_{(0,p)}^{m_j} \rightarrow [0, r)$  is as defined below.

For convenience, we define an operator  $\iota_j^k : [0, 1) \rightarrow [0, 1)$ , for  $j, k \in \mathbb{Z}^+$ , that constructs a new number obtained by concatenating every  $i^{\text{th}}$  digit of a given number, where  $i \equiv j \pmod k$ . More formally, for  $x \in [0, 1)$ ,  $\iota_j^k(x) = \sum_{i=1}^{\infty} ((\lfloor x \times 10^{j+(i-1)k} \rfloor - 10 \lfloor x \times 10^{j+(i-1)k-1} \rfloor) \times 10^{-i})$ .

Also, we define another operator  $\zeta_k : [0, 1)^k \rightarrow [0, 1)$ , for  $k \in \mathbb{Z}^+$  which ‘‘interleaves’’ the digits of  $k$  given numbers in order to produce a new number. More formally, for  $x_0, x_1, \dots, x_{k-1} \in [0, 1)$ ,  $\zeta_k(x_0, x_1, \dots, x_{k-1}) = \sum_{i=0}^{\infty} ((\lfloor x_{k(\frac{i}{k} - \lfloor \frac{i}{k} \rfloor)} \times 10^{1+\lfloor \frac{i}{k} \rfloor} \rfloor - 10 \lfloor x_{k(\frac{i}{k} - \lfloor \frac{i}{k} \rfloor)} \times 10^{\lfloor \frac{i}{k} \rfloor} \rfloor) \times 10^{-(i+1)})$ .

Let  $d$  be the largest integer so that, for all  $x' \in [0, r)$ , we have  $x' \times 10^d < 1$ . For  $x' \in [0, r)$ , let  $x = x' \times 10^d$ . For  $\chi \in \bar{\mathcal{S}}_{(0,p)}^{m_j}$ , define<sup>4</sup>  $\varepsilon(x', \chi) = 10^{-d} \times \zeta_{m_j}(\varepsilon_0(\iota_1^{m_j}(x), \Pi_1(\chi)), \varepsilon_0(\iota_2^{m_j}(x), \Pi_2(\chi)), \dots, \varepsilon_0(\iota_{m_j}^{m_j}(x), \Pi_{m_j}(\chi)))$ , where  $\varepsilon_0 : [0, 1) \times \bar{\mathcal{S}}_{(0,p)} \rightarrow [0, 1)$  is as defined below.

Let  $n \in [0, 1)$  and  $\mathbf{x} \in \bar{\mathcal{S}}_{(0,p)}$ . Furthermore, let  $c = \iota_1^2(n)$  and  $s = \iota_2^2(n)$ . Let  $\mathbf{x} = \langle x^1, x^2, \dots, x^k \rangle$ . We have  $0 \leq k \leq 8$ , because  $p = 8\alpha$ . Also, since  $p = 8r$ , we have  $x^i \times 10^{d-1} < 1$ , for  $1 \leq i \leq k$ . Let  $s' = \zeta_{k+1}(x^1 \times 10^{d-1}, x^2 \times 10^{d-1}, \dots, x^k \times 10^{d-1}, s)$ . If  $c = 0$ , then let  $c' = \frac{k}{10} + 0.09$  else let  $c' = \frac{k}{10} + \frac{c}{10}$ . Finally, define  $\varepsilon_0(n, \mathbf{x}) = \zeta_2(c', s')$ .

Next, we show that  $\mathbf{J}$  satisfies all the axioms of the neuron.

It is immediate that  $\mathbf{J}$  satisfies Axiom (1), since all output spikes in the above construction are at least  $q$  milliseconds apart, and  $q = 2\alpha$ .

We now prove that  $\mathbf{J}$  satisfies Axiom (2). Let  $\chi' \in \bar{\mathcal{S}}_{(0,r_j)}^{m_j}$  and  $\mathbf{x}_0' \in \bar{\mathcal{S}}_{(0,\rho_j)}$  be arbitrary. If  $P_{\mathbf{J}}(\chi', \mathbf{x}_0') = \tau_{\mathbf{J}}$ , then it is immediate from the construction that  $P_{\mathbf{J}}(\chi', \phi) = \tau_{\mathbf{J}}$  which satisfies Axiom (2). On the contrary, if  $P_{\mathbf{J}}(\chi', \mathbf{x}_0') \neq \tau_{\mathbf{J}}$ , from the construction of  $\mathbf{J}$ , we have  $P_{\mathbf{J}}(\chi', \mathbf{x}_0') = 0$ . Also, from the construction we have either  $P_{\mathbf{J}}(\chi', \phi) = 0$  or  $P_{\mathbf{J}}(\chi', \phi) = \tau_{\mathbf{J}}$ . Axiom (2) is satisfied in either case.

Also,  $\mathbf{J}$  satisfies Axiom (3), since it is clear that  $\chi = \phi^{m_j}$  does not satisfy any of the conditions enumerated above. We therefore have  $P_{\mathbf{J}}(\phi^{m_j}, \phi) = 0$ .

Finally, we show that there exists a  $U \in \mathbb{R}^+$  so that for all  $t_1, t_2 \in \mathbb{R}$  and  $\chi_1, \chi_2 \in \mathcal{F}_m$  with  $\Xi_0\sigma_{t_1}(\mathcal{T}(\chi_1)) \neq \Xi_0\sigma_{t_2}(\mathcal{T}(\chi_2))$ , we have  $\Xi_{(0,U)}(\sigma_{t_1}(\mathcal{T}_{\mathbf{J}}(\chi_1) \sqcup \chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}(\mathcal{T}_{\mathbf{J}}(\chi_2) \sqcup \chi_2))$ , where  $\mathcal{T}_{\mathbf{J}} : \mathcal{F}_m \rightarrow \mathcal{S}$  such that for each  $\chi \in \mathcal{F}_m$ ,  $\mathcal{T}_{\mathbf{J}}(\chi)$  is consistent with  $\chi$  with respect to  $\mathbf{J}$ . Let  $U = p + q + r + W$ . Assume  $\Xi_0\sigma_{t_1}(\mathcal{T}(\chi_1)) \neq \Xi_0\sigma_{t_2}(\mathcal{T}(\chi_2))$ . Now, suppose  $\Xi_{(0,0+W)}\sigma_{t_1}(\chi_1) = \phi^m$ , then clearly  $\Xi_{(0,0+W)}\sigma_{t_2}(\chi_2) \neq \phi^m$ , otherwise  $\mathcal{T}(\cdot)$  produces no spike at times  $t_1$  and  $t_2$  respectively on receiving  $\chi_1$  and  $\chi_2$ , by Corollary 3. As a result,  $\Xi_{(0,U)}\sigma_{t_1}(\chi_1) \neq \Xi_{(0,U)}\sigma_{t_2}(\chi_2)$ , which implies the required result. Otherwise, from Proposition 5, it follows that there exist  $V_1, V_2 \in \mathbb{R}^+$  so that  $\Xi_{(0,V_1]}(\sigma_{t_1}(\chi_1)) \neq \Xi_{(0,V_2]}(\sigma_{t_2}(\chi_2))$ . If  $\Xi_{(0,U)}(\sigma_{t_1}(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}(\chi_2))$ , it is immediate that  $\Xi_{(0,U)}(\sigma_{t_1}(\mathcal{T}_{\mathbf{J}}(\chi_1) \sqcup \chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}(\mathcal{T}_{\mathbf{J}}(\chi_2) \sqcup \chi_2))$ . It therefore suffices to prove that if  $\Xi_{[U,V_1]}(\sigma_{t_1}(\chi_1)) \neq \Xi_{[U,V_2]}(\sigma_{t_2}(\chi_2))$  then  $\Xi_{(0,U)}(\sigma_{t_1}\mathcal{T}_{\mathbf{J}}(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}\mathcal{T}_{\mathbf{J}}(\chi_2))$ . Proposition 5 implies that

<sup>4</sup> Recall that the projection operator for spike-train ensembles is defined as  $\Pi_i(\chi) = \mathbf{x}_i$ , for  $1 \leq i \leq m$ , where  $\chi = \langle \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m \rangle$ .

$\Xi_{(V_1, V_1+W)}(\sigma_{t_1}(\chi_1)) = \phi^m$  and  $\Xi_{V_1}(\sigma_{t_1}(\chi_1)) \neq \phi^m$ . Therefore, by Case (1) of the construction,  $\Xi_{(V_1-p)\sigma_{t_1}}\mathcal{T}_J(\chi_1) = \langle V_1 - p \rangle$ . Moreover, since Proposition 5 implies that for all  $t'_1 \in [0, V_1)$ ,  $\Xi_{(t'_1, t'_1+W)}(\sigma_{t_1}(\chi_1)) \neq \phi^m$ , from Case (3) of the construction, we have that for every  $k \in \mathbb{Z}^+$  with  $V_1 - kp > 0$ ,  $\Xi_{(V_1-kp)\sigma_{t_1}}\mathcal{T}_J(\chi_1) = \langle V_1 - kp \rangle$ . Let  $k_1$  be<sup>5</sup> the smallest positive integer, so that  $V_1 - k_1p < U$ . From the previous arguments, we have  $\Xi_{(V_1-k_1p)\sigma_{t_1}}\mathcal{T}_J(\chi_1) = \langle V_1 - k_1p \rangle$ . Also, it is easy to see that  $V_1 - k_1p \geq (q+r)$ . Let  $k_2$  be similarly defined with respect to  $\chi_2$  so that  $\Xi_{(V_2-k_2p)\sigma_{t_2}}\mathcal{T}_J(\chi_2) = \langle V_2 - k_2p \rangle$  and  $V_2 - k_2p < U$ . Now, there are two cases:

1. If  $V_1 - k_1p \neq V_2 - k_2p$ , we now show that  $\Xi_{(0,U)}(\sigma_{t_1}\mathcal{T}_J(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}\mathcal{T}_J(\chi_2))$ , which is the required result. Assume, without loss of generality, that  $V_1 - k_1p < V_2 - k_2p$ . If these two quantities are less than  $p - r$  apart, we have  $\Xi_{(0,U)}(\sigma_{t_1}\mathcal{T}_J(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}\mathcal{T}_J(\chi_2))$ , because by Case (4) of the construction  $\mathcal{T}_J(\chi_1)$  has a spike in the interval  $(V_1 - k_1p - (q+r), V_1 - k_1p - q]$  and by Case (3) of the construction,  $\mathcal{T}_J(\chi_2)$  has no spike in the interval  $(V_2 - k_2p, V_2 - k_2p + p - (q+r))$ . In other words, the spike following the one at  $V_1 - k_1p$  in  $\mathcal{T}_J(\chi_1)$  has no counterpart in  $\mathcal{T}_J(\chi_2)$ . On the other hand, if they are less than  $p$  apart but at most  $p - r$  apart, by similar arguments, it is easy to show that the spike at  $V_2 - k_2p$  in  $\mathcal{T}_J(\chi_2)$  has no counterpart in  $\mathcal{T}_J(\chi_1)$ . Finally, if they are at least  $p$  apart, then  $k_2$  does not satisfy the property that it is the smallest positive integer, so that  $V_2 - k_2p \leq U$ , which is a contradiction.
2. On the contrary, consider the case when  $V_1 - k_1p = V_2 - k_2p$ . We have two cases:
  - (a) Suppose  $k_1 \neq k_2$ . Let  $t'_1$  be the largest positive integer so that  $\Xi_{t'_1}\sigma_{t_1}\mathcal{T}_J(\chi_1) = \langle t'_1 \rangle$  and  $t'_1 < V_1 - k_1p$ . From Case (4) of the construction, we have that  $q \leq (V_1 - k_1p) - t'_1 \leq q + r$ . Let  $t'_2$  be defined likewise, with respect to  $\chi_2$ . Further, let  $n'_1 = (V_1 - k_1p) - t'_1 - q$  and  $n'_2 = (V_2 - k_2p) - t'_2 - q$  and  $n_1 = n'_1 \times 10^d$  and  $n_2 = n'_2 \times 10^d$ . Since  $k_1 \neq k_2$ , it is straightforward to verify that for all  $j$  with  $1 \leq j \leq m_j$ ,  $\iota_1^2(\iota_j^{m_j}(n_1)) \neq \iota_1^2(\iota_j^{m_j}(n_2))$ , for the former number has 9 in the  $(k_1 + 1)^{th}$  decimal place, while the latter number does in the  $(k_2 + 1)^{th}$  decimal place and not in the  $(k_1 + 1)^{th}$  decimal place since  $k_1 \neq k_2$ . Therefore,  $n_1 \neq n_2$  and consequently  $t'_1 \neq t'_2$  which gives us  $\Xi_{(0,U)}(\sigma_{t_1}\mathcal{T}_J(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}\mathcal{T}_J(\chi_2))$ , which is the required result.
  - (b) On the other hand, suppose  $k_1 = k_2$ . Again, we have two cases:
    - i. Suppose, there exists a  $j$  with  $1 \leq j \leq m_j$  and a  $k' \leq k_1$ , so that  $\Xi_{(V_1-k'p, V_1-(k'-1)p]}\Pi_j(\sigma_{t_1}(\chi_1))$  has a different number of spikes when compared to  $\Xi_{(V_2-k'p, V_2-(k'-1)p]}\Pi_j(\sigma_{t_2}(\chi_2))$ . Let  $n_1, n_2$  be defined as before. It is straightforward to verify that  $\iota_1^2(\iota_j^{m_j}(n_1)) \neq \iota_1^2(\iota_j^{m_j}(n_2))$ , because they differ in the  $(k_1 - k' + 1)^{th}$  decimal place<sup>6</sup>. Therefore,  $\Xi_{(0,U)}(\sigma_{t_1}\mathcal{T}_J(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2}\mathcal{T}_J(\chi_2))$ .
    - ii. Now consider the case where for all  $j$  with  $1 \leq j \leq m_j$  and  $k' \leq k_1$ , we have  $\Xi_{(V_1-k'p, V_1-(k'-1)p]}\Pi_j(\sigma_{t_1}(\chi_1))$  have the same number of spikes when compared to  $\Xi_{(V_2-k'p, V_2-(k'-1)p]}\Pi_j(\sigma_{t_2}(\chi_2))$ . Now, by hypothesis, we have  $\Xi_{[U, V_1]}(\sigma_{t_1}(\chi_1)) \neq \Xi_{[U, V_2]}(\sigma_{t_2}(\chi_2))$ . Therefore there must exist a  $1 \leq j \leq m_j$  and  $k' \leq k_1$ , so that there is a point in time where one of the spike-trains  $\Xi_{(V_1-k'p, V_1-(k'-1)p]}\Pi_j(\sigma_{t_1}(\chi_1))$  and  $\Xi_{(V_2-k'p, V_2-(k'-1)p]}\Pi_j(\sigma_{t_2}(\chi_2))$  has a spike, while the other does not. Let  $t'$

<sup>5</sup>  $k_1$  exists because  $U > p$ .

<sup>6</sup> Which in  $n_1$  and  $n_2$  encodes the number of spikes in the interval  $(V_2 - k'p, V_2 - (k' - 1)p]$  on the  $j^{th}$  spike-train of  $\chi_1$  and  $\chi_2$  respectively.

be the latest time instant at which this is so. Also, assume without loss of generality that  $\Xi_{(V_1-k'p, V_1-(k'-1)p]} \Pi_j(\sigma_{t_1}(\chi_1)) = \langle x^1, \dots, x^q \rangle$  has a spike at time instant  $t'$  while  $\Xi_{(V_2-k'p, V_2-(k'-1)p]} \Pi_j(\sigma_{t_2}(\chi_2))$  does not. Let  $p$  be the number so that  $t' = x^p$ . Let  $n_1, n_2$  be defined as before. Also, for each  $h$  with  $1 \leq h \leq k_1$ , let  $r_h$  be the number of spikes in  $\Xi_{(V_1-hp, V_1-(h-1)p]} \Pi_j(\sigma_{t_1}(\chi_1))$ . Each  $r_h$  can be determined from  $n_1$ . Then, it is straightforward to verify<sup>7</sup> that  $\ell_p^{r_1} \ell_{r_1-1}^{r_2} \dots \ell_{r_1-1}^{r_1} \ell_2^{m_1} n_1 \neq \ell_p^{r_1} \ell_{r_1-1}^{r_2} \dots \ell_{r_1-1}^{r_1} \ell_2^{m_1} n_2$ . Therefore,  $n_1 \neq n_2$  and it follows that  $\Xi_{(0,U)}(\sigma_{t_1} \mathcal{T}_J(\chi_1)) \neq \Xi_{(0,U)}(\sigma_{t_2} \mathcal{T}_J(\chi_2))$ .

□

*Some auxiliary propositions used in the proofs of Propositions 2 and 3*

**Proposition 4.** *If  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is time-invariant, then  $\mathcal{T}(\phi^m) = \phi$ .*

*Proof.* For the sake of contradiction, suppose  $\mathcal{T}(\phi^m) = \mathbf{x}_0$ , where  $\mathbf{x}_0 \neq \phi$ . That is, there exists a  $t \in \mathbb{R}$  with  $\Xi_t \mathbf{x}_0 = \langle t \rangle$ . Let  $\delta < \alpha$ . Clearly,  $\sigma_\delta(\phi^m) = \phi^m \in \mathcal{F}_m$ . Since  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is time-invariant,  $\mathcal{T}(\sigma_\delta(\phi^m)) = \sigma_\delta(\mathcal{T}(\phi^m)) = \sigma_\delta(\mathbf{x}_0)$ . Now,  $\sigma_\delta(\mathbf{x}_0) \neq \mathbf{x}_0$  since  $\Xi_{(t-\delta)} \sigma_\delta(\mathbf{x}_0) = \langle t - \delta \rangle$  whereas  $\Xi_{(t-\delta)} \mathbf{x}_0 = \phi$ , for otherwise  $\mathbf{x}_0 \notin \mathcal{S}$ . This is a contradiction. Therefore,  $\mathcal{T}(\phi^m) = \phi$ .

□

**Corollary 3.** *Let  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  be causal, time-invariant and  $W$ -resettable, for some  $W \in \mathbb{R}^+$ . If  $\chi \in \mathcal{F}_m$  has a gap in the interval  $(t, t+W)$ , then  $\Xi_t \mathcal{T}(\chi) = \phi$ .*

*Proof.* Assume the hypothesis of the above statement. One readily sees that  $\Xi_t \mathcal{T}(\chi) = \Xi_{[t, \infty)} \Xi_{(-\infty, t]} \mathcal{T}(\chi)$ . Now, since  $\chi$  has a gap in the interval  $(t, t+W)$  and  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is  $W$ -resettable, we have  $\Xi_{[t, \infty)} \Xi_{(-\infty, t]} \mathcal{T}(\chi) = \Xi_{[t, \infty)} \mathcal{T}(\Xi_{(-\infty, t]} \chi)$ . Further, by definition,  $\Xi_{(t, \infty)} \Xi_{(-\infty, t]} \chi = \Xi_{(t, \infty)} \phi^m$ . Therefore, since  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is causal, it follows that  $\Xi_{[t, \infty)} \mathcal{T}(\Xi_{(-\infty, t]} \chi) = \Xi_{[t, \infty)} \mathcal{T}(\phi^m) = \phi$ , with the last equality following from the previous proposition. Thus, we have  $\Xi_t \mathcal{T}(\chi) = \phi$ .

□

**Proposition 5.** *Let  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  be causal, time-invariant and  $W'$ -resettable, for some  $W' \in \mathbb{R}^+$ . Then for all  $W \in \mathbb{R}^+$  with  $W \geq W'$ ,  $t_1, t_2 \in \mathbb{R}$  and  $\chi_1, \chi_2 \in \mathcal{F}_m$  with  $\Xi_0 \sigma_{t_1}(\mathcal{T}(\chi_1)) \neq \Xi_0 \sigma_{t_2}(\mathcal{T}(\chi_2))$ , where  $\Xi_{(0, 0+W)} \sigma_{t_1}(\chi_1) \neq \phi^m \neq \Xi_{(0, 0+W)} \sigma_{t_2}(\chi_2)$ , there exist  $V_1, V_2 \in \mathbb{R}^+$  so that the following are true.*

1.  $\Xi_{(0, V_1)}(\sigma_{t_1}(\chi_1)) \neq \Xi_{(0, V_2)}(\sigma_{t_2}(\chi_2))$
2.  $\Xi_{(V_1, V_1+W)}(\sigma_{t_1}(\chi_1)) = \phi^m$ ,  $\Xi_{V_1}(\sigma_{t_1}(\chi_1)) \neq \phi^m$  and  $\Xi_{(V_2, V_2+W)}(\sigma_{t_2}(\chi_2)) = \phi^m$ ,  $\Xi_{V_2}(\sigma_{t_2}(\chi_2)) \neq \phi^m$
3. For all  $t'_1 \in [0, V_1)$ ,  $\Xi_{(t'_1, t'_1+W)}(\sigma_{t_1}(\chi_1)) \neq \phi^m$  and for all  $t'_2 \in [0, V_2)$ ,  $\Xi_{(t'_2, t'_2+W)}(\sigma_{t_2}(\chi_2)) \neq \phi^m$ .

*Proof.* Since  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is causal, we have  $\Xi_{[t_1, \infty)} \mathcal{T}(\chi_1) = \Xi_{[t_1, \infty)} \mathcal{T}(\Xi_{(t_1, \infty)} \chi_1)$ . This implies  $\sigma_{t_1}(\Xi_{[t_1, \infty)} \mathcal{T}(\chi_1)) = \sigma_{t_1}(\Xi_{[t_1, \infty)} \mathcal{T}(\Xi_{(t_1, \infty)} \chi_1))$  which gives us  $\Xi_{[0, \infty)} \sigma_{t_1}(\mathcal{T}(\chi_1)) = \Xi_{[0, \infty)} \sigma_{t_1}(\mathcal{T}(\Xi_{(t_1, \infty)} \chi_1))$ . Since  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is time-invariant and  $\sigma_{t_1}(\Xi_{(t_1, \infty)} \chi_1) = \Xi_{(0, \infty)} \sigma_{t_1}(\chi_1) \in \mathcal{F}_m$ , we have

<sup>7</sup> The expression on either side of the inequality is a real number that encodes for the  $p^{\text{th}}$  spike time in the spike-trains  $\Xi_{(V_1-k'p, V_1-(k'-1)p]} \Pi_j(\sigma_{t_1}(\chi_1))$  and  $\Xi_{(V_2-k'p, V_2-(k'-1)p]} \Pi_j(\sigma_{t_2}(\chi_2))$  respectively.

$\Xi_{[0,\infty)}\sigma_{t_1}(\mathcal{T}(\Xi_{(t_1,\infty)}\chi_1)) = \Xi_{[0,\infty)}\mathcal{T}(\Xi_{(0,\infty)}\sigma_{t_1}(\chi_1))$ . In short,  
 $\Xi_{[0,\infty)}\sigma_{t_1}(\mathcal{T}(\chi_1)) = \Xi_{[0,\infty)}\mathcal{T}(\Xi_{(0,\infty)}\sigma_{t_1}(\chi_1))$  which implies  
 $\Xi_0\sigma_{t_1}(\mathcal{T}(\chi_1)) = \Xi_0\mathcal{T}(\Xi_{(0,\infty)}\sigma_{t_1}(\chi_1))$ . Similarly,  $\Xi_0\sigma_{t_2}(\mathcal{T}(\chi_2)) = \Xi_0\mathcal{T}(\Xi_{(0,\infty)}\sigma_{t_2}(\chi_2))$ . Therefore, it follows from the hypothesis that  $\Xi_0\mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1))) \neq \Xi_0\mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_2}(\chi_2)))$ .

Let  $V_1, V_2 \in \mathbb{R}^+$  be the smallest positive real numbers so that  $\Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1))$  and  $\Xi_{(0,\infty)}(\sigma_{t_2}(\chi_2))$  have gaps in the intervals  $(V_1, V_1 + W)$  and  $(V_2, V_2 + W)$  respectively. That such  $V_1, V_2$  exist follows from the fact that  $\chi_1, \chi_2 \in \mathcal{F}_m$ . Since,  $\mathcal{T} : \mathcal{F}_m \rightarrow \mathcal{S}$  is  $W'$ -resettable, it is also  $W$ -resettable for  $W \geq W'$ . It therefore follows that  $\Xi_{(-\infty, V_1]} \mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1))) = \mathcal{T}(\Xi_{(-\infty, V_1]} \Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1)))$  which equals  $\mathcal{T}(\Xi_{(0, V_1]}(\sigma_{t_1}(\chi_1)))$ . This implies that  $\Xi_0 \Xi_{(-\infty, V_1]} \mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1))) = \Xi_0 \mathcal{T}(\Xi_{(0, V_1]}(\sigma_{t_1}(\chi_1)))$  due to which we have  $\Xi_0 \mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_1}(\chi_1))) = \Xi_0 \mathcal{T}(\Xi_{(0, V_1]}(\sigma_{t_1}(\chi_1)))$ . Likewise,  
 $\Xi_0 \mathcal{T}(\Xi_{(0,\infty)}(\sigma_{t_2}(\chi_2))) = \Xi_0 \mathcal{T}(\Xi_{(0, V_2]}(\sigma_{t_2}(\chi_2)))$ . We therefore have  $\Xi_0 \mathcal{T}(\Xi_{(0, V_1]}(\sigma_{t_1}(\chi_1))) \neq \Xi_0 \mathcal{T}(\Xi_{(0, V_2]}(\sigma_{t_2}(\chi_2)))$ . This readily implies  $\Xi_{(0, V_1]}(\sigma_{t_1}(\chi_1)) \neq \Xi_{(0, V_2]}(\sigma_{t_2}(\chi_2))$  and, from the construction, it follows that  $\Xi_{(V_1, V_1+W)}(\sigma_{t_1}(\chi_1)) = \phi^m$ ,  $\Xi_{V_1}(\sigma_{t_1}(\chi_1)) \neq \phi^m$  and  $\Xi_{(V_2, V_2+W)}(\sigma_{t_2}(\chi_2)) = \phi^m$ ,  $\Xi_{V_2}(\sigma_{t_2}(\chi_2)) \neq \phi^m$ , for otherwise  $V_1$  or  $V_2$  would not be the smallest choice of numbers with the said property. Furthermore, for the same reasons, for all  $t'_1 \in [0, V_1)$ ,  $\Xi_{(t'_1, t'_1+W)}(\sigma_{t_1}(\chi_1)) \neq \phi^m$  and for all  $t'_2 \in [0, V_2)$ ,  $\Xi_{(t'_2, t'_2+W)}(\sigma_{t_2}(\chi_2)) \neq \phi^m$ .  $\square$

## References

Gerstner W, Kistler W (2002) Spiking neuron models: Single neurons, populations, plasticity. Cambridge University Press, New York, NY