Solutions to Homework 3

March 17, 2010

Bertsekas 3.3.1: The objective function is $(x-a)^2 + (y-b)^2 + xy$. The Hessian matrix corresponding to this objective function is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ which is convex. Consequently, the global minimum occurs in the interior of the set $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ or on the boundary and this depends on the values of (a, b). Taking derivatives w.r.t. x and y and setting them to zero, we get

$$\begin{aligned} 2(x-a) + y &= 0, \\ 2(y-b) + x &= 0. \end{aligned}$$

The solution is

$$x = \frac{4a-2b}{3}, y = \frac{4b-2a}{3}.$$

Since $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$, $x = 0$ for $a \leq \frac{b}{2}$, $y = 0$ for $b \leq \frac{a}{2}$, $x = 1$ for $a \geq \frac{2b+3}{4}$ and $y = 1$ for $b \geq \frac{2a+3}{4}$.

Bertsekas 3.3.2: We seek to maximize

$$y'x$$

subject to

$$x'Qx \leq 1.$$

Using the method of Lagrange parameters, we get

$$-y + 2\lambda Q x^* = 0$$

which yields

$$x^* = \frac{Q^{-1}y}{2\lambda}.$$

Using the constraint $x'Qx \leq 1$, we get

$$x^{*'} Q x^* = \frac{y'Q^{-1}y}{4\lambda^2} \leq 1$$

which in turn gives $2\lambda \geq \sqrt{y'Q^{-1}y}$. The objective $y'x$ at the optimum x^* becomes $\frac{y'Q^{-1}y}{2\lambda} \leq \sqrt{y'Q^{-1}y}$. Also $(y'x^*)^2 = \frac{(y'Q^{-1}y)^2}{4\lambda^2} = (x^{*'}Qx^*)(y'Q^{-1}y)$ which gives the second inequality. Furthermore since Q is positive definite, $(y'x)^2 = \left[(Q^{-1}y)'(Qx)\right]^2 \leq \left(y'Q^{-1}y\right)(x'Qx)$.

Bertsekas 3.4.2: We seek to maximize

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i}$$

subject to the constraints $\sum_{i=1}^{n} x_i = A$ and $x_i \geq 0$. Setting up the equality constraint via a Lagrange parameter λ, we get

$$\frac{-p_i(s_i + x_i) + p_i x_i + \lambda(s_i + x_i)^2}{(s_i + x_i)^2} = 0$$

from which we get the solution to be

$$x_i = s_i \left(\sqrt{\frac{p_i}{s_i} \frac{1}{\lambda}} - 1 \right).$$

When $\frac{p_i}{s_i} \leq \lambda$, $x_i \leq 0$ which is not permitted. Therefore for all $i = \{m+1, \ldots, n\}$ $x_i = 0$. The index m corresponds to that i for which $\frac{p_i}{s_i} \geq \lambda$.

Bertsekas 3.4.4: Setting up the problem via Lagrange parameters yields the dual function

$$\begin{aligned} q(\mu, \nu) &= \inf_{x \geq 0} \sum_{ij} a_{ij} x_{ij} - \sum_i \mu_i \left(\sum_j x_{ij} - \alpha_i \right) - \sum_j \nu_j \left(\sum_i x_{ij} - \beta_j \right) \\ &= \inf_{x \geq 0} \sum_{ij} \left(a_{ij} - \mu_i - \nu_j \right) x_{ij} + \sum_i \alpha_i \mu_i + \sum_j \beta_j \nu_j \end{aligned}$$

We can make x_{ij} arbitrarily large if $a_{ij} - \mu_i - \nu_j < 0$. Consequently, the dual is

$$\max_{\mu, \nu} \sum_i \alpha_i \mu_i + \sum_j \beta_j \nu_j$$

subject to $a_{ij} - \mu_i - \nu_j \geq 0$. This implies that $\mu_i \leq a_{ij} - \nu_j$ for all j. Again this implies that $\mu_i \leq \min_j (a_{ij} - \nu_j)$. Therefore there is a point $\mu_i^* = -\max_j (\nu_j^* - a_{ij})$ for which $x_{ij} > 0$. The dual problem works in the same manner. You just have to use common sense and work out the situations for which $x_{ij} > 0$ and what it corresponds to in the dual. Please see the solution at `http://www.athenasc.com/nlpsol3.pdf` for more.

Bertsekas 5.5.5: Please see the solution at `http://www.athenasc.com/nlpsol5.pdf`. It is a very difficult problem.

Bertsekas 5.5.8: (a) Let us assume that one extreme point of S is not a permutation matrix. Assume the simplest case wherein two entries are not zero or one. From this, we know that there exists at least two entries on a row (or column) s_{ij} and s_{ik} such that s_{ij} and s_{kj} are greater than zero and sum to one. However, we can continuously modify s_{ij} and s_{kj} such that s_{ij} goes to zero and s_{kj} to one or vice versa. This shows that the matrix is not an extreme point. A more formal proof involves writing this matrix as the convex combination of two permutation matrices one of which has $(s_{ij}, s_{kj}) = (1, 0)$ and the other $(s_{ij}, s_{kj}) = (0, 1)$. (b) Use the result of problem 5.5.5 (c) We have already shown that the $n!$ vertices of this polytope are permutation matrices. Consequently, since every point in the interior and on the convex hull (excluding the vertices) is doubly stochastic, it is easy to see that each such doubly stochastic matrix can be written as a convex combination of two or more permutation matrices.

Bertsekas 5.5.14: The 0-1 linear-integer program is converted into the nonlinear program

$$a'x + c \sum_{i=1}^{n} x_i (1 - x_i)$$

subject to

$$Ax \leq b$$

with $0 \leq x_i \leq 1$ for all $i \in \{1, \ldots, n\}$. The Hessian of this objective function is $H_{ij} = -2c\delta_{ij}$ where δ_{ij} is the Kronecker delta function. Clearly, we can make the Hessian increasingly negative definite as c is increased. Consequently, the objective function becomes more and more concave. Despite this, the constraints ensure that $0 \leq x_i \leq 1$. For a concave objective function, the minima are going to lie on the vertices of the polyhedral set and this ensures that the solutions are integers.

Bertsekas 5.5.15: Here, the new problem is to minimize

$$x'Qx + a'x + c \sum_{i=1}^{n} x_i (1 - x_i)$$

subject to $0 \leq x_i \leq 1$ for all $i \in \{1, \ldots, n\}$. The Hessian of this objective function is $H = 2Q - 2cI$ which is negative definite if c is greater than the smallest eigenvalue of H (largest in magnitude but smallest in value by virtue of being negative). If the Hessian is negative definite, the objective function is concave and by the argument in the preceding question, the minima occur on the vertices of the polyhedral set and this ensures that the solutions are integers.