Solutions to Homework 2

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Bertsekas 3.1.1: (a)

$$x'x + \lambda(x'e - 1)$$

where e is the vector of all ones. The necessary conditions are

$$\begin{aligned} 2x + \lambda e &= 0 \\ \Rightarrow \quad \lambda &= -2/n. \end{aligned}$$

This gives us $x = e/n$.

(b)

$$\sum_{i=1}^{n} x_i + \lambda(x'x - 1).$$

The necessary conditions are

$$\begin{aligned} e + 2\lambda x &= 0 \\ \Rightarrow \quad x = & -e/(2\lambda) \end{aligned}$$

From the constraint $x'x = 1$ we get $\lambda = n/2$ and that $x = -e/n$.

(c)

$$x'x + \lambda(x'Qx - 1).$$

The necessary conditions are

$$\begin{aligned} 2x + 2\lambda Qx &= 0 \\ (Q + \frac{1}{\lambda}I)x &= 0. \end{aligned}$$

From the constraint $x'Qx = 1$, we get $x'x = -\lambda$ which implies that $\|x\| = \sqrt{-\lambda}$. For $\sqrt{-\lambda}$ to be the smallest possible value, x is a scaled eigenvector of Q corresponding to its largest eigenvalue $\alpha = -\frac{1}{\lambda}$ with magnitude $\sqrt{\frac{1}{\alpha}} = \sqrt{-\lambda}$.

Bertsekas 3.1.2: The constrained surface area is

$$2x_1x_2 + 2x_2x_3 + 2x_1x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 3).$$

The necessary conditions are

$$\begin{aligned} 2x_2 + 2x_3 + 2\lambda x_1 &= 0 \quad (1) \\ 2x_1 + 2x_3 + 2\lambda x_2 &= 0 \quad (2) \\ 2x_1 + 2x_2 + 2\lambda x_3 &= 0. \quad (3) \end{aligned}$$

Subtracting (2) from (1), we get $x_1 = x_2$ and by repeating this process, we get $x_1 = x_2 = x_3$. Since we artificially picked the square of the diagonal to be of length 3, we get a unit cube as the solution.

The constrained perimeter is

$$4x_1 + 4x_2 + 4x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 3).$$

From the necessary conditions, we get $x_1 = x_2 = x_3 = -2/\lambda$. Once again, we get a unit cube.

Bertsekas 3.1.5: This is basically principal component analysis (PCA). In the first problem, we minimize $x'Qx$ subject to $x'x = 1$ to get the smallest eigenvalue λ_1 and eigenvector $x = e_1$. For the second problem, we get

$$x'Qx + \mu(e_1'x) + \gamma(x'x - 1)$$

for which the necessary conditions are

$$2(Q + \gamma I)x = -\mu e_1.$$

From the constraint $e_1'x = 0$, we get that γ must be equal to the negative of one of the eigenvalues of Q and that it cannot be λ_1. From further analysis, we see that x must be an eigenvector which is orthogonal to e_1. This analysis can be repeated.

Bertsekas 3.1.8: The constrained problem is

$$\sin x \sin y \sin z + \lambda(x + y + z - \pi)$$

The necessary conditions are

$$\begin{aligned} \cos x \sin y \sin z + \lambda &= 0 \\ \sin x \cos y \sin z + \lambda &= 0 \\ \sin x \sin y \cos z + \lambda &= 0. \end{aligned}$$

Assuming $\lambda \neq 0$ and dividing, we get $\tan x = \tan y = \tan z$ which implies that we have an equilateral triangle.

Bertsekas 3.1.9: (a) The problem is

$$x + y + z + \lambda(xyz - \rho^2(x + y + z)).$$

From the necessary conditions, we get $\lambda yz = \lambda xy = \lambda xz = \rho^2 - 1$. Once again this gives us $x = y = z$.

(b) Let the equation of the line joining a and b be $\alpha x + \beta y + \gamma = 0$. The squared distance of point x from the line is $\frac{(\alpha x+\beta y+\gamma)^2}{\alpha^2+\beta^2}$ where we have taken the coordinates to be (x, y). Setting up a constrained optimization problem, we get

$$\frac{(\alpha x + \beta y + \gamma)^2}{\alpha^2 + \beta^2} + \lambda(x^2 + y^2 - 1).$$

The necessary conditions give us (after some algebra), $x = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $y = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$. If the origin (center of the circle) lies on a line connecting x and the straight line, then the equation of that line has to be $\frac{\beta}{\sqrt{\alpha^2+\beta^2}}x - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}y = 0$ which is perpendicular to the line defined by $\alpha x + \beta y + \gamma = 0$.

(c) Since the points a and b are free, they can be constrained to lie on the circle. In that case, the triangle found in (b) will be equilateral.

Bertsekas 3.1.13: The problem is

$$x'y + \mu(x'x - 1) + \nu(y'y - 1).$$

The necessary conditions are $y+2\mu x = 0$ and $x+2\nu y = 0$. Using the constraints, we get $x'y = -2\mu = -2\nu$ and that $4\mu\nu = 1$ (by solving for y and substituting in the x equation.) Therefore $(x'y)^2 = 1$ which leads to the Schwartz inequality.