3. An interesting surface called the *Moebius strip* can be embedded in the interior of the doughnut $\mathbf{T}$ from Problem 2 above. Let $\mathbf{M}^2$ be the set of all (x_1, x_2, x_3) such that

$$(r-2)^2 + x_3^2 \leq 1 \qquad\qquad x_3 \sin(\theta/2) = (r-2)\cos(\theta/2) \tag{2.115}$$

where (r, θ) are the polar coordinates of (x_1, x_2). This is, in fact, a manifold with boundary. The manifold proper is constructed with strict inequality above. Show that the boundary of $\mathbf{M}^2$ is diffeomorphic to $\mathbf{S}^1$. (If we glue the boundaries of two separate copies of a Moebius strip together we also get a manifold without boundary. This manifold is called the Klein bottle $\mathbf{K}^2$.)

4. Following from Problem 3 above, we note that another manifold with boundary whose boundary is $\mathbf{S}^1$ is the disk $\mathbf{D}^2$. This is the set of all (x_1, x_2) such that $x_1^2 + x_2^2 \leq 1$. As the boundary of $\mathbf{D}^2$ is diffeomorphic to the boundary of $\mathbf{M}^2$ from Problem 3 above, in principle (given four dimensions to do it in), we could glue the boundaries together by fusing diffeomorphic points. If the two surfaces were cut out from paper we could try to tape their boundaries together. However, as we progressed with the taping in three dimensions we would simply run out of room to do it in. In four dimensions there is enough room. Show that the resulting manifold without boundary is diffeomorphic to the projective plane $\mathbf{RP}^2$.

5. Show that the geodesic paths on the sphere $\mathbf{S}^2$ are arcs of great circles found by slicing the sphere with a plane through the center of the sphere.

6. Consider the cylindrical surface in $\mathbf{R}^3$ defined as the set of all (x_1, x_2, x_3) such that $x_1^2 + x_2^2 = 1$ with $-\infty < x_3 < +\infty$. This surface is also represented as $\mathbf{S}^1 \times \mathbf{R}$. Show that the geodesics of $\mathbf{S}^1 \times \mathbf{R}$ are helixes of the form

$$x_1(t) = \cos(at) \qquad x_2(t) = \sin(at) \qquad x_3(t) = bt \tag{2.116}$$

for arbitrary real values a and b.

7. Prove that formulas (2.29) and (2.30) make tangent vector summation and scalar multiplication well defined. That is, show that the equivalence classes of paths defined for $\dot{x}(t_0) + \dot{z}(t_0)$ and $\lambda\, \dot{x}(t_0)$ do not depend upon the coordinate system used. Furthermore, show that if $\dot{y}(t_0) = \dot{x}(t_0)$ and $\dot{w}(t_0) = \dot{z}(t_0)$ then as defined by (2.28) and (2.29) we have

$$\dot{x}(t_0) \;+\; \dot{z}(t_0) \;=\; \dot{y}(t_0) \;+\; \dot{w}(t_0) \tag{2.117}$$

and

$$\lambda\, \dot{x}(t_0) \;=\; \lambda\, \dot{y}(t_0) \tag{2.118}$$