Homework #2 Solutions

1. **Canny edge detector:** Show that the product of the Canny detection and localization criteria

\[
\text{SNR}(f) \times \text{Loc}(f) = \frac{\left| \int_{-W}^{W} G(-x)f(x)dx \right| \left| \int_{-W}^{W} G'(-x)f'(-x)dx \right|}{n_0 \sqrt{\int_{-W}^{W} f^2(x)dx} n_0 \sqrt{\int_{-W}^{W} f'^{(2)}(x)dx}}
\]

is maximized by \( f(x) = G(x) \).

Use the Cauchy-Schwarz inequality \( \int f(x)g(x)dx \leq \int f^2(x)dx \int g^2(x)dx \) with equality at \( f(x) = cg(x) \) and write

\[
\text{SNR}(f) \times \text{Loc}(f) = \frac{\left| \int_{-W}^{W} G(x)f(x)dx \right| \left| \int_{-W}^{W} G'(x)f'(-x)dx \right|}{n_0 \sqrt{\int_{-W}^{W} f^2(x)dx} n_0 \sqrt{\int_{-W}^{W} f'^{(2)}(x)dx}} \leq \frac{1}{n_0} \int_{-W}^{W} G^2(x)dx \int_{-W}^{W} G^2(-x)dx
\]

with equality at \( f(x) = G(x) \).

2. **Canny edge detector:** Rewrite the detection and localization criteria for a filter \( f_w(x) = f(x/w) \). Show that the product of the detection and localization criteria is **invariant** to \( w \).

\[
\text{SNR}(f) \times \text{Loc}(f) = \frac{\left| \int_{-W/w}^{W/w} G(-x)f(x/w)dx \right| \left| \int_{-W/w}^{W/w} G'(x)f'(-x/w)dx \right|}{n_0 \sqrt{\int_{-W/w}^{W/w} f^2(x/w)dx} n_0 \sqrt{\int_{-W/w}^{W/w} f'^{(2)}(x/w)dx}}
\]

\[
= \frac{\left| \int_{-W/w}^{W/w} G(-yw)f(y)dy \right| \left| \int_{-W/w}^{W/w} G'(yw)f'(-y)dy \right|}{n_0 \sqrt{\int_{-W/w}^{W/w} f^2(y)dy} n_0 \sqrt{\int_{-W/w}^{W/w} f'^{(2)}(y)dy}}
\]

So, the product is actually not exactly invariant to \( w \). However, it is approximately invariant.

3. **Level sets:** The differential equation obeyed by \( x(t) \) is

\[
\frac{dx}{dt} = -\frac{x + t}{t + 1}
\]

Assuming that \( \psi(x, t) = ax^2 + btx + ct^2 + dx + et + f \) we get

\[
\frac{\partial \psi}{\partial x} = 2ax + bt + d
\]

and

\[
\frac{\partial \psi}{\partial t} = bx + 2ct + e.
\]
If an embedding can be found such that

\[ \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} \frac{dx}{dt} = 0 \]

\[ \Rightarrow \frac{dx}{dt} = -\frac{\partial \psi}{\partial x} = -\frac{bx + 2ct + e}{2ax + bt + d} = -\frac{x + t}{t + 1}. \]

From this, we can identify \( a = 0, \ b = 1, \ c = 0.5, \ d = 1, \) and \( e = 0 \) giving

\[ \psi(x, t) = xt + 0.5t^2 + x + f. \]

Since we want level sets of \( \psi(x, t) \), these correspond to

\[ xt + 0.5t^2 + x + f = c \]

\[ \Rightarrow x(t) = \frac{(c - f) - 0.5t^2}{t + 1} \]

If we choose \( f = 0 \) and we seek the zero level set, these are

\[ x(t) = \frac{0.5t^2}{t + 1}. \]