Special Session: Impact of Noise on Quantum Algorithms in Noisy Intermediate-Scale Quantum Systems

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***Abstract*—A major challenge in realizing efficient and powerful quantum algorithms is quantum noise. Quantum noise itself is a sophisticated topic that is not seen in the classical domain. In this paper, we explore the impact of noise on quantum algorithms in Noisy Intermediate-Scale Quantum (NISQ) systems. This paper first introduces the origins of quantum noise. Next, it proposes a common treatment to simplify the view of quantum noise. Finally, it presents a case study to investigate the impact of noise on quantum Fourier transform algorithm.**

***Index Terms*—Quantum computing, decoherence, noise**

I. Introduction

Classical computers have evolved to sophisticated machines simply by operating on strings of zeros and ones that represent information. Despite the technological advances, classical computers cannot efficiently solve several classes of important problems, such as optimization and logistics, quantum simulation, and data sampling. Quantum computers provide a promising alternative to solve these problems. Quantum computing uses qubits, which can be in a superposition of zero and one, and can utilize entanglement to perform tasks efficiently. Quantum algorithms make use of these quantum phenomena to offer advantages over classical computing.

Similar to developing classical algorithms, developing quantum algorithms relies on the ability to test and evaluate the corresponding performance. For small designs, evaluation using classical simulation is feasible, including limited noise simulation. Beyond fifty qubits, classical simulation to verify a theoretical quantum algorithm becomes infeasible due to the need to track $\approx 2^{50}$ complex floating point numbers. The only option left is to test the quantum algorithm on a physical Noisy Intermediate-Scale Quantum (NISQ) [1] computer. Access to these stable and larger quantum computers is presently limited, hence classical simulation remains a critical component in evaluating quantum algorithms. Thus, noise modeling is fundamental in classical simulation to understand the underlying behavior of smaller quantum algorithms, and hoping that the observed patterns scale with larger designs.

The most widely used model of quantum computation is the circuit model. In this model, a sequence of unitary quantum gates are applied to a quantum register. Similar to classical circuit model, a universal quantum gate set can be formed that places guarantees on the precision of the circuit implementation [2]. The primary advantage of the circuit model is the transparency and natural extension from classical circuits. A common treatment of noise within the framework of quantum circuit is via the quantum channel [3] which represents a classical uncertainty of a quantum computer to be in a given state. Conceptually, these uncertainties can arise due to errors in initial preparation, in the evolution and gates, or measurement itself.

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In this paper, we explore the impact of noise on quantum algorithms in NISQ systems. Specifically, this paper investigates three important tasks: origins of noise, modeling of noise, and studying the impact of noise on quantum algorithms. The origins of quantum noise itself is a delicate topic, with behavior that is not observed in classical noise. We will introduce an example of quantum noise in Section II, and build intuition to the intricate difference between quantum and classical noise. After introducing the complexity in modeling quantum noise itself, we review simplified models, the quantum channel, and provide a small example to demonstrate the impact of a 1-qubit quantum register subject to noise. We will introduce quantum channels in Section III and their representations. An important observation about quantum channels is that, mathematically, they can be decomposed to various representations, and can always be viewed as an interaction with an environment [4]–[6]. Finally, we present a case study in Section IV to look at the quantum Fourier transform algorithm subject to simple noise models, which further showcases the different nature of quantum noise and an inherent noise-resilience in quantum algorithms.

II. Origins of Quantum Noise

A perfect quantum computer would operate purely as a closed system. However, real systems will suffer from unwanted interactions with an outside environment. The unwanted interactions of a closed system with an environment show up as noise. In general, an open system can be modeled as:

$$H = H_S + H_B + H_{SB} \tag{1}$$

where H_S is the system Hamiltonian describing a quantum computer, H_B describes the evolution of the bath (environment), and H_{SB} describes the interaction between the system and bath.

Equation (1) shows that the time evolution of the quantum computer is not governed by unitary operators. To showcase this point, we review the following example based on the spin-boson model [7]: suppose a particle, which represents our qubit, is placed into an environment that consists of a

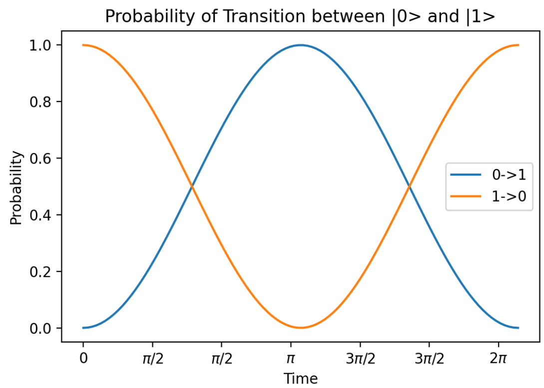

Fig. 1: Proper settings of a controlled external magnetic field, given by (2), can control the state of a particle in order to implement a qubit. In this example, an X-gate, or flip gate, which changes a state from $|0\rangle$ to $|1\rangle$ or $|1\rangle$ to $|0\rangle$ can be implemented by leaving the magnetic field active for π-time. This evolution is noise-free.

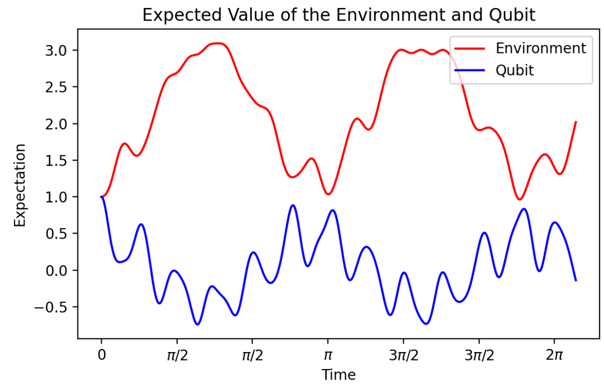

Fig. 2: The qubit no longer simply transitions between 0 and 1 as in Figure 1, instead the evolution is connected with the environment. An expected value of 1 corresponds to $|1\rangle$, and -1 corresponds to $|0\rangle$. If the environment is ignored (not observed, or not modeled), the evolution of the qubit would be unusual, unexpected, and not unitary – this is quantum noise.

collection of particles that "come and go" and interact with our qubit. We briefly introduce the mathematical construction, however the primary intuition lies in Figure 1 and Figure 2.

Suppose the particle is a spin-$\frac{1}{2}$ particle which is controlled by an external magnetic field and defined as:

$$H_S = -\frac{1}{2}\omega_0\sigma_z - \frac{1}{2}\omega_1\sigma_x \cos\omega t + \frac{1}{2}\omega_1\sigma_y \sin\omega t \qquad (2)$$

Given certain settings of the external magnetic field, we can control the state of the particle freely, hence it can serve as a qubit. For example, Figure 1 shows that, under specific magnetic field frequency, we can implement a gate to flip the state of the qubit by leaving the field active for time π.

Now the environment, which is a collection of "come and go" particles can be expressed as a collection of k individual quantum harmonic oscillators:

$$H_B = \sum_k \frac{\hat{p}_k}{2m_k} + \frac{1}{2}m_k\omega_k^2\hat{x}_k^2 = \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \qquad (3)$$

The interaction between the qubit and the harmonic oscillators is given by oscillators' position, coupling strength, and spin energy:

$$H_{SB} = \sigma_z \sum_k \lambda_k x_k(t) \qquad (4)$$

Figure 2 shows an illustrative example of evolving the complete system, H, which leads to unwanted evolution of the qubit when compared to Figure 1. Applying the same frequency and time as in Figure 1 no longer flips the state as one would expect.

III. Modeling of Quantum Noise

At a physical level, reasoning and modeling noise quickly becomes an intricate and difficult study. Proper consideration of thermal and quantum fluctuations or of Markovian and non-Markovian dynamics, must take place in order to grasp the behavior of the open system. In the previous section we discussed the spin-boson model, in which a two-state system interacts with an environment of oscillators. In this model, however, the qubit decoheres in the σ_z eigenbasis, meaning there is no relaxation or excitation – the qubit does not spontaneously decay to the ground state, nor does the ground state suddenly become thermally excited [8]. Moreover, the model only showcases noise on a single qubit, which is not particularly exciting when it comes to quantum computation. For these reasons, when discussing quantum algorithms or error correction, approximations of noise as well as quantum threshold theorems [9] are used.

As mentioned earlier, the evolution of a quantum computer in an environment may no longer be purely described via unitary operators. Instead the evolution of the quantum computer can be captured using the quantum operations formalism, which satisfies a set of physically motivated axioms [10]. A quantum operation $\mathcal{E}$ acts on a density state ρ to produce a new state ρ'. If $\mathcal{E}$ provides a complete description of the quantum process then it is trace-preserving, and commonly referred to as a quantum channel. Similar to classical channels, quantum channels encode a notion of information preservation. To discuss quantum channels, it is convenient to use the operator-sum (Kraus) decomposition [5]: $\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$, where the index i spans all the Kraus operators and $K^\dagger$ is the Hermitian transpose of K.

1) Example of Single-Qubit Algorithm: Consider the following example where the noise introduces a bit-flip error. The Kraus operators are:

$$K_1 = \sqrt{1-p} * I \qquad (5)$$

$$K_2 = \sqrt{p} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (6)$$

K_1 keeps the same state. K_2 flips the state in the computational basis. Suppose we have the following quantum circuit:

$$|0\rangle \;—\; \boxed{R_x(\tfrac{\pi}{4})} \;—\; |\psi\rangle \qquad (7)$$

(a) After R_x gate (b) State after the bit-flip error

Fig. 3: Bloch sphere representation of qubit state. The R_x gate moved the state into a superposition of $|0\rangle$ and $|1\rangle$. The error added uncertainty, which is indicated by an arrow that no longer lies on the edge of the sphere.

The circuit starts in the state $\rho = |0\rangle\langle 0|$, followed by a rotation gate of $\frac{\pi}{4}$, which experiences the bit-flip error. The final state is thus: $K_1 R_x \rho R_x^\dagger K_1^\dagger + K_2 R_x \rho R_x^\dagger K_2^\dagger$. Figure. 3 depicts these steps on the Bloch sphere, where $p = 0.2$. Intuitively, if we continued acting on our qubit with gates that are subject to this bit-flip error, then eventually we would no longer be able to discern $|0\rangle$ or $|1\rangle$, nor between any other states. This is known as a maximally-mixed state, which is proportional to the identity, $\rho_{mix} = \frac{1}{2}I$, and is visualized as the center point in the Bloch sphere.

IV. Impact of Noise on Quantum Algorithms

Given the fundamental wave-like property of constructive and deconstructive interference in quantum mechanics, a natural question arises: could a quantum algorithm be inherently resilient to noise? We first look at a particular example using the quantum Fourier transform algorithm, and then briefly introduce a generic notion to address this question: decoherence-free subspaces.

A. Impact of Noise on Quantum Fourier Transform

1) Algorithm and Visualization: The Quantum Fourier Transform (QFT) is an important component in several quantum algorithms including Shor's Algorithm [11] and Quantum Phase Estimation Algorithm [12]. Similar to the classical Fourier transform, QFT preforms the following mapping:

$$QFT : |x\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} w_N^{xk} |k\rangle \tag{8}$$

where $w_N = e^{2\pi i/N}$. The algorithm can be represented as a quantum circuit, as shown in Figure 4, which preforms QFT on 3-qubits where, for example, the input state $|1\rangle$ is applied in binary form: $|001\rangle$. We do not provide an analysis of QFT here, nor on Fourier transforms, we instead view QFT as a specific mechanism to transform a state of qubits around.

To visualize the process of a n-qubit mechanism, consider 2^n circles containing one marked point which is a representation of the amplitude as a vector in the complex plane. Any n-qubit state can be uniquely represented this way, and conversely any drawing will represent a quantum state assuming normalization – the sum of distances between the marked points and the circle centers is one.

Now, suppose an initial state of 3-qubits to be $|1\rangle$ which is depicted in Figure 5. Applying QFT on this state will

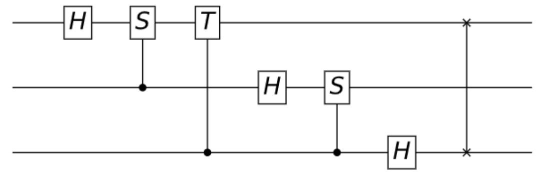

Fig. 4: A circuit depiction of the QFT algorithm acting on 3-qubits, where an input state $|x\rangle$ is expanded in binary form, $|x_0 x_1 \dots x_n\rangle$, where $x_i \in \{0, 1\}$

result in a state that is shown in Figure 6. In Figure 6 a sinusoidal wave emerges that completes one revolution. In fact, given a state $|i\rangle$, the number of revolutions that QFT makes will be exactly i. Moreover, the reverse is critically fundamental in many quantum algorithms: given a state that contains revolutions, applying inverse QFT will give a state $|j\rangle$ which, when measured, indicates that the original state had j revolutions.

2) Quantum Noise: Suppose the quantum computer is subject to the bit-flip noise model as defined earlier. In the worst case, a bit-flip will occur after each gate in Figure 4. Applying this noisy QFT to the state $|1\rangle$ gives a state depicted in Figure 7. At a first glance, it may seem that Figure 7 has no structure that encodes revolutions, and is wrong. However, by re-arranging the circles in Figure 7 we obtain Figure 6, which is the correct answer. Moreover, even if the bit-flip noise is no longer in the worst case, but rather applied to gates at random, a re-ordering of the output will still give the correct result. This is simply a consequence of the bit-flip noise randomly re-arranging the labeling of the basis.

Similarly, if instead of a bit-flip noise the noise is a phase-flip, the sample result has the marked points flipped when compared to Figure 6, as shown in Figure 8. Not all the points are flipped, since the noise can actually cancel with itself throughout the circuit. Moreover, as seen in Figure 8, running the noisy circuit several times yields only two different possible positions of the marked points. The notion of having several potentially different output states is exactly what density operators and quantum channels capture.

B. Decoherence-Free Subspaces

So far the examples have demonstrated that it is possible to obtain the correct result as well as adjust the output result to obtain the correct answer – subject to a simple noise model. But, in order to do post-correction, the results must be analyzed and a re-ordering matrix needs to be constructed and applied. In this section, we review decoherence-free subspaces, in which the quantum algorithm can subjectively avoid noise in the first place [13].

A decoherence-free subspace must satisfy two criteria:

1) The Kraus operators can be partitioned into a good part which is a unitary U, and a noisy part B: $K_i = U \oplus B_i$
2) The initial state can be partitioned in the same way: $\rho = \rho_G \oplus \rho_B$

Then expanding $\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$ gives: $U \rho_G U^\dagger \oplus \sum_i B_i \rho_B B_i^\dagger$. This means that ρ_G will evolve purely unitarily,

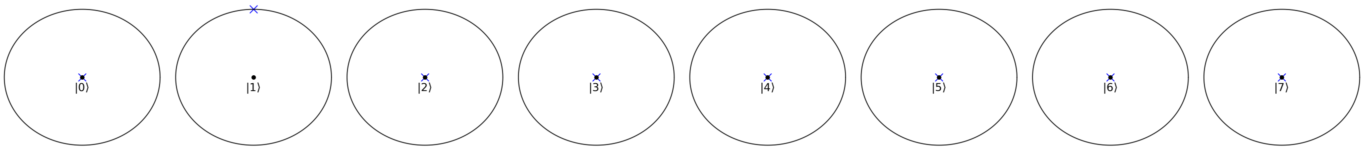

Fig. 5: Graphical representation of a 3-qubit state: $0 \cdot (|0\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle) + 1 \cdot |1\rangle = |1\rangle$

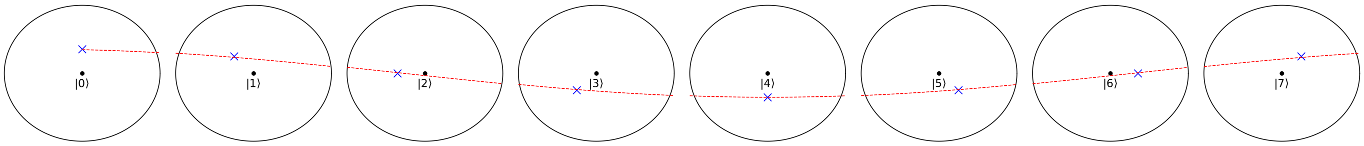

Fig. 6: Graphical representation of the state after applying 3-qubit QFT to $|1\rangle$. The dotted red line assists in visualizing the 1-revolution.

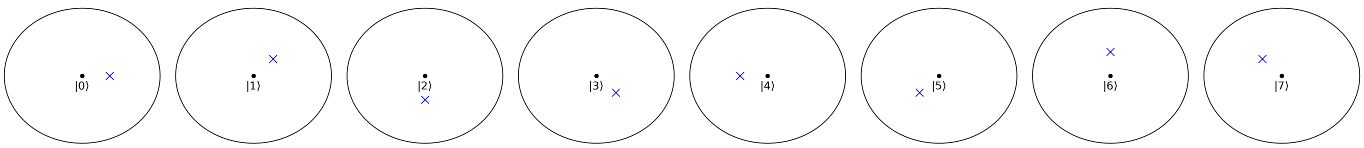

Fig. 7: Graphical representation of the state after applying 3-qubit QFT to $|1\rangle$ with a bit-flip noise.

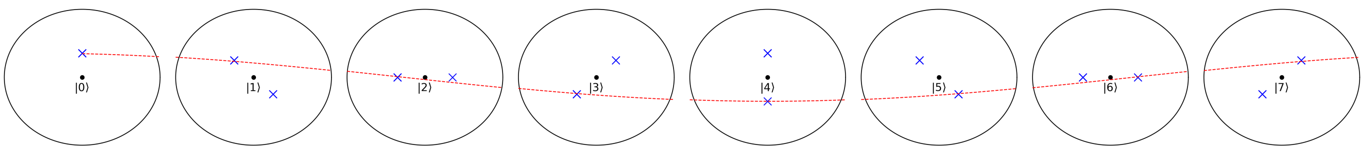

Fig. 8: Graphical representation of the state after applying 3-qubit QFT to $|1\rangle$ with a phase-flip noise.. The algorithm was applied 100 times, with the noise acting at random on each gate. Each marked point in a circle represents one of the 100 possible amplitudes for the appropriate state. In fact, the correct answer, a 1-revolution as in Figure 6, does consistently appear.

hence a quantum algorithm will evolve correctly and the noisy part can simply be ignored. In practice, a designer has to consider two important challenges: (a) finding the decoherence-free subspace, and (b) constructing quantum circuits that will operate solely in this subspace.

V. Conclusion

In this paper, we have explored the impact of noise on quantum algorithms. First, we introduced the origins and complications of quantum noise. Next, we outlined the usage of simplified, mathematically correct, noise models. Finally, we presented a case study using quantum Fourier transform to mitigate specific noise within the construction of a quantum circuit, which is guaranteed by the results from decoherence-free subspaces. It is clear that establishing a quantum circuit that mitigates generic noise becomes infeasible, and moreover, a decoherence-free subspace may not exist at all. Therefore, a different solution must be developed, either by developing a mechanism to correct the errors (e.g., using quantum error correction [14]), or by physically ensuring that errors do not incur in the first place (e.g., using quantum engineering [15]). Given that these solutions have limitations of their own, it is crucial that consideration of noise takes place at all stages in designing and implementing a quantum algorithm.

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