Sharp, quantitative bounds on the distance between a polynomial piece and its Bézier control polygon

D. Nairn * J. Peters‡ † D. Lutterkort ‡

July 10, 1999

Abstract

The maximal distance between a Bézier segment and its control polygon is bounded in terms of the differences of the control point sequence and a constant that depends only on the degree of the polynomial. The constants derived here for various norms and orders of differences are the smallest possible.

In particular, the bound in terms of the maximal absolute second difference of the control points is a sharp upper bound for the Hausdorff distance between the control polygon and the curve segment. It provides a straightforward proof of quadratic convergence of the sequence of control polygons to the Bézier segment under subdivision or degree-fold degree-raising, establishes the explicit convergence constants, and allows analyzing the optimal choice of the subdivision parameter for adaptive refinement of quadratic and cubic segments and yields efficient bounding regions.

1 Curved geometry and control polygons

A widely used, efficient and intuitive way to specify, represent and reason about curved, nonlinear geometry for design and modeling is the control point or control polygon paradigm: for popular representations like the B-spline and the Bernstein-Bézier representation the curve shape is outlined by the broken line connecting the control points. For many applications, e.g. rendering, intersection testing or design, this raises the question just *how well* the control line approximates the exact curved geometry.

This paper gives several simple *quantitative* and exact answers to this question in terms of differences of the Bézier control points and constants that depend only on the degree of the polynomial. The bounds are sharp, i.e. there

*User Services, University of Delaware

†Supported by NSF NYI grant 9457806-CCR, Dept C.I.S.E., CSE Bldg, University of Florida, Gainesville, FL 32611-6120, jorg@cise.ufl.edu

‡Computer Sciences, Purdue University

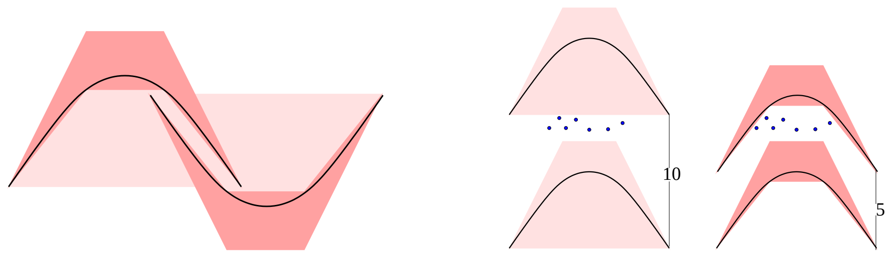

Figure 1: Improved bounds for intersection testing (*left*) and creating tolerance envelopes (*right*). Shaded region corresponds to convex hull, darker portion to new bound derived from the results in this paper.

exist in each case commonly used curves such that the bound is taken on and any reduction of the constant would not yield a bound. In particular the bound in terms of the maximal absolute second difference remains sharp under degree-raising and subdivision, i.e. refinement of the piecewise linear control structure to better approximate the curved geometry. This yields for example a sharp *a priori* bound on the number of subdivision steps needed to bring curve and control polygon within a prescribed Hausdorff distance of one another.

While the focus of this paper is on capturing the essence of the bound and its implications for the general toolkit of computer aided geometric design, the two example scenarios sketched in Figure 1 illustrate the potential use of the new result for a wide range of applications. Combined with the standard min-max bound and an improved bound at the ends of a curve segment, the result confines the curve segment to a region bounded by at most $2d+2$ line segments where d is the degree of the component functions of the curve. Localization of the curve to the convex hull, here depicted as the union of shaded regions gives more conservative estimates than localization to the darker shaded region implied by the new bound of this paper. In Figure 1 (*left*) non-intersection follows immediately from the new bounds, while the convex hull estimate requires several refinements to separate the bounding regions. On the right, the curve and its translate can be chosen closer together while still guaranteeing the inclusion of the given point set.

The new tight bounds reveal the constants that scale the quadratic rate of convergence of the sequence of control polygons to the curve under subdivision and under degree-fold degree-raising. This clearly shows uniform approximation by subdivision to be more efficient than via degree-raising and allows, for segments of low degree, to determine the optimal point of subdivision.

After reviewing prior work, Section 3 represents the technical heart of the paper, a bound for functions in Bernstein-Bézier form. Section 4 shows alternative bounds by varying the choice of norm on the second differences and order of differences. For the remainder, the paper concentrates on the bound in terms of the maximal absolute second difference of the control points. Section 5 extends the max-norm bound to the Hausdorff distance between control polygon and curve segment. Section 6 discusses the bound under subdivision and Section 7 the bound under degree-raising. Section 8 improves the bound at the ends of the segment, Section 9 illustrates the bounds by a gallery of examples and Section 10 draws a few conclusions.

2 Prior Bounds

Two properties lie at the heart of control point representations of curves: the variation diminishing property and the subdivision property. The variation diminishing property, that any line crosses the control polygon at least as often as it does the curve, makes precise the notion that the features of the curve are exaggerated by the control polygon. Variation diminuition also implies the convex hull property, which states that all points on the curve segment are convex combinations of the control points. Thus the convex hull yields a bound on the distance between curve segment and control polygon.

The subdivision property gives a stable way of approximating the curve through a sequence of refinements of the control polygon using fixed-weight, finite averaging. Approximation rates for this process have been established in [1, 2] and by the careful analysis in [14]. Either result yields *qualitative* assurance that the approximation will improve under subdivision, but the corresponding *quantitative* estimates are too coarse for practical use. For example, the estimate in [14] exceeds the bound implied by the convex hull property. In [8], Filip, Magedson and Markot derive bounds for the distance between a curve and the linear interpolant to the end points and Schaback [15] extends and generalizes this approach to more general Hermite interpolants. For a Bézier curve of degree d the bound derived from linear approximation is $d-1$ times the bound derived in this paper. In [16], Sederberg, White and Zundel subtract a circular arc rather than a Hermite interpolant from the curve segment prior to generating a min-max bounding box. The arc offset by the bounding box is called a 'fat arc'.

In [10] upper and lower bounds for the modulus of continuity of polynomial and rational curves in Bézier form are derived. In [7] Farin points out that for rational curves, the convex hull can be tightened to include only rational weight points and end points. A similar projection argument applies to the joint intersection of convex hull and the new tight bound.

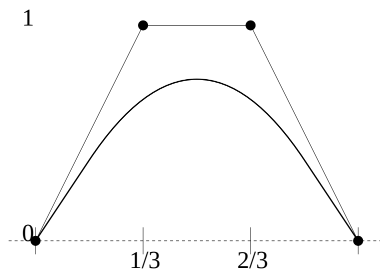

Figure 2: A cubic segment and its control polygon with coefficient sequence $(b_0, b_1, b_2, b_3) = (0, 1, 1, 0)$ and corresponding $\|\Delta_2 b\|_\infty = |\Delta_2 b_1| = |\Delta_2 b_2| = 1$. The bound is taken on at $t = 1/3$ where $\|p(1/3) - \ell(1/3)\| = 1/3 = \mathtt{N}_\infty(3)\|\Delta_2 b\|_\infty$.

3 Bounding functions

This section contains the central estimate for localizing the graph of a function in Bernstein-Bézier form with respect to the control polygon. The estimate is easily computed in terms of a constant $\mathtt{N}_\infty(d)$ that depends only on the degree d and the maximum second difference of the coefficient sequence. Some definitions are in order (c.f. [6], [3]).

A univariate, scalar-valued polynomial p of degree d is in *Bernstein-Bézier form* if

$$p(t) := \sum_{i=0}^{d} b_i B_i^d(t)$$

where $B_i^d(t) := \binom{d}{i}(1-t)^{d-i}t^i$.
The *control polygon* ℓ of p is a broken line connecting the points (t_k, b_k) where the first components $t_k := \frac{k}{d}$ are the Greville abscissae. Its kth segment $\ell_{[t_k, t_{k+1}]}$ on the interval $[t_k, t_{k+1}]$, is defined by

$$\ell_{[t_k,t_{k+1}]}(t) := b_k \frac{t_{k+1} - t}{t_{k+1} - t_k} + b_{k+1} \frac{t - t_k}{t_{k+1} - t_k}.$$

The ith *centered second difference* of the coefficient sequence $b_i, i = 0, \ldots, d$ is abbreviated

$$\Delta_2 b_i := b_{i-1} - 2b_i + b_{i+1} \quad \text{and} \quad \|\Delta_2 b\|_\infty := \max_{0<i<d} |\Delta_2 b_i|.$$

Finally, the maximum absolute difference between p and ℓ on the interval $[s, t]$ is abbreviated as

$$\|p - \ell\|_{\infty,[s,t]} := \max_{u \in [s,t]} |p(u) - \ell(u)|.$$

With these definitions the main result reads as follows.

Theorem 3.1 *The distance from the univariate, scalar-valued, degree d polynomial p to its control polygon ℓ is bounded as*

$$\|p-\ell\|_{\infty,[0,1]} \leq \mathtt{N}_\infty(d)\|\Delta_2 b\|_\infty$$

where

$$\mathtt{N}_\infty(d) := \frac{\lfloor\frac{d}{2}\rfloor\lceil\frac{d}{2}\rceil}{2d}.$$

For example, $[\mathtt{N}_\infty(0),\dots,\mathtt{N}_\infty(8)] = [0,0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{3}{5},\frac{3}{4},\frac{6}{7},1]$.

Proof On the interval $[t_k, t_{k+1}]$, $p(t)-\ell(t) = \sum_i \alpha_{ki}(t)b_i$, where

$$\alpha_{ki}(t) := \alpha^d_{ki}(t) := B^d_i(t) - \begin{cases} k+1-dt & \text{if } i=k \\ dt-k & \text{if } i=k+1 \\ 0 & \text{else} \end{cases}$$

since $t_{k+1}-t_k = t_k - t_{k-1} = 1/d$. The formula for conversion to power form, $\sum_{i=k}^d \binom{i}{k}B^d_i(t) = \binom{d}{k}t^k$, implies linear precision

$$\sum_{i=0}^d \alpha_{ki} = 0 \quad \text{and} \quad \sum_{i=0}^d i\alpha_{ki} = 0.$$

It follows that $\sum_{j=0}^i (i-j)\alpha_{kj} = \sum_{j=i}^d (j-i)\alpha_{kj}$, and hence for $0\leq i\leq d$ and all k

$$\beta_{ki} := \sum_{j=0}^i (i-j)\alpha_{kj} = \begin{cases} \sum_{j=0}^i (i-j)B^d_j & \text{for } 0\leq i\leq k \\ \sum_{j=i}^d (j-i)B^d_j & \text{for } d\geq i\geq k+1. \end{cases}$$

The β_{ki} (cf. Figure 3) are nonnegative second antidifferences of the α_{ki} on $[t_k,t_{k+1}]$. That is, $\beta_{ki}(t)>0$ for $0<i<d$ and

$$\Delta_2\beta_{ki} = \beta_{k,i+1} - 2\beta_{k,i} + \beta_{k,i-1} = \alpha_{k,i} \quad \text{for } 1\leq i\leq d-1.$$

Furthermore, for $k\in\{0,\dots,d-1\}$,

$$\begin{aligned}\sum_{i=1}^{d-1}\beta_{ki}(t) &= \sum_{i=0}^{d-1}\sum_{j=0}^{i}(i-j)\alpha_{kj}(t) = \sum_{j=0}^{d}\sum_{i=j}^{d-1}(i-j)\alpha_{kj}(t) = \sum_{j=0}^{d}\Big(\sum_{i=0}^{d-1-j} i\Big)\alpha_{kj}(t) \\ &= \sum_{j=0}^{d}\binom{d-j}{2}\alpha_{kj}(t) = \sum_{j=0}^{d}\binom{j}{2}\alpha_{kj}(t) = \sum_{j=2}^{d}\binom{j}{2}B^d_j(t) + \frac{k}{2}(k+1-2dt) \\ &= \binom{d}{2}t^2 + \frac{k}{2}(k+1-2dt).\end{aligned}$$

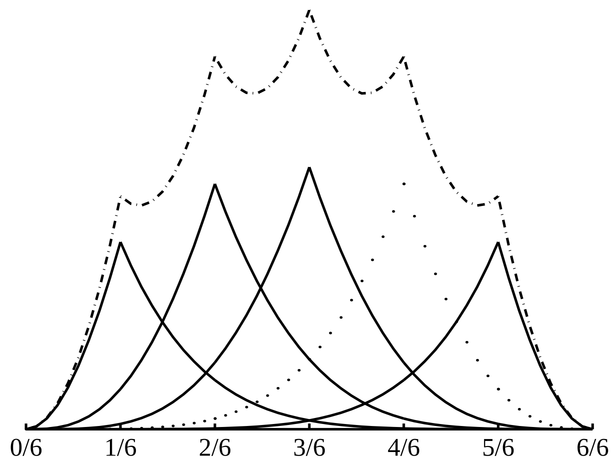

Figure 3: The second antidifferences $\beta_i(t) := \beta_{ki}(t)$ on $t \in [t_k, t_{k+1}] = [k, k+1]/d$, for $d = 6$ and $i = 1, \ldots, 5$ (*solid* except β_4 *dotted*), and their piecewise quadratic sum (*dash-dotted*). The ith peak separates the monotonically increasing part of β_i on $[0, t_i]$ from the decreasing part on $[t_i, 1]$.

On its interval $[t_k, t_{k+1}]$, $\sum_{i=0}^{d} \beta_{ki}(t)$ is a positive quadratic polynomial with positive leading coefficient and therefore takes on its maximum either at t_k or t_{k+1} implying

$$\begin{aligned}
\max_{0 \le k < d} \max_{t_k \le t \le t_{k+1}} \sum_i \beta_{ki}(t) &= \max_{0 \le k < d} \max\{\sum_{i=2}^{d} \binom{i}{2} \alpha_{ki}(t_k), \sum_{i=2}^{d} \binom{i}{2} \alpha_{ki}(t_{k+1})\} \\
&= \max_{0 \le k \le d} \binom{d}{2} \frac{k^2}{d^2} - \binom{k}{2} = \max_{0 \le k \le d} \frac{k}{2d}(d-k) \\
&= \frac{\lfloor d/2 \rfloor \lceil d/2 \rceil}{2d}.
\end{aligned}$$

Abbreviating

$$\|\cdot\|_k := \|\cdot\|_{\infty,[t_k, t_{k+1}]},$$

and assuming without loss of generality that $b_0 = b_d = 0$, the bound follows from

$$\begin{aligned}
\|p-\ell\|_{\infty,[0,1]} &= \max_k \|p-\ell\|_k \\
&= \max_k \|\sum_{i=0}^{d} \alpha_{ki} b_i\|_k \\
&= \max_k \|\sum_{i=1}^{d-1} \Delta_2\beta_{ki} b_i\|_k \\
&= \max_k \|\sum_{i=1}^{d-1} \beta_{ki}\Delta_2 b_i\|_k \\
&\le \|\Delta_2 b\|_\infty \max_k \|\sum_{i=1}^{d-1} \beta_{ki}\|_k \\
&= \frac{\lfloor d/2\rfloor\lceil d/2\rceil}{2d}\|\Delta_2 b\|_\infty .
\end{aligned}$$

⋈

The fourth equation could have been arrived at without setting $b_d = b_0 = 0$ since $\beta_{kd} = \beta_{k0} = 0$. However, this would require defining $\Delta_2\beta_{k0}$ and hence $\beta_{k,-1}$. A key step in the estimate is the inequality which uses the fact that $|\beta_{ki}| = \beta_{ki}$. We will discuss this and alternative inequalities in more detail in the following section.

The constant $\mathtt{N}_\infty(d)$ is not just a better estimate, i.e. $d-1$ times smaller than the previous best estimate in [8], but it is optimal i.e. the single inequality in the proof is sharp for a large class of functions.

Corollary 3.1 *The bound in Theorem 3 is sharp for all degrees.*

Proof If p is the degree-raised representation of a quadratic polynomial then all second differences of the degree-raised representation of p are equal, i.e. for each i, $|\Delta_2 b_i| = \|\Delta_2 b\|_\infty$ since differencing and degree-raising commute. Since the β_{ki} are nonnegative, we have equality throughout the proof. Small perturbations of the coefficients of the degree-raised quadratics yield, for any given degree d, polynomials that asymptotically match the bound. In other words, for any ϵ there are polynomials of degree d that match the bound within ϵ.

⋈

The inequality is not sharp if the sequence of second differences has sign changes, e.g. when the function has inflection points. Examples 1, 2 and 4 of the fourth column of Figure 8 illustrates sharpness, almost sharpness and alternation.

4 Bound alternatives

The estimate in the proof of Theorem 3 is an application of Hölder's inequality

$$\sum_{i=1}^{d-1} \beta_{ki}\Delta_2 b_i \le \|\beta_{ki}\|_p \|\Delta_2 b\|_q, \quad q^{-1}+p^{-1}=1,$$

for fixed argument t, $q=\infty$ and $p=1$. Other choices of p and q lead to alternative estimates of which the case $p=\infty$ and $q=1$ deserves special attention.

Theorem 4.1 *The distance from the univariate, scalar-valued, degree d polynomial p to its control polygon ℓ is bounded by*

$$\|p(t)-\ell(t)\|_{\infty,[0,1]} \le \mathtt{N}_1(d)\|\Delta_2 b\|_1$$

where $t^* := \lceil \frac{d}{2} \rceil / d$ *and*

$$\mathtt{N}_1(d) := \beta_{\lceil \frac{d}{2}\rceil, \lceil \frac{d}{2}\rceil}(t^*) = 2\mathtt{N}_\infty B^d_{\lceil \frac{d}{2}\rceil}(t^*).$$

If d is even, $\mathtt{N}_1(d)$ simplifies to $\binom{d}{\lceil \frac{d}{2}\rceil}\frac{d}{2^{d+2}}$, an expression that is also an upper bound for odd d. $\mathtt{N}_1(d)$ is a slowly growing function with

$$[\mathtt{N}_1(2),...,\mathtt{N}_1(10)] = [0.2500, 0.2963, 0.3750, 0.4147, \\ 0.4688, 0.5036, 0.5469, 0.5782, 0.6152].$$

Proof Fix k. Then for $j<i\le k$, $\beta_{ki} > \beta_{kj}$ since $\beta_{ki}-\beta_{k,i-1} = \sum_{j=0}^{i-1} B^d_j > 0$ and similarly for $j>i>k$, $\beta_{ki}>\beta_{kj}$. Moreover β_{kk} is a monotonically decreasing function on $[t_k, t_{k+1}]$. Hence

$$\begin{aligned}\sup_{t\in[t_k,t_{k+1}]} \|\max_i \beta_{ki}(t)\| &= \sup_{t\in[t_k,t_{k+1}]} \beta_{kk} = \beta_{kk}(t_k) = \sum_{j=0}^{k}(k-j)B^d_j(t_k) \\ &= \frac{(d-k)k}{d} B^d_k(\frac{k}{d}).\end{aligned}$$

We show that the maximum is attained for $k=\lceil \frac{d}{2}\rceil$ and hence

$$\max_k \|\max_i \beta_{ki}\|_k = \mathtt{N}_1(d)$$

as claimed. By symmetry it suffices to show that for $k < \lfloor \frac{d}{2} \rfloor$, i.e. $d-k \geq k' := k+1$

$$\frac{(d-k)k}{d} B_k^d(\frac{k}{d}) \leq \frac{(d-k')k'}{d} B_{k'}^d(\frac{k'}{d}).$$

With $d-k = k+1+a$, $a \geq 0$ the inequality is equivalent to

$$f(a) := \left(\frac{k+1+a}{k+a}\right)^{k+1+a} \leq \left(\frac{k+1}{k}\right)^{k+1}.$$

For $a = 0$, we have equality, and f is a decreasing function in a as can be seen from the derivative

$$f'(a) := K[\ln(1+\frac{1}{k+a}) - \frac{1}{k+a}], K > 0$$

and the estimate $1 + \frac{1}{k+a} < e^{\frac{1}{k+a}}$. ⋈

The above estimate is sharp when the sequence of second differences has one nonzero entry, exactly where $\max_i \beta_{ki}$ attains its maximum. This is the case when the control polygon has the shape of a hat (c.f. row 3 of Figures 8 and 9).

Corollary 4.1 *The bound in Theorem 4.1 is sharp for all degrees.*

Proof The polynomial

$$p := \sum_{j=0}^{\lceil \frac{d}{2} \rceil} j B_j^d + \sum_{j=\lceil \frac{d}{2} \rceil + 1}^{d} (2\lceil \frac{d}{2} \rceil - j) B_j^d$$

has $\Delta_2 b_i = 0$ for all i except for $\Delta_2 b_{\lceil \frac{d}{2} \rceil} = -2$. Therefore

$$|\lceil \frac{d}{2} \rceil - p| = \sum_{j=0}^{\lceil \frac{d}{2} \rceil} (\lceil \frac{d}{2} \rceil - j) B_j^d + \sum_{j=\lceil \frac{d}{2} \rceil + 1}^{d} (j - \lceil \frac{d}{2} \rceil) B_j^d = 2 \sum_{j=0}^{\lceil \frac{d}{2} \rceil} (\lceil \frac{d}{2} \rceil - j) B_j^d.$$

Hence at $t^* = \lceil \frac{d}{2} \rceil / d$, $(\ell - p)(t^*) = 2\beta_{\lceil \frac{d}{2} \rceil, \lceil \frac{d}{2} \rceil}(t^*) = \|\Delta_2 b\|_1 \mathtt{N}_1(d)$ as claimed.

⋈

The norms of the second difference vector in Theorem 3 and Theorem 4.1 can be viewed as measuring an approximation to maximal curvature and total curvature, respectively. Sharpness is obtained in the first case when the curvature is distributed most evenly, in the second case, when it is distributed most unevenly.

Analogously, the average curvature is approximated by $||\Delta_2 b||_2$ yielding a bound (c.f. Figure 8)

$$||p - \ell||_{\infty,[0,1]} \leq \mathtt{N}_2(d)||\Delta_2 b||_2.$$

We state without proof that

$$\mathtt{N}_2(d) := \max_k ||||\beta_{ki}(\cdot)||_2||_k$$

and

$$\begin{aligned}[\mathtt{N}_2(2), \ldots, \mathtt{N}_2(20)] = [&0.2500, 0.2986, 0.3853, 0.4331, 0.5015, 0.5480\\ &0.6079, 0.6530, 0.7079, 0.7517, 0.8032, 0.8458, 0.8946\\ &0.9361, 0.9829, 1.0235, 1.0686, 1.1084, 1.1520].\end{aligned}$$

Another family of bounds, more local to each segment, may be obtained by

replacing the maximum value $\mathtt{N}_\infty(d)$ of $\sum_i \beta_{ki}$ over all intervals $[t_k, t_{k+1}]$ by the maximum $\frac{k(d-k)}{2d}$ over the particular segment.

Choosing nth differences of the control point vector for $n > 2$, e.g. third differences, leads to bounds that include at least one second difference estimate such as the term $\mathtt{N}_\infty(d)|\Delta_2 b_{d-1}|$. In particular in view of the analysis of repeated subdivision, this does not result in better or structurally different bounds to the bounds for second differences. For $n = 0$, we have from the partition of unity

$$||p - \ell||_{\infty,[0,1]} \leq ||p||_{\infty,[0,1]} + ||\ell||_{\infty,[0,1]} \leq 2 \max_i |b_i|,$$

and $1 - B_d^{2d}$ at $1/2$ shows asymptotic sharpness of the bound as $d \to \infty$.

For completeness we state the result for first differences.

Theorem 4.2 *The distance from the univariate, scalar-valued, degree d polynomial p to its control polygon ℓ is bounded by*

$$||p(t) - \ell(t)||_{\infty,[0,1]} \leq \mathtt{L}_\infty(d)||\Delta b||_\infty$$

where $\mathtt{L}_\infty(d) := 2\mathtt{N}_1(d) = 4\mathtt{N}_\infty(d) B^d_{\lceil \frac{d}{2} \rceil}(\lceil \frac{d}{2} \rceil / d)$.

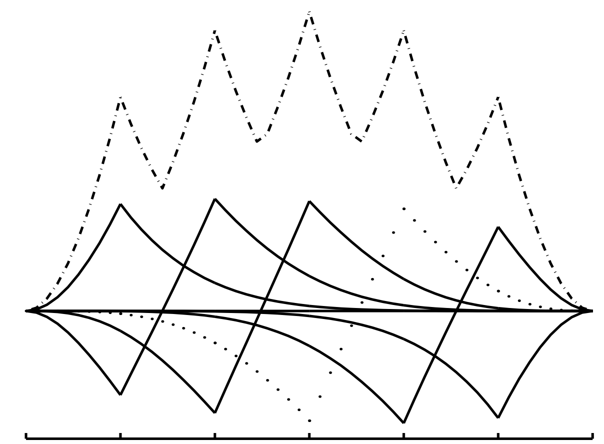

Figure 4: The first antidifferences $\gamma_i(t) := \gamma_{ki}(t)$ on $t \in [t_k, t_{k+1}] = [k, k+1]/d$, for $d = 6$ and $i = 0, \ldots, 5$ (*solid* except γ_3 *dotted*), and their sum (*dash-dotted*).

Proof With $b_0 = b_d = 0$, $\Delta b_i := b_{i+1} - b_i$, and (c.f. Figure 4)

$$\gamma_{ki}(t) := \sum_{j=0}^{i} \alpha_{kj} = \sum_{j=0}^{i} B_j^d - \begin{cases} 0 & \text{if } i < k \\ k+1-dt & \text{if } i = k \\ 1 & \text{if } i > k, \end{cases}$$

we have $\Delta\gamma_{ki} := \gamma_{ki} - \gamma_{k,i-1} = \alpha_{ki}$ and hence

$$\begin{aligned} \|p - \ell\|_{\infty,[0,1]} &= \max_k \| \sum_{i=0}^{d} \alpha_{ki} b_i \|_k = \max_k \| \sum_{i=0}^{d-1} \Delta\gamma_{ki} b_i \|_k = \max_k \| \sum_{i=0}^{d-1} -\gamma_{ki} \Delta b_i \|_k \\ &\leq \|\Delta b\|_\infty \mathrm{L}_\infty(d) \end{aligned}$$

where $\mathtt{L}_\infty(d) := \max_k \| \sum_{i=0}^d |\gamma_{ki}| \|_k$. To determine $\mathtt{L}_\infty(d)$ we observe that $\gamma_{ki} > 0$ for $i < k$ and $\gamma_{ki} < 0$ for $i > k$ and hence

$$\begin{aligned}
\sum_{\substack{i=0\\i\neq k}}^{d} |\gamma_{ki}| &= \sum_{i=0}^{k-1} \gamma_{ki} - \sum_{i=k+1}^{d} \gamma_{ki} = \sum_{i=0}^{k-1}\sum_{j=0}^{i} B_j^d - \sum_{i=k+1}^{d}\left(1 - \sum_{j=0}^{i} B_j^d\right) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i} B_j^d + \sum_{i=k+1}^{d}\sum_{j=i+1}^{d} B_j^d = \sum_{j=0}^{k-1}\sum_{i=j}^{k-1} B_j^d + \sum_{j=k+2}^{d}\sum_{i=k+2}^{j} B_j^d \\
&= \sum_{j=0}^{k-1}(k-j)B_j^d + \sum_{j=k+2}^{d}(j-k-1)B_j^d = \beta_{kk} + \beta_{k,k+1}.
\end{aligned}$$

The missing term, γ_{kk}, changes sign on $[t_k, t_{k+1}]$:

$$\begin{aligned}
\gamma_{kk} &= \sum_{j=0}^{k} B_j^d - (k+1) + dt = \sum_{j=0}^{k} B_j^d - (k+1)\sum_{j=0}^{d} B_j^d + \sum_{j=0}^{d} jB_j^d \\
&= \sum_{j=0}^{k}(j-k)B_j^d + \sum_{j=k+1}^{d}(j-k-1)B_j^d = \beta_{k,k+1} - \beta_{kk}.
\end{aligned}$$

Since the control polygon of $\gamma_{kk} = \sum_{i=0}^d c_i B_i^d$ is monotonically increasing and $c_k = c_{k+1} = 0$, there is a unique zero z_k in $[t_k, t_{k+1}]$ and

$$\sum_{i=0}^{d} |\gamma_{ki}| = \begin{cases} 2\beta_{kk} & t \le z_k \\ 2\beta_{k,k+1} & t > z_k. \end{cases}$$

Since $\sum_{i=0}^d |\gamma_{ki}|$ is a convex, non-negative function

$$\mathtt{L}_\infty(d) = 2\max\{\beta_{kk}(t_k), \beta_{k,k+1}(t_k+1)\} = 2\mathtt{N}_1(d).$$

$\bowtie$

We omit the similar derivation of $\mathtt{L}_1(d)$ such that
$\|p(t) - \ell(t)\|_{\infty,[0,1]} \le \mathtt{L}_1(d)\|\Delta b\|_1$.

5 Bounding the Hausdorff distance

Introduced by Felix Hausdorff in 1914, the Hausdorff metric μ measures the distance of two point sets $\mathcal{L}$ and $\mathcal{P}$. It has been used e.g. in fractal approximation [13] and non-smooth optimization [4], and is defined (c.f. [9], [5]) as

$$\mu(\mathcal{P},\mathcal{L}) := \max\{\sup_{L\in\mathcal{L}} \inf_{P\in\mathcal{P}} \|L-P\|_2, \sup_{P\in\mathcal{P}} \inf_{L\in\mathcal{L}} \|L-P\|_2\}.$$

The two point sets of interest here are the curve segment $\mathcal{P}$ parametrized by p and its control polygon $\mathcal{L}$ parametrized by ℓ. The two numbers whose maximum is the Hausdorff distance, measure respectively the maximum distance of a point on the control polygon to the curve segment and the maximum distance of a point on the curve segment to the control polygon. The Hausdorff distance is independent of the parametrization and is bounded from above by all parametric distance measures:

$$\mu(\mathcal{P},\mathcal{L}) \le \|p-\ell\|_{\infty,[0,1]}.$$

The bound derived earlier for functions is also a sharp bound on the Hausdorff distance between the two point sets.

Lemma 5.1 *The bound*

$$\mu(\mathcal{P},\mathcal{L}) \le \mathtt{N}_\infty(d)\|\Delta_2 b\|_\infty$$

is sharp for the Hausdorff distance of a curve segment $\mathcal{P}$ to its Bézier control polygon $\mathcal{L}$.

Proof Set $x(t)=y(t)=4(1-t)t$. Then the Hausdorff distance 1 is taken on as the distance of the control point b_1 to the curve segment (Figure 5 *left*). and the sharpness proof and perturbation argument of the function case apply directly. ⋈

For an example where, at least in the limit, the maximum distance is taken on as the distance of a point on the curve to the control polygon, consider $x(t)=t$, $y(t)=hq(t)$, where $q(t):=4(1-t)t$. The distance from the curve point $(1/2,h)$ to the nearest point on the control leg is $h-8h^3+O(h^5)$ approaching the Hausdorff bound of h as h goes to zero (Figure 5 *right*).

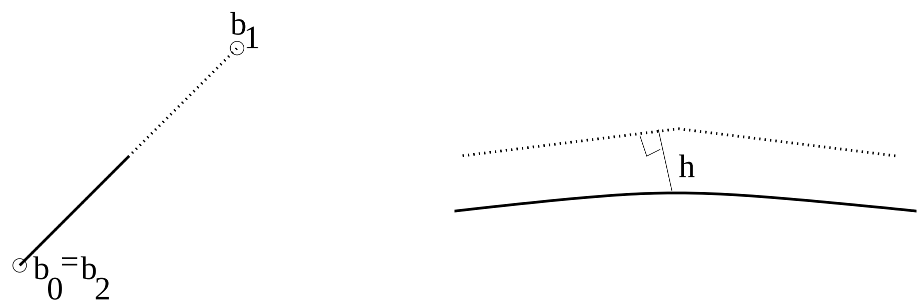

Figure 5: Sharpness of the Hausdorff distance estimate.

6 Bounding subdivision

Refinement, in particular adaptive refinement of the control point sequence to the function or curve can be achieved by creating control polygons for subintervals of the domain. Specifically, we consider

$$p(xt) = p_{[0,x]}(t) := \sum_{i=0}^{d} b_i^x B_i^d(t)$$

the restriction of $p(t) := \sum_{i=0}^{d} b_i B_i^d(t)$ to the interval $[0, x]$, $0 < x < 1$. The coefficients of the restriction can be computed by de Casteljau's algorithm

$$\begin{aligned} & b_i^0 := b_i, i = 0..d \\ & \text{for } j = 1..d \\ & \quad b_i^j := (1-x) b_i^{j-1} + x b_{i+1}^{j-1}, \ i = 0..d-j. \end{aligned}$$

The recurrence expands to

$$b_i^x = b_0^i = \sum_{k=0}^{i} B_k^i(x) b_k^0 = \sum_{k=0}^{i} B_k^i(x) b_k$$

for $i = 0, \ldots, d-2$. Figure 6 illustrates the second equality,

$$\begin{aligned} \Delta_2 b_{i+1}^x &= b_i^x - 2b_{i+1}^x + b_{i+2}^x = x^2(b_0^i - 2b_1^i + b_2^i) \\ &= x^2(\Delta_2 b_1^i) = x^2 \ [\sum_{j=0}^{i} B_j^i(x) \Delta_2 b_{j+1}]. \end{aligned}$$

The bound on the restriction is therefore just a scaled version of the original bound.

Lemma 6.1 *The distance between $p_{[0,x]}(t)$, the restriction of p to the interval $[0, x]$, and $l_{[0,x]}(t)$, the corresponding control polygon, is bounded by*

$$\|p_{[0,x]}(t) - \ell_{[0,x]}(t)\|_{\infty,[0,x]} \leq x^2 \mathtt{N}_\infty(d) \|\Delta_2 b\|_\infty$$

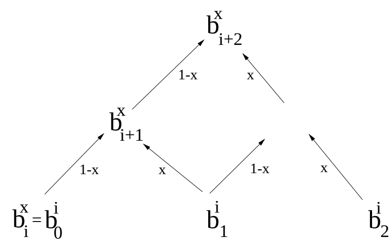

Figure 6: The coefficients b_i^x of the restriction of the polynomial p to $[0,x]$ are obtained as convex combinations of the coefficients b_j^i of the ith step of de Casteljau's algorithm.

where $\|\Delta_2 b\|_\infty$ *is the maximum absolute second difference of the coefficient sequence of* $p_{[0,1]}$.

Since the bound is sharp for any quadratic we have the following corollary.

Corollary 6.1 *The constants* $\mathtt{N}_\infty(d)$ *are sharp under subdivision.*

For example, subdividing $q(t) := 4(1-t)t$ at $0 < x < 1$ into $q_{[0,x]}$ and $q_{[x,1]}$, we get $q_{[0,x]} = 2x \cdot 2(1-t)t + 4(1-x)xt^2$, and

$$\|q_{[0,x]}(1/2) - \ell_{[0,x]}(1/2)\| = |(x + (1-x)x) - 2x| = x^2$$

equals the bound

$$\frac{2}{8}[2(2x) - 4(1-x)x].$$

The next lemma establishes the quadratic rate of convergence of the control polygon to the curve segment under subdivision.

Lemma 6.2 *The distance between the polynomial and its control polygon after* m*-fold subdivision at the local parameter* x *is bounded by*

$$\mathtt{x}^{2m}\mathtt{N}_\infty(d)\|\Delta_2 b\|_\infty \text{ where } \mathtt{x} := \max\{x, 1-x\}.$$

Proof By symmetry, the bound for the polynomial restricted to the interval $[x,1]$ is

$$\|p_{[x,1]}(t) - \ell_{[x,1]}(t)\|_{\infty,[x,1]} \leq (1-x)^2 \mathtt{N}_\infty(d)\|\Delta_2 b\|_\infty$$

and hence the distance of the curve segment to the union of the control polygons of $p_{[0,x]}$ and $p_{[x,1]}$ is bounded by $\mathtt{x}^2\mathtt{N}_\infty(d)\|\Delta_2 b\|_\infty$. ⋈

With the identity $\Delta_2 b^x_{i+1} = x^2 \left[\sum_{k=0}^{i} B^i_k(x)\Delta_2 b_{k+1}\right]$ derived earlier, the problem of *finding the optimal subdivision parameter* x becomes

$$\min_{x\in(0,1)} \max_{i=1,\dots,d-1} \{|\Delta_2 b^x_i|, |\Delta_2 b^{1-x}_i|\}$$
$$= \min_x \max_i \{x^2 |\sum_{k=0}^{i} B^i_k(x)\Delta_2 b_{k+1}|, (1-x)^2 |\sum_{k=0}^{i} B^i_k(1-x)\Delta_2 b_{d-1-k}|\}$$

- For $d = 2$, after scaling by $\Delta_2 b_0$, the problem becomes

  $$\min_x \max\{x^2, (1-x)^2\}$$

  and $x = 1/2$ is optimal.

- For $d = 3$, assuming the curve is not a straight line, the second difference can be normalized by dividing by a nonzero $\Delta_2 b_i$, without loss of generality $\Delta_2 b_2$. We may therefore assume that $\Delta_2 b_1 = 1 + \delta$ and $\Delta_2 b_2 = 1$. The problem becomes

  $$\min_x \max\{x^2|1+\delta|, (1-x)^2, x^2|(1+\delta) - \delta x|, (1-x)^2|1+\delta(1-x)|\}$$

  The numeric solution to the problem is displayed in Figure 7. The limiting optimal value $x_{\pm\infty} \approx 0.43$, is the solution of $(1-x)^3 = x^2$.

A popular criterion for determining the subdivision parameter for adaptive subdivision is the curvature of the Bézier segment. We see that neither for $d = 2$ nor for $d = 3$ is the point of maximum curvature necessarily the distance minimizing subdivision parameter.

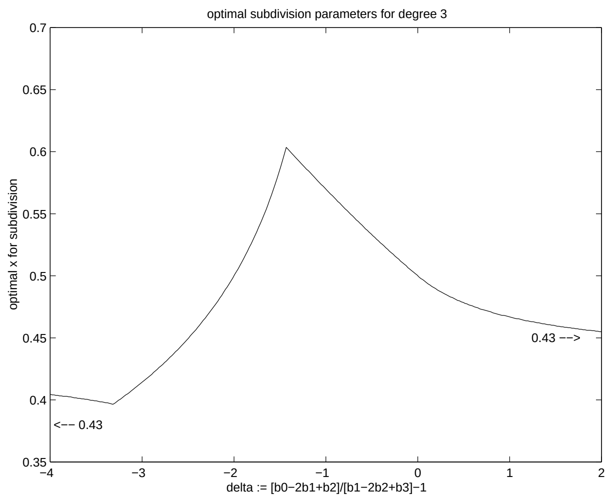

Figure 7: The optimal subdivision parameter x of a cubic as a function of δ, where $\Delta_2 b_0 = 1 + \delta$ and $\Delta_2 b_1 = 1$.

7 Bounding degree-raising

Expressing a polynomial of degree d in Bernstein-Bézier form in the basis B_j^{d+1} by multiplying the polynomial by $B_0^1 + B_1^1 = (1-t) + t$ is called degree-raising. Clearly, the number of coefficients increases by one and since the new coefficients are obtained as convex combinations of the original coefficients, it is possible to show convergence of the sequence of control polygons corresponding to repeated degree-raising to the graph of the polynomial on $[0, 1]$ (see e.g. the analysis in [14]). The next lemma reveals the rate and constant of convergence.

Lemma 7.1 *Let $\ell^{2^k d}$ be the control polygon of the polynomial p of degree d raised to the formal degree $2^k d$. Then*

$$\|p - \ell^{2^k d}\|_{\infty,[0,1]} \le K(d, 2^k d) \mathtt{N}_\infty(d) \|\Delta_2 b\|_\infty, \qquad K(d, 2^k d) \approx 2^{-k}$$

where $\|\Delta_2 b\|_\infty$ is the maximum absolute second difference of the original coefficient sequence.

For the parabola $q(t) := 4(1-t)t$, $K(2, 2^k 2) = (2^k 2 - 1)^{-1}$.

Proof Define the coefficients b_i^{d+1} by

$$(1 - t + t) \sum_{i=0}^{d} b_i^d B_i^d(t) = \sum_{i=0}^{d+1} b_i^{d+1} B_i^{d+1}(t).$$

Differentiating twice yields

$$d(d-1)\sum_{i=0}^{d-2}\Delta_2 b_{i+1}^d B_i^{d-2} = (d+1)d\sum_{i=0}^{d-1}\Delta_2 b_{i+1}^{d+1} B_i^{d-1}.$$

Since degree-raising averages, $\|\Delta_2 b^{d+1}\|_\infty$ attains a maximum when all second differences $\Delta_2 b_i^d$ are equal, implying

$$\|\Delta_2 b^{d+1}\|_\infty \leq \frac{d-1}{d+1}\|\Delta_2 b^d\|_\infty.$$

The distance between the control polygon ℓ^{d+1} of the degree-raised Bernstein representation and p is therefore

$$\|p-\ell^{d+1}\|_{\infty,[0,1]} \leq K(d,d+1)\mathtt{N}_\infty(d)\|\Delta_2 b^d\|_\infty$$

where $K(d,d+1) := \frac{d-1}{d+1}\frac{\mathtt{N}_\infty(d+1)}{\mathtt{N}_\infty(d)}$. Analogously,

$$K(d,2^k d) = \frac{d(d-1)}{2^k d(2^k d-1)}\frac{\mathtt{N}_\infty(2^k d)}{\mathtt{N}_\infty(d)} = \frac{1}{2^k}\frac{1}{d-1/2^k}\begin{cases} d-1 & \text{if } d \text{ is even.} \\ \frac{d^2}{d+1} & \text{if } d \text{ is odd.} \end{cases}$$

⋈

The effect of degree-raising on the bounds is illustrated in rows 5 and 6 of Figures 8 and 9.

Both degree-raising and subdivision generate control polygon sequences that converge to the graph of the function. However, doubling the degree by repeated degree-raising from $2^k d$ to $2^{k+1} d$ requires $\binom{2^{k+1}d+1}{2} - \binom{2^k d+1}{2} \approx 3\binom{2^k d+1}{2}$ additions and multiplications while generating the same number of coefficients via subdivision at midpoints costs only $2^k\binom{d}{2}$ operations. Moreover, the (asymptotic) reduction by $1/2$ implies slower guaranteed convergence than the reduction by $1/4$ of subdivision at the midpoint.

8 Bound improvement at the end points

Since p interpolates, the bound of Theorem 3 can be improved at the end-point $p(0) = b_0$ and, symmetrically, at $p(1) = b_d$. The first-order Taylor expansion of p at 0, $p(0)+p'(0)t$, agrees with the first leg of the control polygon, parametrized by $(1-dt)b_0 + dtb_1$. Hence, for $t \in [0, 1/d]$ and $\xi(t) \in (0,t)$,

$$\begin{aligned} |p(t) - [(1-dt)b_0 + dtb_1]| &= \left|\frac{p''(\xi(t))}{2}t^2\right| \\ &\leq \frac{d(d-1)}{2}\|\Delta_2 b\|_\infty t^2 \\ &\leq \frac{d-1}{2}\|\Delta_2 b\|_\infty t. \end{aligned}$$

In other words, on $[0, 1/d]$, p is confined to a triangle with vertices b_0 and $b_1 \pm \frac{d-1}{2d} \|\Delta_2 b\|_\infty$.

9 Bound examples

The gallery of examples collected in Figures 8 and 9 illustrates (1) sharpness for the $\mathtt{N}_\infty$ bound, (2) approximate sharpness of the $\mathtt{N}_\infty$ bound, (3) sharpness of the $\mathtt{N}_1$ bound, (4) effect of inflection points, (5) more inflection points, (6) degree-raising. In particular, Figure 9 shows the benefits of combining bounds.

10 Conclusion

The explicit bounds on the distance of the control polygon to its Bézier segment presented in this paper facilitate a constructive, quantitative derivation of the fundamental piecewise linear control and approximation properties of the Bernstein-Bézier representation. The proof technique applies to uniform (B-)splines [11], splines with arbitrary knot sequences, and extends to several variables not just by tensoring. Applications of the bounds for splines over arbitrary knot sequences are given in [12].

Acknowledgements: The presentation benefited from the careful reading of T. Kunkle, V. J. Srinivasan, U. Reif and the referees.

References

[1] Cohen, E., and Schumaker, L. L. Rates of convergence of control polygons. *Comput. Aided Geom. Design 2*, 1-3 (1985), 229–235. Surfaces in CAGD '84 (Oberwolfach, 1984).

[2] Dahmen, W. Subdivision algorithms converge quadratically. *JCAM 16* (1986), 145–158.

[3] de Boor, C. W. B-form basics. In *Geometric Modeling: Applications and New Trends* (1987), G. F. ed., Ed., SIAM, Philadelphia.

[4] Dem'yanov, V. F., and Vasil'ev, L. V. *Nondifferentiable Optimization.* Optimization Software, Inc., 4 Park Avenue, New York, New York, 1987, pp. 11–20.

[5] Falconer, K. *Fractal Geometry*, vol. 28. John Wiley and Sons, 1990, pp. 113–117.

[6] Farin, G. *Curves and Surfaces for Computer Aided Geometric Design*, 1 ed. Academic Press Inc. (London) Ltd., 1988.

[7] Farin, G. Tighter convex hulls for rational Bézier curves. *CAGD 10* (1993), 123–125.

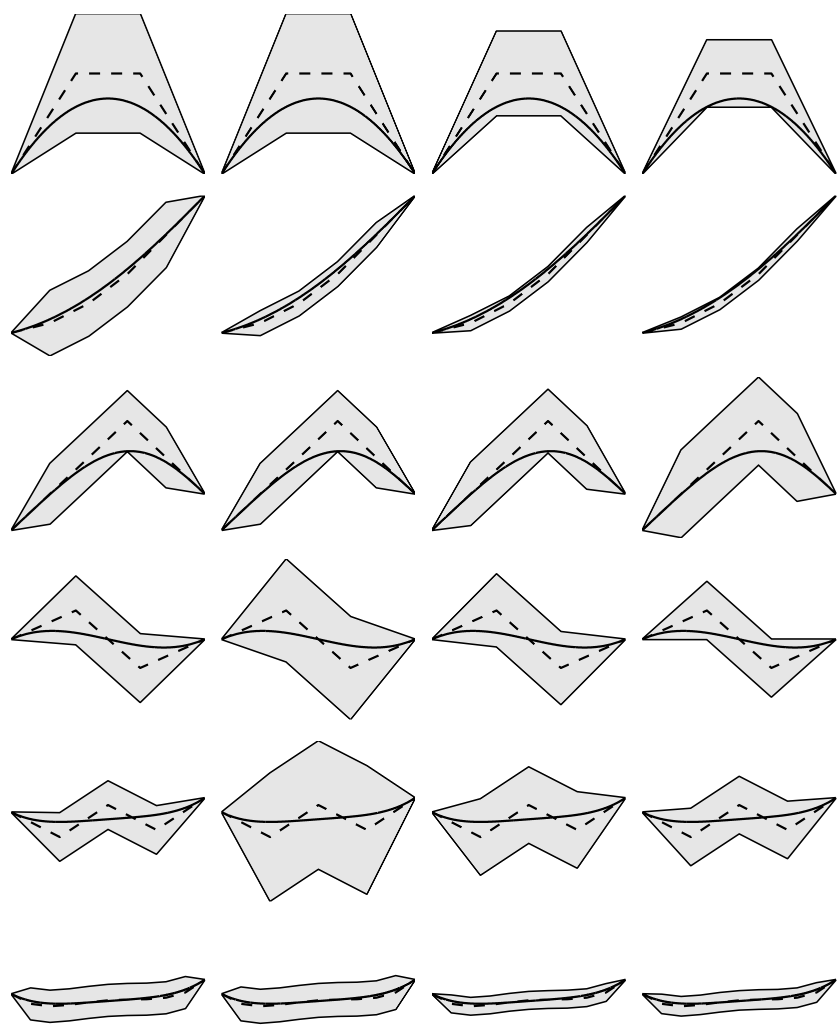

Figure 8: The control polygon is *dashed*, the bounding region *shaded*. (from *top* to *bottom*:) (1) $\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}$, (2) $\begin{bmatrix}0 & 1 & 3 & 6 & 10 & 14\end{bmatrix}$, (3) $\begin{bmatrix}0 & 1 & 2 & 3 & 2 & 1\end{bmatrix}$, (4) $\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}$, (5) $\begin{bmatrix}0 & -7 & 2 & -5 & 4\end{bmatrix}$, (6) the same polynomial raised to degree 10. (from *left* to *right*:) bounds implied by (a) the $\mathtt{L}_\infty$ bound, (b) the $\mathtt{N}_1$ bound, (c) the $\mathtt{N}_2$ bound, (d) the $\mathtt{N}_\infty$ bound.

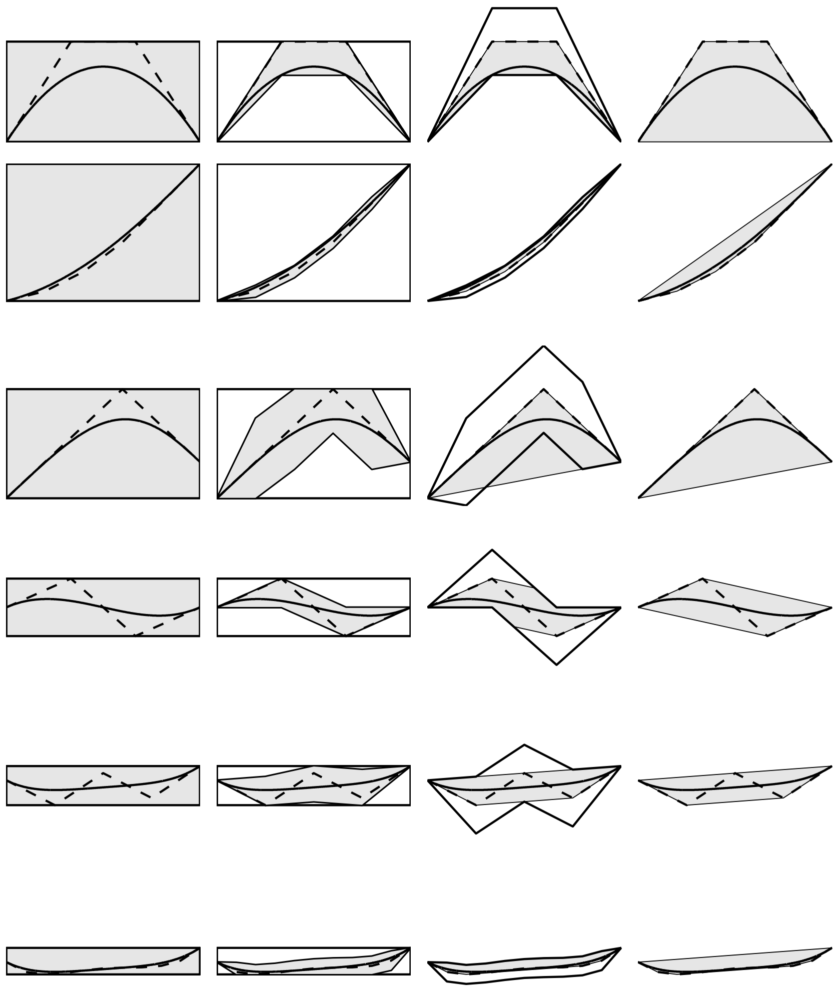

Figure 9: The control polygon is *dashed*, the bounding region *shaded*. (from *top* to *bottom*:) functions as in Figure 8. (from *left* to *right*:) bounds implied by (a) the min-max bound (b) the $\mathtt{N}_\infty$ bound clipped against the min-max bound, (c) the $\mathtt{N}_\infty$ bound clipped against the convex hull, (d) the convex hull,

[8] Filip, D., Magedson, R., and Markot, R. Surface algorithms using bounds on derivatives. *CAGD 3*, 4 (1986), 295–311.

[9] Hausdorff, F. *Set theory.* Chelsea, 1978. translated from the German.

[10] Hermann, T. On a tolerance problem of parametric curves and surfaces. *Comput. Aided Geom. Design 9*, 2 (1992), 109–117.

[11] Lutterkort, D., and Peters, J. The distance between a uniform (B)-spline and its control polygon. Tech. Rep. 98–013, Dept CISE, University of Florida, September 1998.

[12] Lutterkort, D., and Peters, J. Smooth paths in a polygonal channel. In *Proceedings of the 15th annual symposium on Computational Geometry, Miami Beach, Florida* (June 13-16 1999), acm press.

[13] Petrushev, P. P., and Popov, V. A. *Approximation of Real Functions.* Cambridge University Press, 1987, pp. 263–260.

[14] Prautzsch, H., and Kobbelt, L. Convergence of subdivision and degree elevation. *Advances in Computational Mathematics 2* (1994), 143–154.

[15] Schaback, R. Error estimates for approximations from control nets. *Comput. Aided Geom. Design 10*, 1 (1993), 57–66.

[16] Sederberg, T. W., White, S. C., and Zundel, A. K. Fat arcs: A bounding region with cubic convergence. *Computer Aided Geometric Design 6*, 3 (1989), 205–218.