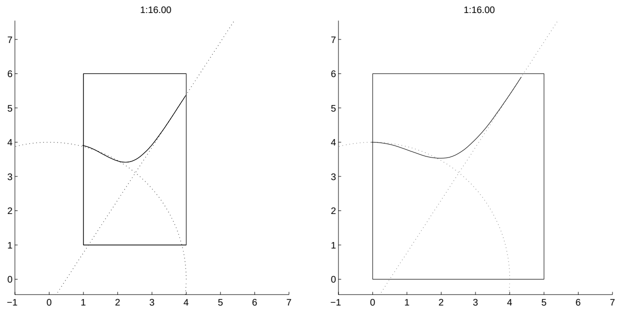

Figure 7: Blends under variation of blend volume size.

the shape of its zero set can be inferred from the 3D array of coefficients (cf. the preceding discussion of zero sheets). The blend can be modified by scaling the array-entries other than the boundary sets, e.g. by local averaging, by changing the scale λ, or by changing the blend volume(cf. Figure 7).

5.4 Compatibility with existing shape description techniques.

Combining the blend volume-approach with a set-theoretic modeling environment is straightforward: subtract the volume covered by the blend volume and add the blend volume. The 3D examples consist of clipped parametric or implicit primitives outside the blend volume while the data inside are specified by a Boolean expression over the coefficient arrays.

6 Conclusion

Smooth surfaces as the zero set of a piecewise polynomial function on a fixed-grid are capable of modeling free-form objects (see Figure 8 and also [17], [18]). However, this representation is clearly less efficient for many tasks than parametric spline-based surfacing. Our implementation is therefore specific to local blending.

The general approach offers a number of degrees of freedom that may still be explored: apart from the boundary set, the 3D array defining the blend surface can be initialized independently of the primary surfaces. In particular, non-constant scaling may be applied.

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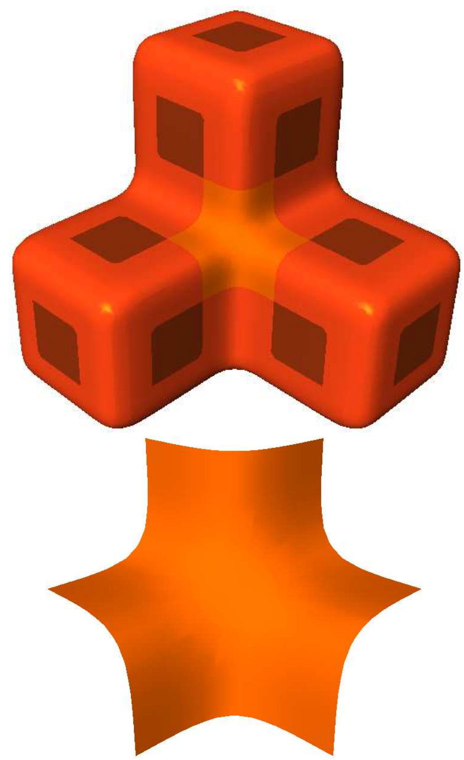

Figure 8: A curvature continuous higher-order saddle point (monkey saddle) modeled with surfaces of algebraic degree 4.

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Figure 9: Uniform univariate splines

7 Box splines

Box splines represent a generalization of univariate spline theory to several variables. Since box splines were introduced by de Boor and DeVore [8] a rich theory has been developed and collected in the "box spline book" [3] which serves as reference for the following exposition of these piecewise polynomial functions.

The box spline M_Ξ in s variables is defined by the $s \times n$ matrix Ξ (pronounced Xi) with columns in $R^s \backslash 0$. For the purposes of this paper we may assume that the first s columns of Ξ form the identity matrix I. This yields the following inductive definition of the box spline. If $\Xi = I$, then M_Ξ is the function that is 1 on the unit cube and 0 elsewhere:

$$M_I(x) := \begin{cases} 1, & \text{if } x \in [0..1)^s, \\ 0, & \text{else.} \end{cases}$$

This box spline is piecewise constant, has degree zero and is discontinuous. If $\Xi \cup \xi$ is any matrix formed from Ξ by the addition of the column $\xi \in R^s$, then the box spline $M_{\Xi\cup\xi}$ is given by the convolution equation

$$M_{\Xi\cup\xi}(u) = \int_0^1 M_\Xi(u - t\xi).$$

For $s = 1$ this is exactly the B-spline construction by convolution(c.f. Figure 9).

7.1 Box spline properties

The box spline has the following properties.

(i) M_Ξ is positive and its shifts sum to one: $\sum_{\alpha \in Z^s} M_\Xi(\cdot - \alpha) = 1$.

(ii) The support of M_Ξ is $\Xi[0..1)^s$, i.e. the set sum of the columns contained in Ξ.

(iii) M_Ξ is piecewise polynomial of degree $n - s$. That is, each convolution in another direction ξ increases the degree by one.

(iv) M_Ξ is $\rho{-}2$ times continuously differentiable, where ρ is the minimal number of columns that need to be removed from Ξ to obtain a matrix whose columns do not span R^s.

(v) M_Ξ reproduces all polynomials of degree $m := \rho - 1$ and none of degree higher than $n - s$.

(vi) The L^p approximation-order of the spline space $S :=$ span $(M(\cdot - \alpha))$ is ρ. That is with the refinement of the lattice $x \to hx$, $h < 1$, $\text{dist}(f, \sum a(\alpha) M_\Xi((\cdot - \alpha)/h)) = O(h^\rho)$ for all sufficiently smooth f.

Thus the n columns of Ξ, which may be interpreted as directions in R^s, determine the support of the piecewise polynomial and its continuity properties. Understanding the number ρ requires an analysis of the independent submatrices of Ξ.

7.2 Box spline examples

We develop three examples relevant to this paper.

1. The well-known univariate uniform cubic B-spline has the matrix (direction set)

$$\Xi := \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 9 shows in order the characteristic function of the 1-dimensional cube and its repeated convolution in the direction 1 yielding the linear 'hat' function, the quadratic and finally the cubic B-spline. We determine the characteristic numbers as

$$s = 1, n = 4, \text{ and } \rho = 4$$

since all elements of the set have to be removed to make it nonspanning in R^s. The degree of the B-spline pieces is $n - s = 3$ and the continuity is of order $\rho - 2 = 2$ as expected. The cubic spline formed as a linear combination of B-splines is guaranteed to at least reproduce polynomials of degree 3 and none of degree higher than 3. The approximation order is 4.

2. The bivariate box spline M_Ξ based on the matrix

$$\Xi := \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

is called Zwart-Powell element [29], [20], [21]. It is displayed in Figure 12 (lower right). The characteristic numbers are

$$s = 2, n = 4, \text{ and } \rho = 3$$

and hence the element is of degree 2 and its polynomial pieces are connected C^1. Since $n - s = 2 = \rho - 1$ linear combinations of the ZP-element reproduce exactly all

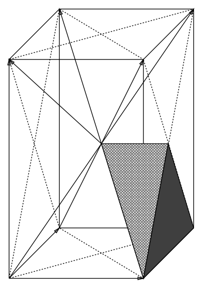

Figure 10: The 7 directions of the box spline and its domain tetrahedra

quadratic polynomials; that is any quadratic $q(x, y)$ can be written as

$$q(x,y) = \sum_{\alpha \in Z^2} a(\alpha) M_\Xi((x,y) - \alpha).$$

The Zwart-Powell element stands out among the low-degree box-splines defined over the plane, in that it has maximal smoothness equal to the degree minus one and is piecewise polynomial over a regular triangulation.

3. The 7-direction box spline is a similar serendipity element among the trivariate box splines. It is based on the direction matrix

$$\Xi := \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The seven directions defined by the colums of the matrix cut R^3 into a symmetric regular arrangement of tetrahedra. The characteristic numbers of the 7-direction box spline are

$$s = 3, n = 7, \text{ and } \rho = 4.$$

Thus the polynomial piece defined over each tetrahedron is of degree $n - s = 4$ and splines formed as a linear combination of shifts of the box spline are $C^{\rho-2} = C^2$. Elements of the spline space reproduce all cubics in three variables (and some additional polynomials of degree four) and the approximation order is 4.

7.3 Box spline subdivision

To quickly approximate any box spline we may use subdivision. Since the shifts of the box spline M_Ξ form a nonnegative, local partition of unity, a spline formed as a linear combination of shifts of the box spline is a finite convex combination of its coefficients $a(\alpha)$. To the extent that the local variation of the coefficients is small, the coefficients $a(\alpha)$ approximate the spline well. This is the basis for fast algorithms for graphic display and rendering. The key observation is that the variation of the coefficients is reduced when the spline is expressed in terms of box splines corresponding to the refined lattice $\frac{1}{2}Z^s$:

$$\begin{array}{cc} 4a & 4b \\ & \\ 4c & 4d \end{array} \quad \rightarrow \quad \begin{array}{cccc} a & a & b & b \\ a & a & b & b \\ c & c & d & d \\ c & c & d & d \end{array}$$

$$\rightarrow_{1/2} \begin{array}{cccc} . & . & . & . \\ aa & ab & bb & . \\ ac & bc & bd & . \\ cc & cd & dd & . \end{array} \quad \rightarrow_{1/4} \begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & aabc & abbd & . \\ . & accd & bcdd & . \end{array}$$

Figure 11: Four-direction box spline subdivision.

$$\sum_{j \in Z^s} a(j) M(x - j) = \sum_{k \in \frac{1}{2} Z^s} a_{1/2}(k) M(2(x - k)).$$

The successive computation of a sequence of refined coefficients $a_1, a_{1/2}, \ldots$ is called a **subdivision algorithm:** We compute $a_{h/2}$ from a_h for α on the finer mesh $\frac{h}{2} Z^s$. First set

$$a_{h/2}(\alpha) := \begin{cases} 2^s a_h(\alpha), \text{if } \alpha \in hZ^s \\ 0, \text{else} \end{cases}.$$

Then average in each of the directions in Ξ. That is for each $\xi \in \Xi$ compute, careful not to overwrite still needed values,

$$a_{h/2}(\alpha) \leftarrow (a_{h/2}(\alpha) + a_{h/2}(\alpha - \xi/2))/2.$$

Under mild assumptions on the matrix Ξ that are satisfied by all three box splines defined above, the sequence of control points converges quadratically to the spline [3], (30)Theorem, page 169. The sequence of array entries for the subdivision of a spline with coefficients a, b, c, d and the ZP-element as M is displayed in Figure 11.

To illustrate the effect of subdivision as approximate evaluation, we choose one box-spline coefficient (at the origin) non-zero, and all other coefficients equal zero, i.e.

$$a(\alpha) = \begin{cases} 1, \text{ if } \alpha_1 = \alpha_2 = 0 \\ 0, \text{ else} \end{cases} \qquad \alpha := (\alpha_1, \alpha_2).$$

Then

$$\sum_\alpha a(\alpha) M_\Xi((x,y) - \alpha) = M_\Xi(x,y),$$

and the spline represents just a single basis function. Figure 12 below shows four steps of subdivision on the spline coefficients. The central spike is of height 1.

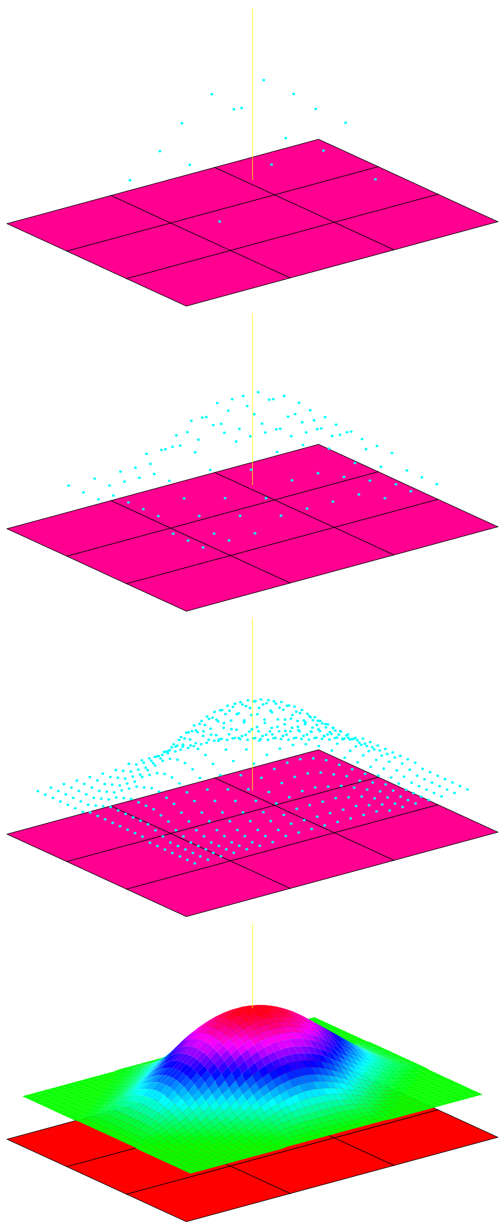

Figure 12: The Zwart-Powell element M_Ξ approximated using 4 steps of subdivision. The point cloud are the coefficients generated by the refinement.

In priciple, one can convert any piecewise polynomial in box-spline form into any other piecewise polynomial representation such as the power form or the Bernstein form. For example, in the Bernstein-Bézier form, the Zwart element is represented by 28 quadratic pieces with coefficients 1/2, 1/4, 1/8 and 0.

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