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Spline surfaces from irregular control meshes

Surface splines extend the definition of B-splines (NURBS) and box splines to allow the modeling of free-form surfaces following the outlines of possibly irregular control meshes, where a mesh point may have n neighbors and a mesh panel m sides. On regular meshes, where every mesh point is surrounded by quadrilateral panels, C^1 smooth surface splines reproduce biquadratic non-uniform tensor-product B-splines and 4-direction box splines. Linear and rational quadratic surface features can be represented exactly. Blend ratios, analogous to knot spacings, allow a direct manipulation of the change of the surface normal.

1. Surface splines introduced

Analogous to the well-known (B-)splines, a *surface spline* is defined by (cf. Figure 1_{left})

- mesh points $\mathcal{M}$,
- mesh connectivity $\uparrow$,
- blend ratios (knot spacings) a_i,

where the connectivity is consistent with that of a bivariate surface but can otherwise be irregular with n-valent mesh points and m-sided, not necessarily planar, mesh panels. The surface spline can be represented in terms of polynomial pieces, called patches. Figure 1_{middle} displays the patches as a net of Bernstein-Bézier coefficients (see e.g. [2]). These are evaluated to obtain the output surface Figure 1_{right}. The patches can be chosen to be either 3-sided, 4-sided or a mixture of both (Figure 2). Figure 3 illustrates how a surface follows its control mesh. The sharp corner is an intended feature realized by setting the blend ratios at the mesh point to zero. This freedom in adjusting blend ratios gives local control of the surface curvature. Setting the ratio to zero even allows a jump in the normal direction as intended for example for the back face and the claws of the plastic gripper element shown in Figure 4_{middle}. The effect of setting all blend ratios to zero is illustrated in Figure 9_{right}. Surface splines with the same mesh connectivity and blend ratios form a linear vector space.

2. Related work.

Apart from tensor-product B-splines and box splines [1], the closest relatives of surface splines are G-splines and generalized subdivision algorithms. The corresponding literature is surveyed in [3]. Surface splines may be viewed both as G-splines, since they are geometrically smooth, and as generalized subdivision surfaces where the subdivision proceeds in two stages. Modification of the surface at multiple resolutions can be based on the subdivision of the Bernstein-Bézier representation. For more information see also http://www.cs.purdue.edu/people/jorg on the world wide web.

3. Piecewise polynomial representation of surface splines

Translating the control mesh into a piecewise polynomial representation, specifically the Bernstein-Bézier representation, is an affine transformation. Let C be the vector of the m control points and B the vector of n Bernstein-Bézier coefficients, then there exists a $n \times m$ matrix A whose row-entries sum to one and are positive and vary with the blend ratios such that

$$B = A * C \tag{1}$$

In the case of a C^1 surface construction, m is typically forty times smaller than n illustrating the compression rate of the surface spline over the Bernstein-Bézier representation. The action of the matrix A on the mesh control points is graphically illustrated in Figure 5. Since the row-entries sum to one and are positive, the process is one of averaging or edge and corner cutting. The blend ratios decide on the depth of the cut. For example, in the cube displayed in

Figure 5_{left}, the ratios decide how far the top quadrilateral shrinks towards its centroid. The grey region in Figure 5_{middle} indicates the nine Bernstein-Bézier coefficients of a biquadratic patch resulting from the cut after inserting averages. A small cubic perturbation of this polynomial yields a 4-sided bicubic patch. Similarly, splitting the grey panel into four 3-sided panels, a cubic perturbation yields a smooth surface consisting of four 3-sided total degree cubic patches (please see [3] for detailed formulas).

4. Representation of piecewise polynomials by surface splines

If the mesh is regular, then the process of transforming Figure 5_{left} to Figure 5_{middle} is equivalent to the well-known knot insertion followed by a basis conversion of the B-spline into Bernstein-Bézier form. Thus, where the mesh is regular, already the biquadratics join smoothly with their neighbor patches. In fact, it is easy to show (prompted by a question by Gabor Renner) that any biquadratic non-uniform tensor-product B-spline can be represented by surface splines. Moreover, planar and rational quadratic surface features can be represented exactly in this framework (Figure 6).

5. Blend ratios

Perhaps the most intriguing feature of the representation are the *blend ratios* which influence shape and curvature of the surface in a predictable manner. Figure 7 illustrates local adjustment options at vertices and edges while Figure 8 shows how the Gaussian curvature can be locally manipulated by adjusting the blend ratios in different directions. Figure 9 shows that any two of convex hull, smoothness and interpolation properties can be achieved for an input mesh. This is optimal in the sense that no surface construction, in fact not even a curve construction, can achieve all three properties simultaneously for all data sets.

6. Interpolation

Interpolation can be achieved in three ways: by solving a sparse system of equations analogous to spline interpolation; by a local modification of the mesh obtained in Figure 5_{left} and by setting blend ratios to zero. The latter generally destroys smoothness. Isolated quadratic curves defined by the a path of mesh control points, in particular global boundary curves, can be interpolated.

7. References

1 C. W. de Boor, K. Höllig, S. Riemenschneider: *Box splines, Springer Verlag, NY, 1994.*
2 G. Farin*: Curves and surfaces for computer aided geometric design, Academic Press, 1993.*
3 J. Peters C^1*-surface splines, SIAM J. of Numerical Analysis* **32**-*2 (1995), 645-666.*

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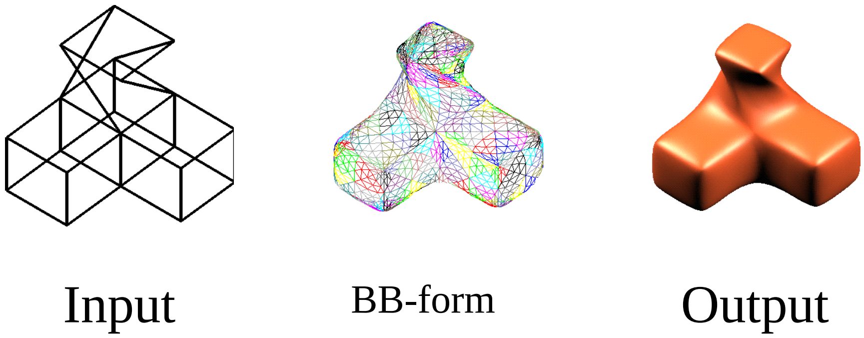

Fig.1: Definition

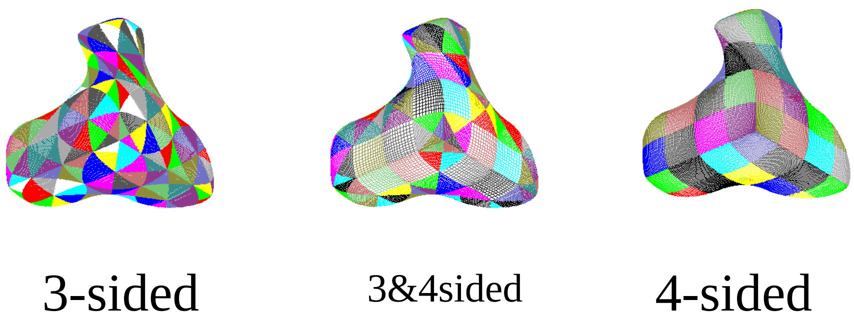

Fig.2: Alternative Representations

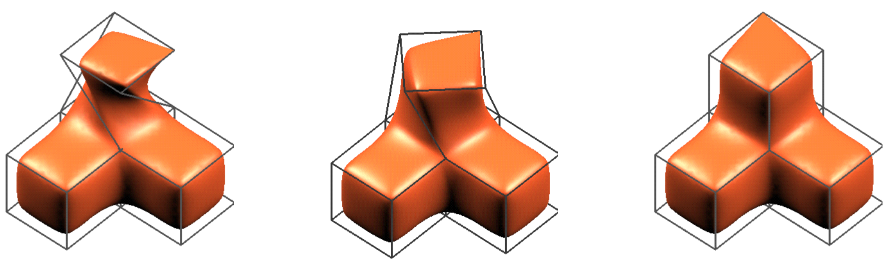
Fig.3: Surface follows the outlines of the control mesh

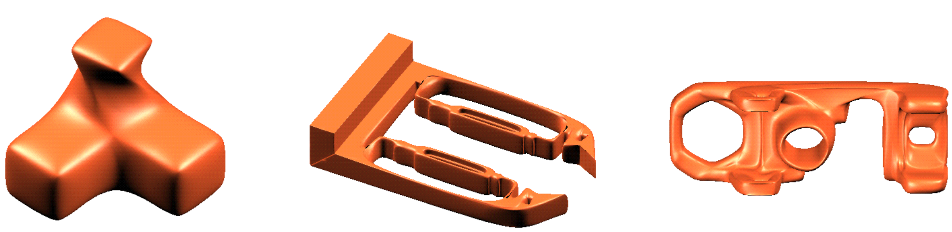
Fig.4: Complex meshes can be smoothed with locally adjustable blend ratios; Claws and attachment of the gripper feature sharp edges

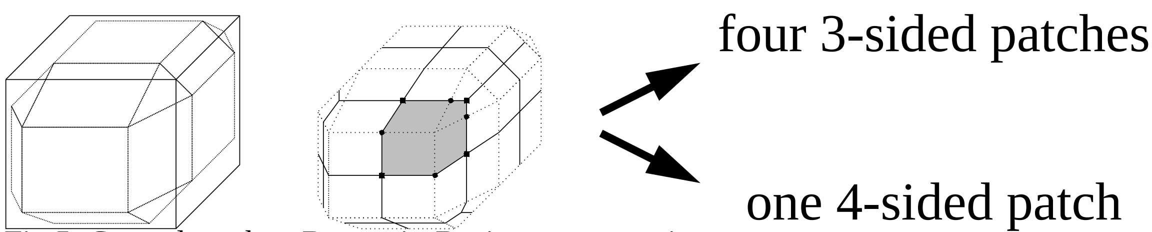

Fig.5: Control mesh to Bernstein-Bezier representation

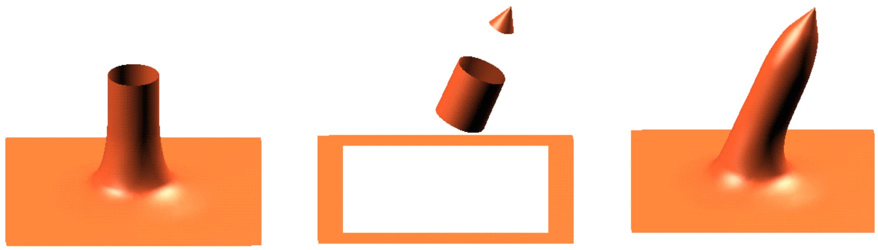
Fig.6: Reproduction of planes, cones and cylinders

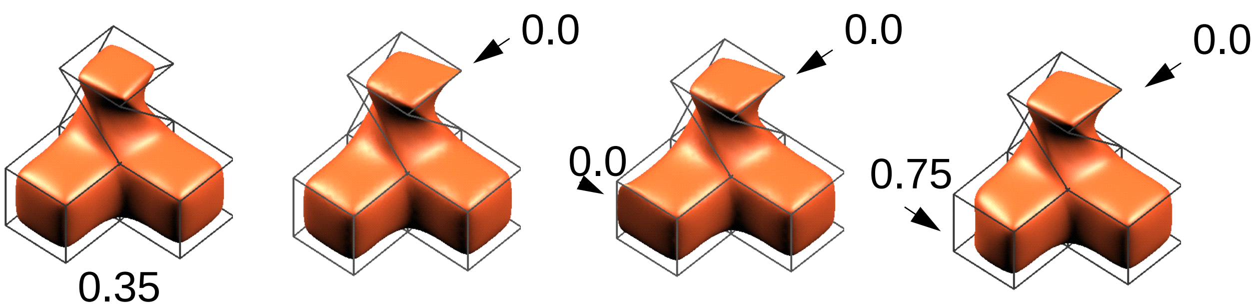

Fig.7: Shape and blend ratios (Global starting ratio is 0.35)

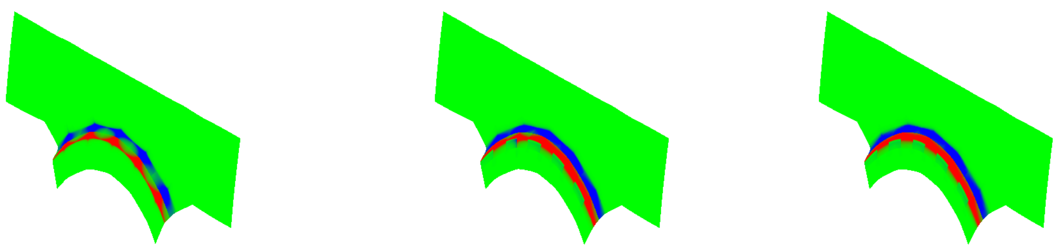
Fig.8: Adjusting the Gaussian curvature of a car fender

Fig.9: Interpolation, Convexity and smoothness