Interpolation Regions for Convex Cubic Curve Segments

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Abstract. Given a family of planar convex cubic curve segments with fixed end points and tangents, subregions of the plane are characterized in which additional points can be interpolated by at least one member of the family. The region for a second additional point is a remarkably thin double crescent.

§1. Motivation and Approach

Taking the freedom of (re)parametrization into account, $2d$ points in general position are required to pin down a real-valued, planar polynomial curve of parametric degree d: each point implies two constraints but offers the choice of one parameter value at which the curve interpolates. Two degrees of freedom have to be subtracted, since curve and degree are invariant under linear reparametrization. Conversely however, given $2d$ data points in the plane there may not exist an interpolating polynomial curve nor, if it exists, does it follow that the curve is unique. Whether it exists and exists uniquely, depends on the relative position of the points. For example, a parabolic arc is uniquely specified by four points in general position (to be compared to three for a quadratic function and five for a conic). The question as to *where* the points have to lie with respect to one another poses a nonlinear problem both in parametric and implicit representation. In the parametric representation both (re)parametrization and coefficients of the curve are unknown while in the implicit representation additional nonlinear restrictions are necessary to ensure that the curve segment is real valued and consists of just one component. As a second motivation, de Boor, Höllig, and Sabin [1] (see also the conjecture in [2]) show that a single cubic curve segment can usually but not always match six data: location, tangent direction and curvature at the end points. This is the limiting case of the study in this paper as two intermediate interpolation points approach the end points whose location and tangent direction is fixed. Given that order $O(h^6)$ approximation in [1] works for a large range of data, it is remarkable how small the convex-interpolation region for the second of two intermediate points turns out to be. Thirdly, given the frequent use of quadratic and cubic curve segments in geometric modeling, understanding their basic interpolation properties is fundamental. Applications arise for example from the need of constraint solvers to completely characterize the space of all solutions to a given set of constraints on its design primitives (see e.g. [3]). Typically, one needs to know what data can be matched by a *finite* number of curve segments and be able to enumerate those that are free of cusps, inflections and loops and join other segments with prescribed tangent to attain a smooth transition. Note that the question as to what data can be matched by one curve segment is in contrast to the well-known interpolation problem for spline curves. In spline interpolation the issue of completely understanding the interpolation properties of curves is usually side-stepped by using more degrees of freedom than data. That is, it is customary to use one curve segment per data point, additional degrees of freedom to join the pieces smoothly, and some rule, say chord length parametrization, to discard the remaining, nonlinear degrees of freedom.

For completeness, Section 2 characterizes interpolation regions for quadratics. Section 3 focuses on convex cubic segments. For cubics the direct, symbolic treatment of the interpolation question is no longer possible, because this would require solving two equations cubic in two unknowns. The key to the investigation is to treat the curves as bivariate maps $c(t, v)$, with one parameter, t, fixing the (re)parametrization, while the other, v, serves to traverse the particular curve. The two parameters t and v can and are treated symmetrically in the course of the proofs. As befits the notion of an interpolation region, the answer is given graphically in the spirit of [4] (see also [6, 7, 8]) Representing the curve segment in Bernstein-Bézier form,

$$\sum_{i=0}^{d} c_i \binom{d}{i} (1-v)^{d-i} v^i, \quad c_i \in R^2,$$

we may choose a convenient coordinate system since the results are affinely invariant; that is, the fixed coordinate system can be mapped to the coordinate system of interest and the interpolation regions are the affine image of the regions for the fixed coordinate system. Thus we choose the first point of the segment to be $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the last to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Also by affine mapping, we can choose the parameter values v at the end points to be 0 and 1 respectively. Additional points through which the curve segment should pass for a parameter value $v = t$ in the open interval $(0..1)$ are called *intermediate points.* To exhibit symmetries in the problem we use

$$u := 1 - v, \qquad s := 1 - t.$$

Note that $(a..b)$ represents an open interval and $[a..b]$ a closed interval.

§2. Interpolation Regions for Quadratic Plane Curves

As a warm-up exercise, we consider the case of parabolic arcs already discussed in [5]. Consider the family of curve segments, $c(t, v)$, whose members are indexed by the parameter $t \in (0..1)$ and are quadratic in the parameter $v \in [0..1]$. With the appropriate choice of coordinate system, c has the Bernstein-Bézier representation

$$c(t, v) := \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^2 + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} 2uv + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v^2, \quad v \in [0..1].$$

To complete the choice of coordinate system we choose the origin to be the first intermediate point excluding the trivial case of an intermediate point on the line segment from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which corresponds to the line segment itself as interpolation region. With t the value of v at which the quadratic $c(t, \cdot)$ interpolates the origin, $t(2sb_1(t) + t) = 0$ and $s(2tb_2(t) + s) = 0$ has to hold. Since $0 < t < 1$, we may solve for

$$b_1(t) = -\frac{t}{2s}, \qquad b_2(t) = -\frac{s}{2t}.$$

After substitution, we see that c is a bivariate rational map of degree $2, 1$ in the numerator and $0, 1$ in the denominator. Since the parameters t and v are still free, we may attempt to interpolate a fourth point (x, y) by solving the equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v^2 - \frac{t}{s}uv \\ u^2 - \frac{s}{t}uv \end{bmatrix}.$$

The result is summarized in Figure 2 and the following proposition.

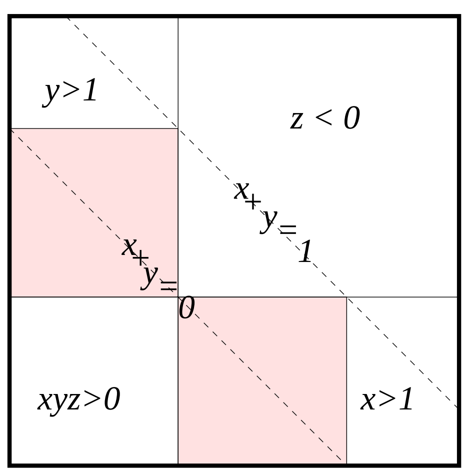

Fig. 2. A quadratic curve segment interpolating $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in order can interpolate the point $\begin{bmatrix} x \\ y \end{bmatrix}$ if and only if $\begin{bmatrix} x \\ y \end{bmatrix}$ falls into the shaded region. Here $x + y + z = 1$.

Proposition 2. *Let* $c : (0..1) \times [0..1] \mapsto \mathbb{R}^2$ *be a family of parabolic curve segments with*

$$c(t,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c(t,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c(t,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then a member $c(t^*, \cdot)$ *of the family can interpolate a point* $\begin{bmatrix} x \\ y \end{bmatrix}$ *if and only if* $x < 0$ *and* $0 < y < 1$ *or, symmetrically,* $y < 0$ *and* $0 < x < 1$*. If it exists, the curve* $c(t^*, \cdot)$ *is unique.*

Proof: Since $s = 1 - t$ and t are both nonzero, the interpolation constraint is equivalent to

$$xs = v^2 s - tuv = v(s-u) \qquad yt = u^2 t - suv = u(t-v).$$

Since $t, v \in (0..1)$, $x \geq 1$ implies $xs > vs > v(s-u)$. Therefore $x < 1$ and, by the symmetric argument, $y < 1$ has to hold. With $z := 1 - x - y$, the two linear-quadratic equations in two unknowns have the formal solutions, $\sigma \in \{-1, 1\}$,

$$v^*_\sigma = \frac{x + \sigma\sqrt{-xyz}}{x+y}, \quad t^*_\sigma = \frac{xz + \sigma\sqrt{-xyz}}{(x+y)z}.$$

The second intermediate point is real, and t^* and v^* are in their respective intervals if and only if

$$xyz < 0 \qquad z > 0$$

and σ is the sign of y. Finally checking that the limit is well defined as $x + y$ goes to zero establishes t^* and v^* as a solution. ■

Note that the prescribed order of traversal of the points rules out the second parabolic interpolant enumerated in [5].

§3. Interpolation Regions for Convex Cubic Plane Curves

We now consider the 2-parameter family of cubic plane curve segments, $c(t, v)$ that interpolate given locations and tangent directions at the end points and are convex. As before, we characterize the regions where one, respectively two intermediate points can be interpolated. Even for the restricted cubics, the problem is considerably harder than in the quadratic case, since a direct solution of the second intermediate point requires finding the joint roots of two bivariate cubic rational functions which in general can not be done symbolically.

Ignoring for now the case of parallel tangents, we can use the end points $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the intersection-point of the end-tangents as our coordinate system. That is, we write

$$c(t, v) := \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^3 + \begin{bmatrix} 0 \\ b_1(t) \end{bmatrix} 3u^2v + \begin{bmatrix} b_2(t) \\ 0 \end{bmatrix} 3uv^2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v^3.$$

Any other convex configuration of end points and finite tangent intercept is affinely related to the above choice. It is easy to see that a member of the family $c(t, v)$ interpolates the point (x, y) for $t \in (0..1)$ if and only if

$$b_2(t) = \frac{x - t^3}{3st^2}, \qquad b_1(t) = \frac{y - s^3}{3s^2t}.$$

If $b_1 b_2 < 0$ then the curvature at the end points of the segment is of opposite sign and if both b_1 and b_2 are negative then the curvature at the midpoint and either end point is of opposite sign (cf. Figure 3.1).

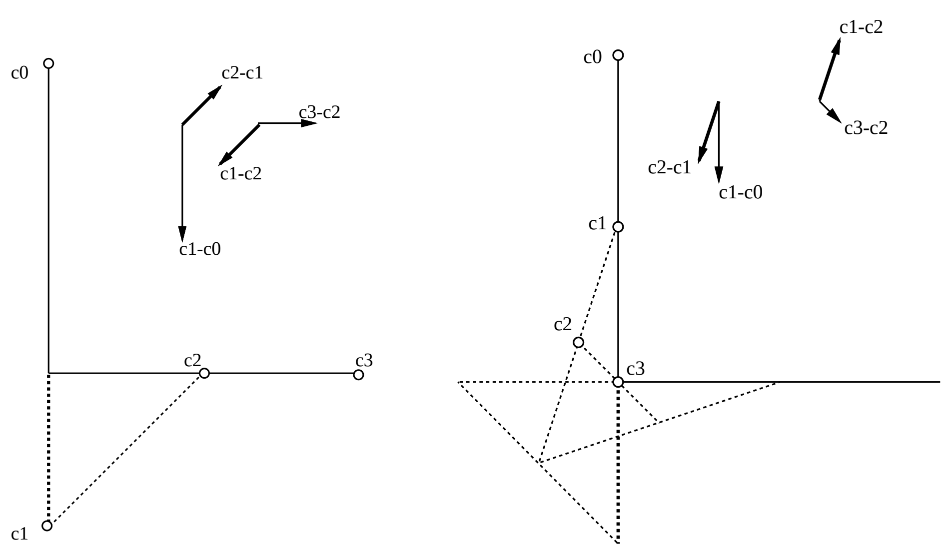

Fig. 3.1: The sign of the curvature at endpoints of the cubic segment with Bernstein coefficients $c0, c1, c2, c3$ is defined by the relative orientation of the first two difference vectors. The first of the two vectors is rendered bold.
If $b_1 b_2 < 0$ (left) then the orientation at the endpoints is opposite.
If $b_1 < 0$ and $b_2 < 0$ (right) then the orientation at end and midpoint differs.
The sketch shows the control polygon after subdivision at the midpoint.

For the curve to be convex we therefore require $0 < b_1(t) < 1$ and $0 < b_2(t) < 1$. Then t is constrained by

$$t^3 \le x \le 3st^2 + t^3, \quad s^3 \le y \le 3s^2t + s^3.$$

We can interpret the inequalities as yielding for a given parameter value t a $3st^2 \times 3s^2t$ rectangle of points that can be interpolated by curves from the family. We determine the minimal parameter value $\underline{t}$ and the maximal parameter value $\bar{t}$ to a given interpolation point (x, y) as

$$\underline{t} := \max\{1 - y^{\frac{1}{3}}, r_1\}, \quad \bar{t} := \min\{x^{\frac{1}{3}}, 1 - r_2\}$$

where r_1 is the least real root in $[0..1]$ of $-2r^3 + 3r^2 - x$ and r_2 is the least real root in $[0..1]$ of $-2r^3 + 3r^2 - y$. If the curve $c(\underline{t}, \cdot)$ interpolates the y-component of the interpolation point with the (least possible) parameter value $v = s^3$, then $b_1 = 0$ and hence $c(\underline{t}, \cdot) = [\,*, u^3\,]$. This implies that $c(\underline{t}, \cdot)$ reaches each y-level with the least possible parameter value.

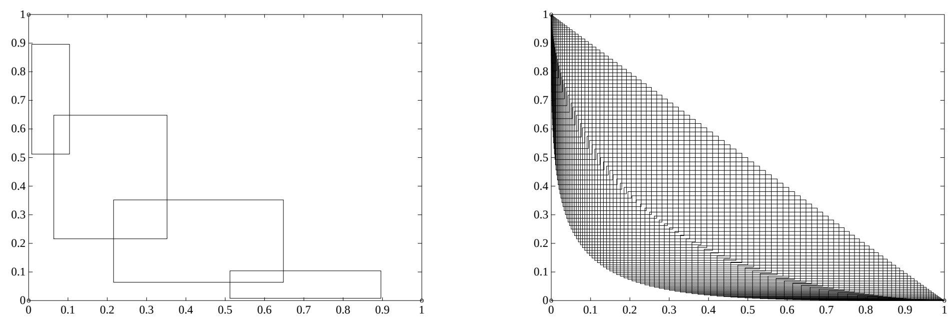

Fig. 3.2: (Left) Regions of points reachable at time $t = .2, .4, .6, .8$. (Right) The union of the reachable regions for $0 \le t \le 1$ is the region in which an intermediate point can be interpolated by a convex cubic with fixed end points and tangents.

We now consider the interpolation region for the second intermediate point. The interpolation region forms a simply connected region, shaped like a *double crescent* and bounded by a small number of specific curves from the cubic family of interpolants (cf. Figure 3.3). Following the direct approach of Section 2, one might attempt to set up and solve two additional interpolation equations. However, since the polynomials involved are cubic in the two parameters t and v, these equations are not solvable symbolically. The key to characterizing the interpolation region is an analysis of the regularity of the bivariate map c.

Lemma 3.2. *The rank of the Jacobian $Dc(t, v)$ on the segment $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$, respectively on the segment $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, depends only on t and not on v.*

On either segment there are at most four real values of t such that $|Dc| = 0$. These are the zeros of

$$f(t) := t^4 - 2t^3(1+x-y) + t^2(1+5x-y) - 4tx + x(1-y).$$

Proof: We compute

$$\frac{\partial c(t,v)}{\partial v} = \frac{1}{(1-t)^2t^2}\begin{bmatrix} sv(2t^3-3t^2v-2x) \\ tu(2s^3-3s^2u-2y) \end{bmatrix}$$
$$\frac{\partial c(t,v)}{\partial t} = \frac{(1-v)v}{(1-t)^2t^2}\begin{bmatrix} \frac{v}{t}(t^3-3tx+2x) \\ \frac{u}{s}(s^3-3sy+2y) \end{bmatrix}$$

to get

$$|Dc(t,v)| = \frac{\partial c(t,v)}{\partial v} \times \frac{\partial c(t,v)}{\partial t} = k * f(t)$$

where $k := (v-t)\frac{3u^2v^2}{s^4t^4}$ is nonzero from one end point to the first intermediate point. ■

We can now characterize the convex cubic interpolation of two intermediate points.

Theorem 3.3. *Let $c : (0..1)\times[0..1] \mapsto \mathbb{R}^2$ be a family of cubic curve segments that interpolate*

$$c(t,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c(t,t) = \begin{bmatrix} x \\ y \end{bmatrix}, \quad c(t,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and have tangent directions $(0,-1)$ and $(1,0)$ at $v=0$ and $v=1$ respectively. Then the region in which a second intermediate point can be interpolated is simply connected and bounded by a subset of the six curves

$$c(\underline{t},\cdot), c(t_1,\cdot), c(t_2,\cdot), c(t_3,\cdot), c(t_4,\cdot), c(\overline{t},\cdot),$$

where $t_1,\ldots,t_4$ are the zeros of $f(t)$.

Proof: As a subset of the interpolation region for one intermediate point, the interpolation region for two intermediate points is bounded. Consider a point $c(t',v')$ on the boundary. Then $c(t',v')$ is an extreme point of the curve $c(\cdot,v')$, and hence either $t' \in \{\underline{t},\overline{t}\}$ or $Dc(t',v')$ is singular and hence t' is one of t_1 through t_4. Thus any point on the boundary of the region belongs to one of the six curves. Since the iso-v curves are continuous and connect at $v=0$ and $v=1$, they must fill the region in between with a continuum of curves. ■

The detailed example in the next section illustrates that indeed any of the singular and extremal curves can be part of the boundary of the region.

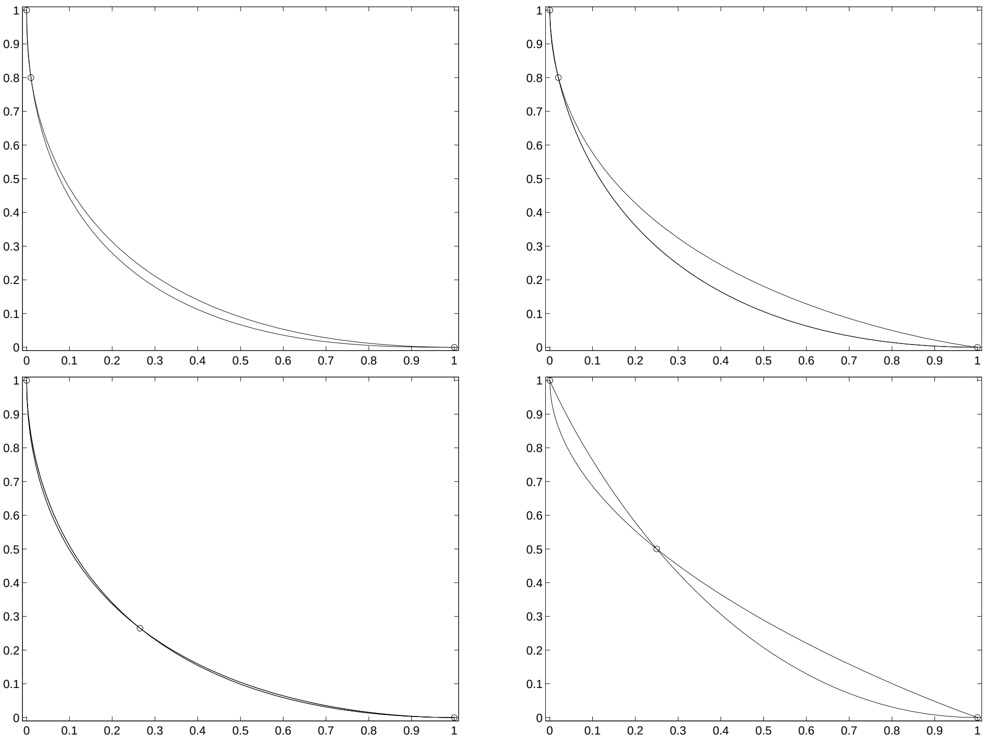

Fig. 3.3: Interpolation regions of a convex cubic for a second intermediate point. Regions in the figures have zero one or two extreme t curves as boundaries. The top figures show the effect of even a small change in the x coordinate of the intermediate point.

Given that the order six approximation of [1] usually works, it is remarkable how small the interpolation region for the second of two points turns out to be when we restrict the curve to be free of inflections and loops.

Finally, we consider the case of parallel tangents. If all points and tangents lie on the same line then so does the interpolation region as may be shown, for example, by the convex hull property of the Bernstein-Bézier representation of the curve. If the tangents are parallel and point in the same direction so that the segment can be convex, we may choose the coordinate system so that for fixed p and $a > 0$,

$$c(t,v) := \begin{bmatrix} 0 \\ 0 \end{bmatrix} u^3 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 3u^2 v + \begin{bmatrix} 1 \\ p+a \end{bmatrix} 3uv^2 + \begin{bmatrix} 1 \\ p \end{bmatrix} v^3.$$

To interpolate the point (x, y) at $0 < v = t < 1$, $x = t^2(3(1-t)+t)$ and hence $0 < x < 1$ has to hold and additionally

$$0 < a = \frac{y - 3s^2 t - ps^3}{3st^2} - p$$

and hence $y > ps^3 + p3st^2 + 3s^2 t$ for some $0 < t < 1$.

§4. A Cubic Example

To see that indeed any of the six curves can bound the interpolation region, we look at interpolation regions for $x = y$. Since $x = y$ the bounds on $b_1(t)$ and $b_2(t)$ imply $0 \le x \le \frac{1}{2}$ (in fact $x \ge \frac{1}{5}$ must hold) and the four zeros of $|Dc|$ away from the end points and the first intermediate point are

$$r_{1..4} := \frac{1}{2}(1 \pm \sqrt{1 - 8x \pm 4\sqrt{x(5x-1)}}).$$

The third power of the only root of $1 - 3t^2 + t^3$ in $[0..1]$, $\hat{x} \approx 0.278067$, is the upper bound for x. Since $1 - 8x + 4\sqrt{x(5x-1)}$ vanishes at $x = \frac{1}{4}$ and since $1 - 8x - 4\sqrt{x(5x-1)}$ is either negative or imaginary, $|Dc|$ has either

$$\begin{cases} 0 & \text{real roots if } x < \frac{1}{4} \\ 1 & \text{real root if } x = \frac{1}{4} \\ 2 & \text{real roots if } \hat{x} \ge x > \frac{1}{4}. \end{cases}$$

If $x < \frac{1}{4}$, Theorem 3.3 implies that $c(\underline{t}, \cdot)$ and $c(\overline{t}, \cdot)$ bound the double crescent. If $\hat{x} > x > \frac{1}{4}$, the two real solutions are $\frac{1}{2}(1 \pm \sqrt{1 - 8x + 4\sqrt{x(5x-1)}})$ and the bounding curves are singular curves that change with x. The pattern for curves of fixed t parameter crossing one another as the first intermediate point is moved from $x = y = \frac{1}{5}$ to $x = y = \frac{1}{2}$ is laid out below, followed by graphs of the singular and extremal curves in a small subsection of the crescent.

$x = y$						
	a	b	c	d	e	f
$\vert$	a	b	d	c	e	f
$\vert$	a	d	b	e	c	f
$\vert$	a	d	e	b	c	f
$\vert$	a	e	d	c	b	f
$\vert$	e	a	d	c	f	b
$\vert$	e	d	a	f	c	b
$\vert$	e	d	f	a	c	b
$\vert$	e	f	d	c	a	b
$\downarrow$	f	e	d	c	b	a

Table 4.1: Ordering of six members $c(t, \cdot)$ of the family of convex cubics that interpolate (x, x) for x increasing from 0.2 to 0.5.

We conclude with the observation that, as opposed to the quadratic case, $2d$ data, endpoints, tangents and two intermediate points, do not uniquely determine a convex cubic. Using Maple with 50 digits, the curves $c(t_1, \cdot)$ and $c(t_2, \cdot)$ with

$$t_1 := .521942452\ldots, t_2 := .603367345\ldots,$$

interpolate $\begin{bmatrix} 0.255 \\ 0.255 \end{bmatrix}$ and for $v_1 := .316479578\ldots$, and $v_2 := .406193943\ldots$,

$$c(t_1, v_1) = c(t_2, v_2) = \begin{bmatrix} .091 \\ .5 \end{bmatrix}.$$

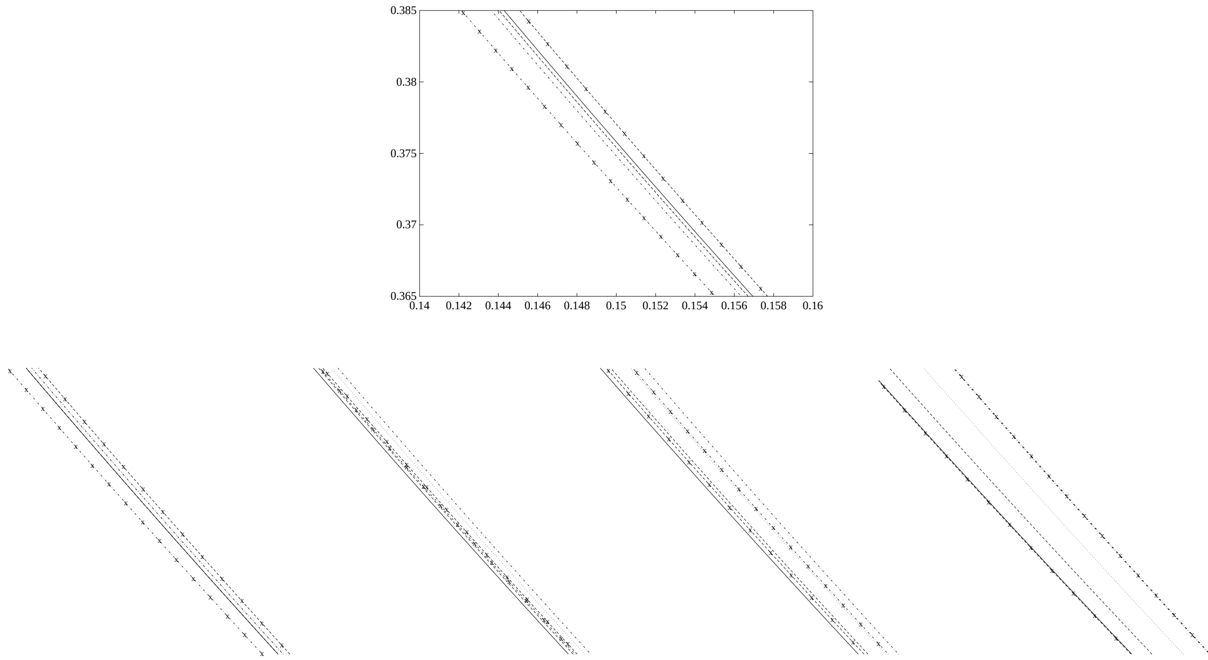

Fig. 4.1. Enlarged detail showing the reversal of the ordering of the curves in one crescent as $x = y$ increases.

However, the curves traced out differ since the normalized tangents at the common point differ in the fourth digit

$$\frac{D_v c(t_1, v_1)}{\|D_v c(t_1, v_1)\|} = \begin{bmatrix} .39727\ldots \\ -.91769\ldots \end{bmatrix}, \frac{D_v c(t_2, v_2)}{\|D_v c(t_2, v_2)\|} = \begin{bmatrix} .39769\ldots \\ -.91751\ldots \end{bmatrix}.$$

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