A characterization of connecting maps as nonlinear roots of the identity

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Abstract. In order to define the smoothness of a piecewise polynomial surface, the domains of adjacent pieces must be related to one another by connecting maps; such maps reparametrize the surface pieces by mapping the domains of adjacent pieces to a joint domain. We characterize the subclass of connecting maps that can be used to surround a point by three or more pieces. The characterization of connecting maps for second order continuity suggests a lower bound on the degree of any curvature continuous surface assembled from polynomial pieces.

§1. Motivation

A popular approach to modeling smooth parametric surfaces is to assemble them from polynomial patches $p_k : \Omega_k \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$. To determine the smoothness of transition from one patch to its neighbor, the domains of adjacent patches must locally be mapped to a joint domain so that directions of differentiation are well defined. Thus connecting maps $\phi_k : \mathbb{R}^2 \mapsto \Omega_k$ play a central role in the construction of smooth parametric surfaces, affecting for example the polynomial degree and the shape of the surfaces. Of particular interest for constructions is the subclass of connecting maps that can be used to smoothly surround a point by three or more patches. The paper characterizes this subclass.

When three or more patches join smoothly at a common point, the pairwise continuity constraints between the patches form a circular system. Correspondingly, the composition of all n connecting maps must map any initial domain to itself and must agree with the identity map, id, at the preimage of the common point up to the given order of continuity (cf. [2, Theorem

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7.1]). This motivates viewing the connecting maps as roots of the identity. In particular, if all connecting maps at the point are equal, we call them *uniform roots* of the identity.

The connecting-map across an edge between two vertices must Hermite interpolate the connecting-maps at the vertices. Based on the characterization of uniform and special non uniform second-order roots of the identity we prove that the Hermite interpolant cannot be linear, and hence that the formal degree of a curvature continuous surface built from polynomial pieces exceeds the degree of the boundary curves by four.

The paper is organized as follows. Section 2 formalizes the notion of connecting maps and gives a closed form expression for the constraint on their composition. Section 3 looks at linear roots of the identity and identifies uniform linear roots as rotations. Sections 4 and 5 characterize second-order uniform and special non-uniform roots. This characterization is used in Section 6 to derive a lower bound on the degree of curvature continuous piecewise polynomial surfaces.

§2. Roots of the identity

To formalize the constraint on the connecting maps, let $k = 1..n$ and denote by ϕ_k the connecting map between the domain $\Omega_{k-1} \subset \mathbb{R}^2$ of the $(k-1)$st patch p_{k-1} and the domain Ω_k of the adjacent patch p_k. Concretely, let Ω_k have two consecutive edges aligned with the unit vectors e_1 and e_2. Then ϕ_k maps e_2 to e_1 in a neighborhood of $\mathbf{0} = (0,0)$. Circularity implies that $\Omega_0 = \Omega_n$ and that $\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$ maps a neighborhood of $\mathbf{0}$ to itself. Since we are only interested in the values of functions at $\mathbf{0}$, we write the name of a function when we really mean that function's value at $\mathbf{0}$. Let $D_i := \frac{\partial}{\partial x_i}$ be the derivative in the direction of the ith unit vector e_i. Thus

$$J_r\phi := (D_1^m D_2^n \phi)_{m+n\le r}$$

is an ordered collection of Taylor coefficients of a connecting-map ϕ expanded at $\mathbf{0}$ up to the rth Taylor term. The composition constraint on admissible ϕ_l is

$$J_r\ \mathrm{id} = J_r(\circ_{l=1}^n \phi_l) := J_r(\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1), \tag{C}$$

where $\circ$ is the symbol for composition. Note that by the above convention, both sides of Constraint C are evaluated at $\mathbf{0}$. We normalize ϕ_l such that $\phi_l(0,0) = \mathbf{0}$ for $l = 1..n$. Hence, for $r = 0$, C is

$$\mathbf{0} = \circ_{l=1}^n \phi_l. \tag{2.0}$$

Denote the components of any connecting map ϕ as $\phi^{[1]}$ and $\phi^{[2]}$ and define the derivative of ϕ to be the Jacobian, $D\phi := \begin{pmatrix} D_1\phi^{[1]} & D_2\phi^{[1]} \\ D_1\phi^{[2]} & D_2\phi^{[2]} \end{pmatrix}$. For $r = 1$ and $i \in \{1,2\}$, since the derivative of the identity map at $\mathbf{0}$ is the identity matrix and the ith column of the identity matrix is the ith unit vector e_i, the additional constraints are, by the chain rule,

$$e_i = D_i(\circ_{l=1}^n \phi_l) = (\prod_{l=1}^n D\phi_l)e_i \tag{2.1}$$

For $r = 2$, and $i, j \in \{1, 2\}$, since id has no quadratic terms, the chain rule and the product rule yield

$$\mathbf{0} = D_j D_i(\circ_{l=1}^n \phi_l) = \sum_{k=1}^n (\prod_{l<k} D\phi_l) D^2\phi_k \Big((\prod_{l>k} D\phi_l)e_i, (\prod_{l>k} D\phi_l)e_j \Big). \tag{2.2}$$

Each of the two components of the Hessian $D^2\phi_k\Big(,\Big)$ is a bilinear form with two vector-valued arguments.

Since Ω_k must share an edge with $\phi_k(\Omega_{k-1})$, it is reasonable to stipulate that Ω_k and $\phi_k(\Omega_{k-1})$ share a coordinate direction corresponding to the common edge. This implies that the edge is traced with a common orientation and parameter v:

$$\phi(0, v) = \begin{bmatrix} v \\ 0 \end{bmatrix} \tag{A$_1$}$$

and that the transversal derivative (with respect to u) of $\phi^{[2]}$ is constant for varying v:

$$D_1 D_2 \phi^{[2]} = 0. \tag{A$_2$}$$

§3. Uniform linear roots of the identity

We first consider the case $r = 1$, the characterization of the linear components of the connecting-maps. For now, we assume that the connecting maps are *uniform*, that is $\phi_l = \phi$. Since the neighborhood of the origin is to be covered exactly once, the linear part of ϕ is a rotation by $\theta := 2\pi/n$.

Proposition 3. *If $\phi_l = \phi$, $l = 1..n$, and (A$_1$) holds then (2.0) and (2.1) hold if and only if*

$$\phi = \begin{bmatrix} \phi^{[1]} \\ \phi^{[2]} \end{bmatrix} := \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \textit{higher order terms}$$

Proof: The assumption $\phi(0, v) = (v, 0)$ implies $D\phi = \begin{bmatrix} u_1 & 1 \\ u_2 & 0 \end{bmatrix}$. Since $D\phi$ has to have the eigenvalues $e^{\pm\iota\theta}$, $\iota := \sqrt{-1}$, of a rotation matrix,

$$e^{\pm 2\iota\theta} - u_1 e^{\pm\iota\theta} - u_2 = 0$$

must hold. From this constraint, $u_1 = 2\cos\theta$, $\quad u_2 = -1$ follows.

Conversely, the linear part of ϕ can be diagonalized:

$$D\phi := \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -e^{-\iota\theta} & -e^{\iota\theta} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\iota\theta} & 0 \\ 0 & e^{\iota\theta} \end{bmatrix} \begin{bmatrix} -e^{-\iota\theta} & -e^{\iota\theta} \\ 1 & 1 \end{bmatrix}^{-1}$$

implying that

$$\begin{aligned} D(\circ_{k=1}^{l}\phi) &= \prod_{k=1}^{l} D\phi = \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix}^{l} \\ &= \begin{bmatrix} -e^{-\iota\theta} & -e^{\iota\theta} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\iota\theta} & 0 \\ 0 & e^{\iota\theta} \end{bmatrix}^{l} \begin{bmatrix} -e^{-\iota\theta} & -e^{\iota\theta} \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{F(1)} \begin{bmatrix} F(l+1) & F(l) \\ -F(l) & -F(l-1) \end{bmatrix} \end{aligned}$$

where

$$F(l) := e^{\iota\theta l} - e^{-\iota\theta l} = 2\iota\sin(\theta l).$$

If $l = n$, then $\theta l = 2\pi$ and hence

$$D(\circ_{k=1}^{n}\phi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as required for a first-order connecting-map by condition (2.1). ∎

§4. Uniform quadratic roots of the identity

We now apply the calculus of the previous section to the case $r = 2$. In particular, we want to characterize the uniform non-linear connecting maps that satisfy (2.0–2.2).

Theorem 4. *If $\phi_l = \phi$, $l = 1..n$ and (A_1) and (A_2) hold, then (2.0), (2.1) and (2.2) hold if and only if*

$$\begin{aligned} \phi := &\begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &+ \frac{1}{2}\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x_1 & a \\ a & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_1 \\ &+ \frac{1}{2}\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_2 \\ &+ h.o.t. \end{aligned}$$

for certain constants x_1, x_2, a.
If $n > 3$, then x_1, x_2, a can be chosen independently and arbitrarily.
If $n = 3$, then $x_1 = x_2 = -2a$ must hold.

Proof: The proof is structured as follows. First we express (2.2) in terms of $F(l) := 2\iota\sin(\theta l)$ and use the diagonalization derived in the previous section.

Then we show that the sums of $F(l)$ that multiply the constants x_1, x_2, and a vanish except if $n = 3$. The case $n = 3$ is analyzed separately.

Since $D_2^2\phi = \mathbf{0}$ and $D_1 D_2 \phi^{[2]} = 0$ by (A_1) and (A_2), the kth summand of the right hand side (2.2) can be expressed using the constants $A_{ij}(k)$ and $B_{ij}(k)$:

$$(D\phi)^{k-1} D^2\phi\Big((D\phi)^{n-k}e_i, (D\phi)^{n-k}e_j\Big) \tag{4.1}$$
$$= F^{-3}(1)\begin{bmatrix} F(k) & F(k-1) \\ -F(k-1) & -F(k-2)\end{bmatrix}(D_1^2\phi A_{i,j}(k) + D_1 D_2\phi B_{i,j}(k)).$$

The constant $A_{i,j}(k)$ is the entry in the ith row and jth column of the matrix $A(k)$ which tabulates all possible combinations of the first entry of $(\prod_{l=k+1}^n D\phi)e_i$ and $(\prod_{l=k+1}^n D\phi)e_j$ since these multiply $D_1^2\phi$. Since $F(0) = F(n) = F(n/2) = 0, F(n+l) = F(l)$ and $F(-l) = -F(l)$,

$$\begin{aligned} A(k) &:= \begin{bmatrix} F(n-k+1) \\ F(n-k)\end{bmatrix}[\, F(n-k+1) \quad F(n-k)\,] \\ &= \begin{bmatrix} F^2(n-k+1) & F(n-k)F(n-k+1) \\ F(n-k)F(n-k+1) & F^2(n-k)\end{bmatrix} \\ &= \begin{bmatrix} F^2(k-1) & F(k)F(k-1) \\ F(k)F(k-1) & F^2(k)\end{bmatrix}. \end{aligned}$$

Similarly $B_{i,j}(k)$ is the i,j entry of

$$\begin{aligned} B(k) &:= \begin{bmatrix} F(n-k+1) \\ F(n-k)\end{bmatrix}[\, -F(n-k) \quad -F(n-k-1)\,] \\ &\quad + \begin{bmatrix} -F(n-k) \\ -F(n-k-1)\end{bmatrix}[\, F(n-k+1) \quad F(n-k)\,]. \\ &= -\begin{bmatrix} 2F(k)F(k-1) & F(k+1)F(k-1) + F^2(k) \\ F(k+1)F(k-1) + F^2(k) & 2F(k)F(k+1)\end{bmatrix}. \end{aligned}$$

Combining all the summands, we have $D_j D_i(\circ_{l=1}^n \phi) =$

$$(F(1))^{-3}\sum_{k=1}^n \begin{bmatrix} F(k)x_1 + F(k-1)x_2 \\ -F(k-1)x_1 - F(k-2)x_2\end{bmatrix} A_{i,j}(k) + a\begin{bmatrix} F(k) \\ -F(k-1)\end{bmatrix} B_{i,j}(k).$$

Now we note that the multipliers of a, x_1 and x_2 are of the form

$$\sum_{k=1}^n \sin(k\theta)\sin((k+l)\theta)\sin((k+m)\theta)$$
$$= \alpha_1\sum_{k=1}^n \sin(k\theta) + \alpha_2\sum_{k=1}^n \sin^3(k\theta) + \alpha_3\sum_{k=1}^n \cos(k\theta) + \alpha_4\sum_{k=1}^n \cos^3(k\theta)$$

for constants α_1, α_2 α_3 and α_4.
Claim: If $n > 3$, then

$$\sum_{k=1}^{n} \sin(k\theta) = \sum_{k=1}^{n} \cos(k\theta) = \sum_{k=1}^{n} \sin^3(k\theta) = \sum_{k=1}^{n} \cos^3(k\theta) = 0.$$

proof of claim: Let $\Im a$ denote the imaginary part of a and $\Re a$ the real part of a. Then

$$\sum_{k=1}^{n} \sin(k\theta) = \sum_{k=0}^{n-1} \Im\, e^{\iota k\theta} = \Im\,(\frac{e^{\iota n\theta}-1}{e^{\iota\theta}-1}) = 0,$$

since $n\theta = 2\pi$ and similarly $\sum_{k=1}^{n} \cos(k\theta) = 0$. Since $4\cos^3\alpha = \cos 3\alpha + 3\cos\alpha$,

$$\sum_{k=1}^{n} \cos^3(k\theta) = \frac{1}{4}\sum_{k=0}^{n-1} \Re\, e^{\iota k3\theta} = \Re\, \frac{e^{\iota n3\theta}-1}{e^{\iota 3\theta}-1} = 0$$

and similarly $\sum_{k=1}^{n} \sin^3(k\theta) = 0$.
end of proof of claim

The claim proves the theorem for $n > 3$. If $n = 3$, then $\sum_{k=1}^{3} \cos^3(k\theta_3) = \Re 3 \neq 0$ and we need to analyze (2.2), $\mathbf{0} = D_j D_i(\phi \circ \phi \circ \phi)$, in detail. We now list the three cases $i = j = 1$, $i \neq j$, and $i = j = 2$ of (2.2), one per column, and use the fact that $F(0) = 0$ and $F(2) = -F(1)$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix} a + \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} x_2.$$

We see that (2.2) holds if and only if $x_1 = x_2 = -2a$. ■

§5. Non uniform quadratic roots of the identity

Next, we characterize first order uniform, but second order non uniform connecting-maps that satisfy (2.2) for the special case of fourth roots.

Proposition 5. *If $n = 4$ and $J_1\phi_l = J_1\phi$ for $l = 1..4$, then (2.0–2.2) hold if and only if*

$$\begin{aligned} \phi_k := & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ & + \frac{1}{2} [u \quad v] \begin{bmatrix} x_{1,k} & a_k \\ a_k & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_1 \\ & + \frac{1}{2} [u \quad v] \begin{bmatrix} x_{2,k} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e_2 \\ & + h.o.t. \end{aligned}$$

and

$$a_1 = a_3, \quad a_2 = a_4, \quad x_{i,j} = -x_{i,j+2}, \quad \text{for } i,j \in \{1,2\},$$

Proof: We follow the structure of the proof of Theorem 4. Since $F(1) = 1 = -F(3)$, $F(0) = F(2) = F(4) = 0$,

$$A(k) := \begin{bmatrix} F^2(k-1) & 0 \\ 0 & F^2(k) \end{bmatrix}, \quad B(k) := (-1)^{k+1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Setting

$$a := \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad x_1 := \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ x_{1,3} \\ x_{1,4} \end{bmatrix}, \quad x_2 := \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ x_{2,4} \end{bmatrix}$$

the equations 2.2 and 4.1 simplify to

$$D_1 D_2(\circ_{l=1}^4 \phi) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} a$$

$$D_1 D_1(\circ_{l=1}^4 \phi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_2$$

$$D_1 D_2(\circ_{l=1}^4 \phi) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} x_2$$

Setting the expressions to zero proves the claim. ■

The next section uses the following simple extension of the Proposition.

Corollary 5.2. *If $n = 8$ and every odd connecting map is the identity, then Proposition 5 applies to the even numbered connecting maps.*

§6. Degree bounds for curvature continuous surfaces

We now apply the theorems developed in Sections 4 and 5 to estimate the minimal degree of polynomial pieces necessary for building free-form surfaces that follow the outline of an irregular mesh. A mesh is irregular if neither the degree of its vertices nor the number of vertices to a mesh cell is restricted. To improve our chances of fitting a low degree surface we may decrease the combinatorial complexity of the input mesh by inserting a midpoint on every edge and connecting the midpoints of a cell to its centroid (see [3]). After this refinement every original vertex is surrounded by vertices of degree four and all cells are quadrilateral. To mimic the quartic C^2 box spline with directions $e_1, e_1, e_2, e_2, e_1+e_2, e_1-e_2$ and keep the *total degree* of the surface pieces low, we split each quadrilateral into four triangles and connect the patch domains by the identity across the splitting edges.

Since we are interested in a worst case analysis, we can cook up data. In particular, let P be one of the midpoints generated above surrounded by 8 patches and with original mesh point neighbors P_i, $i = 1..4$. We may assume that P_1 and P_3 have different degree, but that the data at both points are *locally symmetric*. That is, the data relevant to the determination of each connecting map are indistinguishable under rotation. Thus any rotation invariant construction must use uniform roots of the identity at P_1 and P_3. At P, the data are not locally symmetric, but the degrees of freedom at P are maximal when the linear part of the connecting map is uniform. We therefore proceed under the assumption that the connecting maps at P are uniform up to first order. Let ϕ_i be the connecting map associated with the edge PP_i and $\lambda_i(v) := (D_2\phi_i^{[1]})(0,v)$. Then by Corollary 5.2 either λ_1 or λ_3 has to be at least quadratic if (2.0–2.2) are to hold. For if both λ_i were linear, then their derivative at P is determined differently depending on the number of patches meeting at P_i and thus $a_1 \neq a_3$.

Now consider the two patches $p(u,v)$ and $q(u,v)$ with a common boundary curve $\gamma(u)$ such that $\gamma(0) = P_1$ and $\gamma(1) = P$. Let $\phi := \phi_1$ be the connecting map between p and q, $\lambda := \lambda_1$ be quadratic and choose the data such that symmetry implies $D_2\phi_i^{[2]} \equiv -1$. If d is the degree of γ then the left hand side of the G^1 constraints (cf. [1])

$$\lambda D_1\gamma = D_2p + D_2q \tag{G$_1$}$$

is of degree $d-1+2$ and hence D_2p and D_2q are formally of degree $d+1$. We say formally, since D_2p and D_2q could be degree-raised polynomials. The terms λD_1D_2p and λD_1D_2q in the G^2 constraints

$$D_2^2p - D_2^2q - \lambda D_1D_2p + \lambda D_1D_2q = \frac{1}{2}(D_2q - D_2p)D_2^2\phi^{[2]} \tag{G$_2$}$$

are therefore of degree $d+1-1+2$. Unless we have cancellation, D_2^2p and D_2^2q must therefore be of degree $d+2$ and, if none of the intermediate polynomials are degree-raised, p and q must be of degree $d+4$.

Matching this bound, a curvature continuous surface spline that generalizes the C^2 box spline with directions $e_1, e_1, e_2, e_2, e_1+e_2, e_1-e_2$ and boundary curves of degree $d=4$ has recently been developed and implemented by the author.

References

1. Gregory, J.A., Geometric continuity, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker (eds.), Academic Press, New York, 1989, 353–371.
2. Hahn, J, Geometric continuous patch complexes, Computer-Aided Geom. Design **6** (1989), 55–67.
3. Peters, J., C^1 free-form surface splines SIAM J. Numer. Anal. **x** (199x), xx–xx.

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