Bézier Nets, Convexity and Subdivision on Higher Dimensional Simplices

Tim Goodman
Department of Mathematics
and Computer Sciences
The University of Dundee
Dundee DD1 4HN, Scotland

Jörg Peters [1]
Department of Computer Sciences
Purdue University
W-Lafayette, IN 47907-1398
USA

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1 Introduction

It is common to represent a polynomial on an interval or triangle by its Bézier points, since this is useful both in theory and in applications such as computer aided geometric design. One useful feature is that the appropriate piecewise linear interpolant of the Bézier points, the so-called Bézier net, reflects the shape of the polynomial curve or surface. For the univariate case the total positivity properties of the Bézier representation give rise to many such shape preserving properties, see [5]. For the bivariate case, total positivity is not available, but it is shown in [2] that convexity of the Bézier net implies convexity of the polynomial surface. This work also implies (though it is not stated there) that the Bernstein polynomial $B_d f$ of a function f on a triangle preserves a stronger form of convexity, which is related to the triangle.

Another useful feature of the Bézier representation is that it provides an approximation to the polynomial. If the interval or triangle is subdivided, then a new piecewise linear approximation can be gained by combining the Bézier nets over the constituent pieces. Under suitable successive subdivision, these composite Bézier nets will converge to the polynomial surface and can be computed by efficient algorithms [10]. A particular algorithm for regular subdivision of a triangle is given in [4]. This type of regular subdivision preserves the convexity of the Bézier net [6], (which is not true of subdivision procedures in general [8]), and thus gives an efficient means of gaining a sequence of convex, piecewise linear surfaces converging to a convex polynomial surface.

Curiously most of the above results have not previously been extended to more than two dimensions. One paper which considers this problem is [3], where it is shown that for a polynomial defined on a simplex in any dimension, convexity of the Bézier net implies convexity of the polynomial. In this paper we extend the remaining results. In more than two dimensions the situation is interesting in that there are many triangulations over which one can define the Bézier net, depending on the ordering of the vertices. We study such triangulations in Section 2. While these are the same triangulations as considered in [3], we study them in more detail so as to gain, in particular, explicit necessary and sufficient conditions for the convexity of the Bézier net, which were not given in [3]. These conditions are derived in Section 3, where they are used to show that the Bernstein polynomial $B_d f$ of a function f on a simplex preserves a stronger form of convexity, that takes the generating directions of the simplex into account. In Section 4, we derive an efficient algorithm for computing the Bézier points on a regular subdivision of a simplex, generalizing the algorithm in [4]. Finally, in Section 5, we show that this subdivision process preserves the convexity of the Bézier net.

Throughout the paper we shall employ the standard multi-index notation referred to by greek letters: for $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$,

$$|\, \boldsymbol{\alpha} \,| := \alpha_1 + \ldots + \alpha_n, \qquad \alpha! := \alpha_1! \cdots \alpha_n!$$

and for $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

A particular multi-index $\boldsymbol{\epsilon}^j$ is defined as

$$\epsilon_i^j := \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Depending on the context $\mathbf{V} := [V_0, \ldots, V_m]$ means either that $\mathbf{V}$ is a matrix with columns $V_0, \ldots, V_m$ or that $\mathbf{V}$ is a simplex with vertices $V_0, \ldots, V_m$.

2 Triangulation

Let $\mathbf{V}$ be a simplex in $\mathbb{R}^n$ with affinely independent vertices $V_0, \ldots, V_m$ in $\mathbb{R}^n$, $1 \leq m \leq n$. For $k \geq 1$, we define a decomposition of $\mathbf{V}$ into subsimplices called the 'standard triangulation' and denoted by

$$T_{k,m} = T_{k,m}(\mathbf{V}) = T_k(V_0, \ldots, V_m)$$

as follows. Each multi-index $\boldsymbol{\alpha} \in \{0, \ldots, m\}^k$, where $\alpha_1 \leq \ldots \leq \alpha_k$ gives rise to a point

$$U_{\boldsymbol{\alpha}} := k^{-1}(V_{\alpha_1} + \ldots + V_{\alpha_k})$$

in $T_{k,m}$. A simplex $\mathbf{U}$ lies in $T_{k,m}$ if and only if it has vertices $U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}$ such that

$$\boldsymbol{\alpha}^j = \boldsymbol{\alpha}^{j-1} + \boldsymbol{\epsilon}^{\ell_j}, \qquad j = 1, \ldots, m, \quad \ell_j \in \{1, \ldots, k\}, \tag{S1}$$

$$\alpha_r^m = \alpha_{r+1}^0, \qquad r = 1, \ldots, k-1. \tag{S2}$$

Since $|\boldsymbol{\alpha}^m| = |\boldsymbol{\alpha}^0| + m$, constraint (S2) implies $\alpha_1^0 = 0$ and $\alpha_k^m = m$. Note that $T_{k,m}$ depends on the ordering of the vertices. Now given a simplex $\mathbf{U}$ in $T_{k,m}$, we can define a map

$$F_{\mathbf{U}} : \{1, \ldots, m\} \to \{1, \ldots, k\} \qquad j \mapsto F_{\mathbf{U}}(j) = \ell_j.$$

That is, $F_{\mathbf{U}}(j)$ is the unique index $\ell_j = r$ for which $\alpha_r^j = \alpha_r^{j-1} + 1$. The map $\mathbf{U} \mapsto F_{\mathbf{U}}$ is a bijection from $T_{k,m}$ to $\{1, \ldots, k\}^{\{1,\ldots,m\}}$. This implies that $|T_{k,m}| = k^m$, which is also confirmed by the fact that for each simplex $\mathbf{U}$ in $T_{k,m}$, vol $\mathbf{U} = k^{-m}$ vol $\mathbf{V}$. To clarify the definitions we give some examples. For simplicity we denote the simplex with vertices $U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}$ by $\langle \boldsymbol{\alpha}^0, \ldots, \boldsymbol{\alpha}^m \rangle$.

Example 1: $m = 2, k = 2$.

$\mathbf{U}$	$F_{\mathbf{U}}(1)$	$F_{\mathbf{U}}(2)$
$\langle 00, 01, 02 \rangle$	2	2
$\langle 01, 02, 12 \rangle$	2	1
$\langle 01, 11, 12 \rangle$	1	2
$\langle 02, 12, 22 \rangle$	1	1

Example 2: $m = 3, k = 2$.

$\mathbf{U}$	$F_{\mathbf{U}}(1)$	$F_{\mathbf{U}}(2)$	$F_{\mathbf{U}}(3)$
$\langle 00, 01, 02, 03\rangle$	2	2	2
$\langle 01, 02, 03, 13\rangle$	2	2	1
$\langle 01, 02, 12, 13\rangle$	2	1	2
$\langle 01, 11, 12, 13\rangle$	1	2	2
$\langle 02, 03, 13, 23\rangle$	2	1	1
$\langle 02, 12, 13, 23\rangle$	1	2	1
$\langle 02, 12, 22, 23\rangle$	1	1	2
$\langle 03, 13, 23, 33\rangle$	1	1	1

Example 3: $m = 2, k = 3$.

$\mathbf{U}$	$F_{\mathbf{U}}(1)$	$F_{\mathbf{U}}(2)$
$\langle 000, 001, 002\rangle$	3	3
$\langle 001, 002, 012\rangle$	3	2
$\langle 001, 011, 012\rangle$	2	3
$\langle 002, 012, 022\rangle$	2	2
$\langle 011, 012, 112\rangle$	3	1
$\langle 011, 111, 112\rangle$	1	3
$\langle 012, 022, 122\rangle$	2	1
$\langle 012, 112, 122\rangle$	1	2
$\langle 022, 122, 222\rangle$	1	1

Note that $T_k(V_0, \ldots, V_m)$ is unchanged under a cyclic permutation or reversal in the order of $(V_0, \ldots, V_m)$. Thus for given vertices $\{V_0, \ldots, V_m\}$, there are at most $\frac{1}{2}m!$ different standard triangulations. For $m = 2$, the standard triangulation is unique, but for $m \geq 3$, $T_{k,m}$ may depend on the ordering of the vertices. As an example, consider $m = 3, k = 2$. The corner simplices $\langle 00, 01, 02, 03\rangle$, $\langle 01, 11, 12, 13\rangle$, $\langle 02, 12, 22, 23\rangle$, $\langle 03, 13, 23, 33\rangle$ are clearly permuted on permuting the vertices. The remaining simplices are

$$\langle 01, 02, 03, 13\rangle, \quad \langle 01, 02, 12, 13\rangle, \quad \langle 02, 03, 13, 23\rangle, \quad \langle 02, 12, 13, 23\rangle. \tag{2.1}$$

Interchanging V_0 and V_1 changes the simplices to

$$\langle 01, 12, 13, 03\rangle, \quad \langle 01, 12, 02, 03\rangle, \quad \langle 12, 13, 03, 23\rangle, \quad \langle 12, 02, 03, 23\rangle. \tag{2.2}$$

while interchanging V_1 and V_2 changes the simplices to

$$\langle 02, 01, 03, 23\rangle, \quad \langle 02, 01, 12, 23\rangle, \quad \langle 01, 03, 23, 13\rangle, \quad \langle 01, 12, 23, 13\rangle. \tag{2.3}$$

Note that all 12 simplices in (2.1) - (2.3) are distinct.

We now investigate when a simplex $\mathbf{U} := [U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}]$ in $T_{k,m}$ shares a common face with vertices $\{U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}\} \backslash \{U_{\boldsymbol{\alpha}^p}\}$ with another simplex $\mathbf{W} := [W_{\boldsymbol{\beta}^0}, \ldots, W_{\boldsymbol{\beta}^m}]$ in $T_{k,m}$. There are three possible cases.

$0 < p < m$. In this case, $\boldsymbol{\beta}^j = \boldsymbol{\alpha}^j$ for $j \neq p$ and $\mathbf{W}$ satisfies $F_{\mathbf{W}}(p) = F_{\mathbf{U}}(p+1)$ and $F_{\mathbf{W}}(p+1) = F_{\mathbf{U}}(p)$ and hence $F_{\mathbf{U}}(p) \neq F_{\mathbf{U}}(p+1)$; $F_{\mathbf{W}}(j) = F_{\mathbf{U}}(j)$ otherwise.

$p = 0$. In this case, $\boldsymbol{\beta}^j = \boldsymbol{\alpha}^{j+1}$ for $j = 0..m-1$ and $\mathbf{W}$ satisfies $F_{\mathbf{W}}(j) = F_{\mathbf{U}}(j+1), j = 1..m-1$ and, by (S2), $F_{\mathbf{W}}(m) = F_{\mathbf{U}}(1) - 1$ and hence $F_{\mathbf{U}}(1) \neq 1$.

$p = m$. Here $\boldsymbol{\beta}^j = \boldsymbol{\alpha}^{j-1}$ for $j = 1..m$ and $\mathbf{W}$ satisfies $F_{\mathbf{W}}(j) = F_{\mathbf{W}}(j-1), j = 2..m$ and $F_{\mathbf{W}}(l) = F_{\mathbf{U}}(m) + 1$ and hence $F_{\mathbf{U}}(m) \neq m$.

We next consider triangulating the simplices in $T_{k,m}$. Such subtriangulations will be needed in Section 5.

Lemma 2.1 *Take $k, l \geq 2$ and let $\mathbf{U}$ in $T_k(V_0, \ldots, V_m)$ have vertices $U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}$, where $\boldsymbol{\alpha}^0, \ldots, \boldsymbol{\alpha}^m$ satisfy (S1) and (S2). Then $T_l(U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m})$ is the restriction of $T_{kl}(V_0, \ldots, V_m)$ to $\mathbf{U}$.*

For any multi-index $\boldsymbol{\beta} \in \{0, \ldots, m\}^l$, with $\beta_1 \leq \ldots \leq \beta_l$, we define $B_{\boldsymbol{\beta}} := l^{-1}(U_{\boldsymbol{\alpha}^{\beta_1}} + \ldots + U_{\boldsymbol{\alpha}^{\beta_l}})$ and the multi-index $\boldsymbol{\beta}^{\cup} \in \{0, \ldots, m\}^{kl}$ whose components are the components of $\boldsymbol{\alpha}^{\beta_1}, \ldots, \boldsymbol{\alpha}^{\beta_l}$, arranged in increasing order. If $\boldsymbol{\beta}^{\cup} := (\alpha_1, \ldots, \alpha_{kl})$, then, $B_{\boldsymbol{\beta}^{\cup}} = (kl)^{-1}(V_{\alpha_1} + \ldots + V_{\alpha_{kl}})$. Now take a simplex $\mathbf{W}$ in $T_l(U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m})$, with vertices $B_{\boldsymbol{\beta}^0}, \ldots, B_{\boldsymbol{\beta}^m}$ such that (S1) and (S2) hold (with $\boldsymbol{\alpha}$ replaced by $\boldsymbol{\beta}$ and k by l). It follows that (S1) and (S2) hold with $\boldsymbol{\alpha}$ replaced by $\boldsymbol{\beta}^{\cup}$ and k by kl. Thus $\mathbf{W}$ is in $T_{kl}(V_0, \ldots, V_m)$. Hence $T_l(U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m})$ is the restriction of $T_{kl}(V_0, \ldots, V_m)$ to $\mathbf{U}$. $\square$

The standard triangulation is also well-defined on any lower dimensional simplex spanned by consecutive vertices in $\mathbf{V}$.

Lemma 2.2 *For $1 \leq r \leq m-1$, the restriction of $T_k(V_0, \ldots, V_m)$ to the simplex with vertices $V_0, \ldots, V_r$ is $T_k(V_0, \ldots, V_r)$.*

It is sufficient to prove the result for $r = m-1$. The restriction of $T_k(V_0, \ldots, V_m)$ to $[V_0, \ldots, V_{m-1}]$ consists of

$$\{[U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^{m-1}}] : [U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}] \in T_k(V_0, \ldots, V_m),\ \alpha_k^i \leq m-1, i = 0..m-1\}.$$

But these are all simplices $[U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^{m-1}}]$ for which $\boldsymbol{\alpha}^i \epsilon \{0, \ldots, m-1\}^k, \alpha_1^i \leq \ldots \leq \alpha_k^i$, satisfy (S1), (S2) with m replaced by $m-1$, i.e. they form $T_k(V_0, \ldots, V_{m-1})$. $\square$

Since $T_k(V_0, \ldots, V_m)$ is unchanged under a cyclic permutation of $(V_0, \ldots, V_m)$, we can replace $(V_0, \ldots, V_r)$ in Lemma 2.2 by any cyclically consecutive subsequence of $(V_0, \ldots, V_m)$.

3 Convexity

If $\mathbf{V} = [V_0, \ldots, V_m]$ is a simplex, we say a point $\mathbf{x}$ in $\mathbf{V}$ has barycentric coordinates $\mathbf{u} = (u_0, \ldots, u_m) \in \mathbb{R}^{m+1}$ (with respect to $\mathbf{V}$) if

$$\mathbf{x} = \mathbf{V}\mathbf{u} \quad \sum_{i=0}^{m} u_i = 1.$$

If p is a polynomial of degree d on $\mathbf{V}$, we say that p has Bézier coefficients

$$b(\gamma), \qquad \text{where } \gamma \in \mathbb{Z}_+^{m+1}, \text{ and } |\gamma| = d$$

with respect to $\mathbf{V}$ if

$$p(\mathbf{x}) = \sum_{|\gamma|=d} \frac{d!}{\gamma!} b(\gamma) \mathbf{u}^{\gamma}, \mathbf{x} \in \mathbf{V}, \tag{3.1}$$

where $\mathbf{x}$ has barycentric coordinates $\mathbf{u}$. We define the Bézier net $\hat{p}$ of p with respect to $\mathbf{V}$ to be the unique continuous function on $\mathbf{V}$ which is linear or each element of $T_{d,m}(\mathbf{V})$ and satisfies

$$\hat{p}\left(\frac{1}{d}\mathbf{V}\gamma\right) = b(\gamma), \quad |\gamma| = d. \tag{3.2}$$

It is shown in [3] that if $\hat{p}$ is convex in $\mathbf{V}$, then so is p. However, [3] does not give explicit conditions for $\hat{p}$ to be convex and these we shall now give. We define

$$\epsilon^{-1} := \epsilon^m, \quad V_{-1} := V_m$$

and, for V in $\mathbb{R}^n$, the directional derivative $D_V := \sum_1^n v_i \frac{\partial}{\partial x_i}$.

Theorem 3.1 *For a polynomial p of degree d defined on the simplex $\mathbf{V} := [V_0, \ldots, V_m]$, the following are equivalent.*

a) The Bézier net $\hat{p}$ of p is convex.

b) The Bézier coefficients $b(k)$ of p satisfy

$$b(\gamma + \epsilon^i + \epsilon^{j-1}) + b(\gamma + \epsilon^j + \epsilon^{i-1}) \geq b(\gamma + \epsilon^{i-1} + \epsilon^{j-1}) + b(\gamma + \epsilon^i + \epsilon^j) \tag{3.3}$$

$$\text{for all } 0 \leq i < j \leq m \text{ and } \gamma \in \mathbb{Z}_+^{m+1} \text{ with } |\gamma| = d - 2.$$

c) The Bézier coefficients of $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} p$ are ≤ 0 for all $0 \leq i < j \leq m$.

As in the proof of Lemma 2.5 of [3], the convexity of $\hat{p}$ is equivalent to the convexity of $\hat{p}$ restricted to the union of each pair of simplices in $T_{d,m}$ that share a common face of dimension $m-1$. We show that one needs only consider a 2D slice of the abutting simplices and hence the equivalence of a) and b) follows from Inequality (2.2b) of Lemma 2.5 in [3].

Suppose that $\mathbf{U}$ and $\mathbf{W}$ in T_d have a common face of dimension $m-1$. Then, in all three cases enumerated at the end of Section 2, the points with barycentric coordinates

$$d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i-1}+\boldsymbol{\epsilon}^{j-1}) \text{ and } d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j})$$

for some i,j with $0 \le i < j \le m$ and $|\boldsymbol{\gamma}| = d-2$ are contained in both simplices while those with coordinates

$$d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j-1}) \text{ and } d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i-1}+\boldsymbol{\epsilon}^{j})$$

lie in just one of the simplices. Conversely, take any $\boldsymbol{\gamma}$ with $|\boldsymbol{\gamma}| = d-2$, and $0 \le i < j \le m$. We show that there are simplices $\mathbf{U}$ and $\mathbf{W}$ in $T_{d,m}$ with a common face of dimension $m-1$, so that both simplices contain $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i-1}+\boldsymbol{\epsilon}^{j-1})$ and $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j})$ while $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j-1})$ and $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i-1}+\boldsymbol{\epsilon}^{j})$ each lie in just one of the simplices. First suppose $i \ge 1$. Let

$$dU_{\boldsymbol{\alpha}} = V_{\alpha_0} + \ldots + V_{\alpha_d} = \sum_{l=0}^{m} \gamma_l V_l + V_{i-1} + V_{j-1}, \text{ where } \alpha_0 \le \ldots \le \alpha_d$$

i.e. the point $U_{\boldsymbol{\alpha}}$ has barycentric coordinates $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i-1}+\boldsymbol{\epsilon}^{j-1})$. We define $\mathbf{U} := [U_{\boldsymbol{\alpha}^0},\ldots,U_{\boldsymbol{\alpha}^m}]$ so that $\boldsymbol{\alpha}^{p-1} = \boldsymbol{\alpha}$ and $F_{\mathbf{U}}(l) = 1$, $l = 1..p-1$ by defining $p := \alpha_1 + 1$ and

$$\alpha_1^l := l, \quad \alpha_k^l := \alpha_k, \text{ for } l = 0..p-1,\ k = 2..d.$$

Let $U_{\boldsymbol{\alpha}^p}$ and $U_{\boldsymbol{\alpha}^{p+1}}$ have barycentric coordinates $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j-1})$ and $d^{-1}(\boldsymbol{\gamma}+\boldsymbol{\epsilon}^{i}+\boldsymbol{\epsilon}^{j})$ respectively. We complete the definition of $\mathbf{U}$ by defining $F_{\mathbf{U}}$ so that

$$F_{\mathbf{U}}(l-1) \le F_{\mathbf{U}}(l), l = p+2..m \text{ and } \alpha_d^m = m, \alpha_r^m = \alpha_{r+1}^0 = \alpha_{r+1}, r = 1..d-1.$$

Example 4: Illustration of the construction in Theorem 3.1 for $m = 3$, $d = 2$, $i = 1$, $j = 2$ and of the standard indices $\boldsymbol{\alpha}^i$ and barycentric indices $\boldsymbol{\gamma}^i$ (in parenthesis) of $T_{2,3}$.

$$\begin{array}{lcll}
\boldsymbol{\gamma} & = & & (0000) \\
\boldsymbol{\alpha} & = & 01 & (1100) \\
p & = & 1 & \\
\boldsymbol{\alpha}^0 & = & 01 & (1100) \\
\boldsymbol{\alpha}^1 & = & 11 & (0200) \\
\boldsymbol{\alpha}^2 & = & 12 & (0110) \\
\boldsymbol{\alpha}^3 & = & 13 & (0101) \\
r & = & 1 & \alpha_1^3 = \alpha_2^0 = 1 \\
 & & & \\
\boldsymbol{\beta}^1 & = & 02 & (1010)
\end{array}$$

Choosing $\mathbf{W} := [W_{\boldsymbol{\beta}^0}, \ldots, W_{\boldsymbol{\beta}^m}]$ such that $\boldsymbol{\beta}^l = \boldsymbol{\alpha}^l, l \neq p$, and $\boldsymbol{\beta}^p$ with barycentric coordinates $d^{-1}(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^j)$ defines $\mathbf{U}$ and $\mathbf{W}$ that satisfy (S1) and (S2). For $i = 0$, we define $\mathbf{U}$, $\mathbf{W}$ similarly. It now follows from the proof of Lemma 2.5 of [3], specifically Inequality 2.26, that a) is equivalent to b).

Equation 3.1 implies that the Bézier coefficients of $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} p$ are

$$d(d-1)\{b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^j) - b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^j) - b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^{j-1}) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^{j-1})\},$$

$|\boldsymbol{\gamma}| = d - 2$, and hence b) and c) are equivalent. □

Defining $\boldsymbol{\epsilon}^{m+1} := \boldsymbol{\epsilon}^0$, we see that for $m = 2$ and $i = 0, 1, 2$ the inequalities 3.3 become

$$b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^i) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^{i+1}) \geq b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^i) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^{i+1}),$$

while for $m = 3$ and $i = 0..3$, they become,

$$b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^i) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^{i+1}) \geq b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^{i-1} + \boldsymbol{\epsilon}^i) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^{i+1}),$$

$$b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^1) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^2 + \boldsymbol{\epsilon}^3) \geq b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^3) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^2),$$
$$b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^2) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^3) \geq b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^2) + b(\boldsymbol{\gamma} + \boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^3),$$

in both cases for all $|\boldsymbol{\gamma}| = d - 2$. For $m = 2$ this is (29) - (31) of [2], while for $m = 3$ this is (2.11) and (2.20) of [3].

Now for affinely independent points $V_0, \ldots, V_m$ in $\mathbb{R}^n, 1 \leq m \leq n$, we say a real-valued function f defined on $\mathbf{V} := [V_0, \ldots, V_m]$ is *strongly convex* with respect to $\mathbf{V}$ if for any $h > 0$ and $0 \leq i < j \leq m$,

$$\begin{aligned} f(\mathbf{x} + hV_i + hV_{j-1}) \quad &+ \quad f(\mathbf{x} + hV_{i-1} + hV_j) \geq f(\mathbf{x} + hV_{i-1} + hV_{j-1}) \\ &+ \quad f(\mathbf{x} + hV_i + hV_j) \end{aligned} \tag{3.4}$$

for any $\mathbf{x}$ for which this is defined, where $V_{-1} = V_m$. For $m = 1$ this is equivalent to convexity of f. For $m = 2$ it is defined in [7] and is independent of the order of V_0, V_1, V_2. For $m \geq 3$, the definition depends on the ordering of $V_0, \ldots, V_m$.

Lemma 3.2 *If f is in $C^2(\mathbf{V})$, then f is strongly convex wrto. $\mathbf{V}$ if and only if*

$$D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} f \leq 0, \quad 0 \leq i < j \leq m. \tag{3.5}$$

Integrating $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} f$ over the planar parallelogram with vertices

$$\mathbf{x} + hV_i + hV_{j-1}, \quad \mathbf{x} + hV_{i-1} + hV_j, \quad \mathbf{x} + hV_{i-1} + hV_{j-1}, \quad \mathbf{x} + hV_i + hV_j,$$

Inequality 3.5 implies Inequality 3.4. Conversely letting $h \to 0$ in Inequality 3.4 gives Inequality 3.5.□

Now for a function f on $\mathbf{V}$, we define the Bernstein polynomial $B_d f$ of f of degree d (with respect to $\mathbf{V}$) by

$$(B_d f)(\mathbf{x}) := \sum_{|\gamma|=d} \frac{d!}{\gamma!} f\left(\frac{1}{d}\mathbf{V}\gamma\right) \mathbf{u}^{\gamma}, \quad \mathbf{x} \in \mathbf{V},$$

where $\mathbf{x}$ has barycentric coordinates $\mathbf{u}$.

Lemma 3.3 *If f is strongly convex wrto. $\mathbf{V}$, then $B_d f$ is convex.*

From the definition of strong convexity, we see that the Bézier coefficients of $B_d f$ satisfy Inequality 3.3 and hence the Bézier net of $B_d f$ with respect to $V_0, \ldots, V_m$ is convex, by Theorem 3.1. It follows from Theorem 2.2 of [3] that $B_d f$ is convex.□

Theorem 3.4 *If f in $C^0(\mathbf{V})$ is strongly convex wrto. $\mathbf{V}$, then f is convex.*

By Lemma 3.3, $B_d f$ is convex for all d. But it is well-known that $B_d f$ converges uniformly to f as $d \to \infty$, [9], and hence f is convex.□

From Lemma 3.2, we immediately have

Corollary 3.5 *If f is in $C^2(\mathbf{V})$ and satisfies Inequality 3.5, then f is convex.*

Finally we have the following stronger form of Lemma 3.3.

Theorem 3.6 *If f is strongly convex wrto. $\mathbf{V}$, then so is $B_d f$.*

We have seen that the Bézier coefficients of $B_d f$ satisfy (3.3). By Theorem 3.1, the Bézier coefficients of $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} B_d f$ are ≤ 0 for all $0 \leq i < j \leq m$ and hence $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} B_d f \leq 0$. By Lemma 3.2, $B_d f$ is strongly convex with respect to $V_0, \ldots, V_m$.□

4 Subdivision

For $0 \leq j \leq m$, we define the operator E_j as follows:

$$a : \mathbb{R}^{m+1} \to \mathbb{R}, \quad E_j a(\mathbf{v}) := a(\mathbf{v} + \boldsymbol{\epsilon}^j).$$

Then Equation 3.1 can equivalently be written as

$$p(\mathbf{x}) = \left(\sum_{j=0}^{m} u_j E_j\right)^d b(\mathbf{0}), \tag{4.1}$$

where $\mathbf{x}$ has barycentric coordinates $\mathbf{u}$ (cf. [1]). For $\mathbf{V}$ as before, let $\mathbf{V}^*$ be a simplex with vertices $V_i^*, i = 0..m$ that have barycentric coordinates $\mathbf{u}^i$ with respect to $\mathbf{V}$.

Theorem 4.1 *If a polynomial p of degree d has Bézier coefficients $b(\boldsymbol{\gamma})$ with respect to $\mathbf{V}$, then it has Bézier coefficients $b^*(\boldsymbol{\gamma})$ with respect to $\mathbf{V}^*$, where*

$$b^*(\boldsymbol{\gamma}) = \prod_{i=0}^{m} \left(\sum_{j=0}^{m} u_j^i E_j \right)^{\gamma_i} b(\mathbf{0}). \tag{4.2}$$

Take a point $\mathbf{x}$ with barycentric coordinates $\mathbf{u}$ with respect to $\mathbf{V}^*$. Then

$$\mathbf{x} = \sum_{i=0}^{m} u_i V_i^* = \sum_{i=0}^{m} u_i \sum_{j=0}^{m} u_j^i V_j = \sum_{j=0}^{m} V_j \sum_{i=0}^{m} u_i u_j^i.$$

So $\mathbf{x}$ has barycentric coordinates $\mathbf{v}$ with respect to $\mathbf{V}$, where $v_j = \sum_{i=0}^{m} u_i u_j^i$, $j = 0, \ldots, m$. Then

$$\begin{aligned}
p(\mathbf{x}) &= \left(\sum_{j=0}^{m} V_j E_j \right)^d b(\mathbf{0}) \\
&= \left\{ \sum_{j=0}^{m} \sum_{i=0}^{m} u_i u_j^i E_j \right\}^d b(\mathbf{0}) \\
&= \left\{ \sum_{i=0}^{m} u_i \sum_{j=0}^{m} u_j^i E_j \right\}^d b(\mathbf{0}) \\
&= \left(\sum_{i=0}^{m} u_i E_i \right)^d b^*(\mathbf{0}).
\end{aligned}$$

The last equality follows from the multi-nomial theorem and Definition 4.2: if $\boldsymbol{\gamma} = \boldsymbol{\epsilon}^i$, then (4.2) implies $\sum_{j=0}^{m} u_j^i E_j b(\mathbf{0}) = E_i b^*(\mathbf{0})$. □

We remark that this generalizes Theorem 8 of [2], though the proof here is simpler. We now derive a formula for the Bézier coefficients with respect to the simplices in the standard triangulation $T_2(V_0, \ldots, V_m)$ of $\mathbf{V}$.

Theorem 4.2 *Suppose that p is a polynomial of degree d with Bézier coefficients $b(\boldsymbol{\gamma})$ with respect to $\mathbf{V}$. We extend b in an arbitrary manner to define $b(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma} \in \frac{1}{2}\mathbb{Z}_+^{m+1}$, $|\boldsymbol{\gamma}| = d$. For $0 \le i \le j \le m$, we define the averaging operator:*

$$A(i,j)b(\boldsymbol{\gamma}) := \frac{1}{2} \left\{ b(\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\epsilon}^i - \frac{1}{2}\boldsymbol{\epsilon}^j) + b(\boldsymbol{\gamma} - \frac{1}{2}\boldsymbol{\epsilon}^i + \frac{1}{2}\boldsymbol{\epsilon}^j) \right\}.$$

Then with respect to the simplex $\mathbf{U} = \langle i_0 j_0, \ldots, i_m j_m \rangle$ in T_2, p has Bézier coefficients

$$\tilde{b}(\boldsymbol{\gamma}) := (A(i_0, j_0)^{k_0} \cdots A(i_m, j_m)^{k_m} b) \left(\sum_{l=0}^{m} k_l \frac{1}{2} (\boldsymbol{\epsilon}^{i_l} + \boldsymbol{\epsilon}^{j_l}) \right).$$

Here $A(i,j)^k = (A(i,j))^k$.

By Theorem 4.2,

$$\tilde{b}(\boldsymbol{\gamma}) = \frac{1}{2^d} \prod_{l=0}^{m} (E_{i_l} + E_{j_l})^{k_l}\, b(\mathbf{0}). \tag{4.3}$$

For $0 \le i < m$ we define the operator

$$A_i a(\mathbf{v}) := a\left(\mathbf{v} + \frac{1}{2}\boldsymbol{\epsilon}^i\right).$$

Then for $0 \le i \le j \le m$, $\frac{1}{2}(E_i + E_j) = A_i A_j A(i,j)$. Since all operators commute, (4.3) gives

$$\begin{aligned}\tilde{b}(\boldsymbol{\gamma}) &= \prod_{l=0}^{m} A(i_l, j_l)^{k_l} \prod_{l=0}^{m} A_{i_l}^{k_l} A_{j_l}^{k_l} b(\mathbf{0}) \\ &= \prod_{l=0}^{m} A(i_l, j_l)^{k_l}\, b\left(\sum_{l=0}^{m} k_l \frac{1}{2}(\boldsymbol{\epsilon}^{i_l} + \boldsymbol{\epsilon}^{j_l})\right).\end{aligned}$$

□

As an example, take $m = 3$ and $\mathbf{U} = [00, 01, 02, 03]$, which is the unique simplex in T_2 containing V_0. Then, since $A(0,0)$ is the identity, we have

$$\tilde{b}(i,j,k,l) = (A(0,1)^j (A(0,2)^k (A(0,3)^l b)\left(i + \frac{1}{2}(j+k+l), \frac{1}{2}j, \frac{1}{2}k, \frac{1}{2}l\right).$$

In particular, $d = 4, i = 2, j = k = 1, l = 0$, gives

$$\begin{aligned}\tilde{b}(2,1,1,0) &= (A(0,1)A(0,2)b)\left(3, \frac{1}{2}, \frac{1}{2}, 0\right) \\ &= \frac{1}{2}A(0,1)b\left(\frac{7}{2}, \frac{1}{2}, 0, 0\right) + \frac{1}{2}A(0,1)b\left(\frac{5}{2}, 1, 0\right) \\ &= \frac{1}{4}\left\{b(4,0,0,0) + b(3,1,0,0) + b(3,0,1,0) + b(2,1,1,0)\right\}.\end{aligned}$$

5 Convexity and Subdivision

We have just obtained a formula for expressing the Bézier coefficients of a polynomial p with respect to a simplex $\mathbf{V}$ in terms of the Bézier coefficients of p with respect to a simplex in the subdivision $T_{2,m}$ of $\mathbf{V}$. More generally we can consider the Bézier coefficients with respect to the simplices of $T_{k,m}$ for any $k \ge 2$. Piecing together the Bézier nets of p with respect to all of these simplices will then give a piecewise linear function over $\mathbf{V}$. To be precise we define the *composite Bézier net* $\hat{p}_k$ of a polynomial p of degree d with respect to $T_{k,m}(\mathbf{V})$ as follows. For any simplex $\mathbf{U} := [U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^n}]$ in $T_k(V_0, \ldots, V_m)$ the restriction of $\hat{p}_k$ to $\mathbf{U}$ equals the Bézier net of p with respect to $\mathbf{U}$. Then Lemma 2.1 implies that $\hat{p}_k$ is a continuous function which is linear on each element of $T_{kd,m}(\mathbf{V})$.

Theorem 5.1 *If the Bézier net of a polynomial p with respect to $V_0, \ldots V_m$ is convex, then for any $k \geq 2$ the composite Bézier net of p with respect to $T_k(V_0, \ldots V_m)$ is also convex.*

The proof follows ideas that were put forth for the case $m = 2$ in [11]. Suppose that the Bézier net of p with respect to $V_0, \ldots, V_m$ is convex. Then by Theorem 3.1, the Bézier coefficients of $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} p$ with respect to $\mathbf{V}$ are ≤ 0 for all $0 \leq i < j \leq m$. Now let $\mathbf{U}$ be any simplex in $T_k(V_0, \ldots, V_m)$. It follows from Theorem 4.1 that the Bézier coefficients of $D_{V_j - V_{j-1}} D_{V_i - V_{i-1}} p$ with respect to $\mathbf{U}$ are also ≤ 0 for all $0 \leq i < j \leq m$. Let $\mathbf{U} := [U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}]$ with $\boldsymbol{\alpha}^0, \ldots, \boldsymbol{\alpha}^m$ satifying (S1) and (S2). Then (S1)and (S2) imply a bijection $T : \{1, \ldots m\} \to \{1, \ldots, m\}$ so that for $j = 1, \ldots, m$,

$$U_{\boldsymbol{\alpha}^j} - U_{\boldsymbol{\alpha}^{j-1}} = k^{-1}(V_{T(j)} - V_{T(j)-1}), \tag{5.1}$$

and $U_{\boldsymbol{\alpha}^0} - U_{\boldsymbol{\alpha}^m} = k^{-1}(V_0 - V_m)$. Therefore, with $\boldsymbol{\alpha}^{-1} := \boldsymbol{\alpha}^m$, the Bézier coefficients of

$$D_{U_{\boldsymbol{\alpha}^j} - U_{\boldsymbol{\alpha}^{j-1}}} D_{U_{\boldsymbol{\alpha}^i} - U_{\boldsymbol{\alpha}^{i-1}}} p$$

with respect to $\mathbf{U}$ are ≤ 0 for all $0 \leq i < j \leq m$. By Theorem 3.1, the Bézier net of p with respect to $U_{\boldsymbol{\alpha}^0}, \ldots, U_{\boldsymbol{\alpha}^m}$ is convex, i.e. $\hat{p}_k$ is convex on $\mathbf{U}$.

Since subdivision creates subpolynomials over the elements of $T_{k,m}$ that represent the same polynomial, the subpolynomials of two elements that share a common face of dimension $m-1$ meet C^1 and hence the conditions for convexity of $\hat{p}_k$ across the face are satisfied with equality. □

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