Splines and unsorted knot sequences

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Dedicated to Carl de Boor on his 75th birthday

Abstract

The definition of a B-spline is extended to unordered knot sequences. The added flexibility implies that the resulting piecewise polynomials, named U-splines, can be negative and locally linearly dependent. It is therefore remarkable that linear combinations of U-splines retain many properties of splines in B-spline form including smoothness, polynomial reproduction, and evaluation by recurrence.

1 Introduction

A univariate spline of degree d consisting of $n + d$ polynomial pieces is defined by n B-spline coefficients and $n + d + 1$ real-valued scalars t_j, called knots. By convention, these knots are listed in non-decreasing order. It is natural to ask what advantages or drawbacks arise if we drop this convention and define splines using an unsorted knot sequence. Unsorted knot sequences appear, for example, as a special case of splines with complex knots [GT11] and arise in interpolation problems for linear reproduction, as shown in Example 3.2 of this paper. One approach to splines, polar forms [Ram89], lists ordering-independence as one of its fundamental properties; however, all detailed discussions and lecture notes assume either sorted knots or non-repeated knots. Of the many interactive online illustrations of splines only a small number, four at current counting, allows manipulating the knots. Of these, all but one either block knots from passing one another or re-sort the knot sequence (e.g. the nice applet [KMRS13]). Only one illustration [Kra13] allows for full flexibility of knots, but encounters singularities. Apart from applications, the motivation for exploring U-splines is to probe and re-enforce the notions, proofs and properties underlying splines.

A first hint of what splines for unordered knot sequences might look like arose from a Gedankenspiel by Carl de Boor, Malcolm Sabin and the author a decade ago, that, for $t_j > t_{j+1}$, the analogue of a B-spline of degree 0 might be defined as the negative of the characteristic function of the interval $[t_{j+1} \,.\,.\, t_j]$. Ten years on, this ansatz is substantiated by U-splines.

Overview. In Section 2, U-splines are defined for unsorted, but 'collocated' knot sequences. A collocated knot sequence repeats knot values only consecutively or sufficiently spaced apart. As is customary for B-splines, the ith U-spline of degree d is associated with the subsequence $t_{i:i+d+1} := (t_i, t_{i+1}, \ldots, t_{i+d+1})$, but now of the unsorted knot sequence $t_{0:n+d}$. A key observation, Lemma 2.1, is that any U-spline can be expressed as a multiple of the B-spline with the same, but sorted subsequence of knots. Alternatively we derive, again for unsorted, collocated knot sequences, a U-spline of higher degree by recurrence from U-splines of lower degree. The recurrence is that of B-splines, but it starts with signed characteristic functions.

In Section 3 we characterize the space spanned by U-splines. To this end, we need to 'complete' finite knot sequences and clarify, for U-splines, the notion of 'basic interval' appearing, in deceptively simple form, when analyzing splines in B-spline form. On such a basic interval, the U-splines form a partition of unity, Marsden's identity holds, we have linear precision and we can compute derivatives via differences of coefficients. But since U-splines may take on negative values, they can be locally linearly dependent.

Section 4 shows that a slight variant of de Boor's algorithm evaluates splines with collocated, unsorted knots. Prior to evaluation, any U-spline whose first knot equals its last knot has to be removed and, in the final step of the modified algorithm, the values of multiple intervals containing the point of evaluation have to be summed. Knot insertion is well-defined but does not imply variation diminuition as for B-splines. Finally, Section 5 gives an example of knots that are not collocated and for which no two-term evaluation recurrence with convex weights can correctly evaluate the associated spline. This points to collocated sequences as a maximal practical generalization of non-decreasing knot sequences.

2 Splines with unsorted, collocated knots

We start by characterizing an important sub-class of all unsorted knot sequences.

Definition 2.1 (collocated knot sequence) *Let $t_{0:n+d}$ be an unsorted sequence of real scalars called* knots *hereafter. This knot sequence is d-*collocated *if*

$$\forall \quad 0 < j \leq d, \qquad t_i = t_{i+j} \;\Rightarrow\; t_i = t_{i+1} = \cdots = t_{i+j}. \tag{L}$$

That is, if any knot of a collocated sequence is repeated after fewer than d entries then all intermediate knots must have the same value.

Whenever the degree d is clear from the context, we simply say that the knot sequence is collocated. A collocated sequence appears to be necessary to establish basic properties and apply algorithms that make splines useful (see (1), (9), (18), (21), Section 5 of this paper).

For collocated knot sequences, we define U-splines analogous to B-splines via a table of *divided differences* $\Delta(t_{i:j})$. For a sufficiently smooth univariate real-valued function h with kth derivative $D^k h$,

$$\Delta(t_i)h := h(t_i),$$

$$\Delta(t_{i:j})h := \begin{cases} (\Delta(t_{i+1:j})h - \Delta(t_{i:j-1})h)/(t_j - t_i), & \text{if } t_i \neq t_j, \\ \frac{D^{j-i}h}{(j-i)!}(t_i), & \text{if } t_i = t_j, \end{cases}, j \in \{i+1, \dots, i+d+1\}. \tag{1}$$

Since for collocated knots, $t_i = t_j$ implies that $t_i = t_{i+1} = \cdots = t_j$, the definition is consistent in that the second case of (1) is the limiting case of the first when $t_i \to t_j$.

Definition 2.2 (U-spline from divided differences) *Let $t_{i:i+d+1}$ be an unsorted, d-collocated knot sequence. Then the piecewise polynomial of degree d defined by*

$$U(x|t_{i:i+d+1}) := (t_{i+d+1} - t_i)\ \Delta(t_{i:i+d+1})(\max\{(\cdot - x), 0\})^d, \tag{2}$$

is called a U-spline.

If $t_{i:j}$ is a non-decreasing sequence then its knots are automatically collocated and $U(x|t_{i:j})$ is a B-spline $B(x|t_{i:j})$ as defined in [Boo01, IX(2)].

The constructive definition of divided differences is typically arranged in the form of a divided difference table as in recurrence (1). But an alternative definition of $\Delta(t_{i:j})h$ exists, as the leading coefficient of the polynomial that interpolates h at the knots $t_{i:j}$ [CdB80, p42]. This equivalent definition is clearly invariant under re-ordering of $t_{i:j} := (t_i, \dots, t_j)$. Therefore, together with (2), we have the following useful Lemma 2.1, first suggested in this form by Carl de Boor.

Lemma 2.1 (re-ordering of knots) *If $s_{i:j}$ is any re-ordering of $t_{i:j}$ then*

$$(s_j - s_i)U(x|t_{i:j}) = (t_j - t_i)U(x|s_{i:j}). \tag{3}$$

In particular, for any re-ordering $s_{i:j}$ of $t_{i:j}$, and end-knots p and q of the knot sequence,

$$U(x|p, t_{i:j}, q) = U(x|p, s_{i:j}, q), \tag{4}$$

$$U(x|p, t_{i:j}, q) = -U(x|q, t_{i:j}, p), \tag{5}$$

$$U(x|q, t_{i:j}, q) = 0. \tag{6}$$

Equation (6) explains why Property (L) need not rule out U-splines whose first and last knots are equal: when dealing with linear combinations of U-splines, the contributions of U-splines of the form $U(x|p, \dots, q)$ with $p = q$ are anyhow zero.

Since a U-spline with a sorted, non-decreasing knot sequence is a B-spline with that sequence, Lemma 2.1 implies that we can express any U-spline with non-constant sequence $t_{i:j}$ as a B-spline

$$U(x|t_{i:j}) = \frac{t_j - t_i}{s_j - s_i} B(x|s_{i:j}), \text{ where } s_{i:j} := \operatorname{sort}(t_{i:j}). \tag{7}$$

Therefore, the space of U-splines is spanned by the B-splines $B(x|s_{i:j})$ and it inherits the piecewise polynomiality and continuity of these B-splines.

Sorting the whole knot sequence $\mathbf{t}$ of a spline can however introduce subsequences of length $d+2$ that do not occur when separately sorting the knot subsequences of $\mathbf{t}$. Consider for example $\mathbf{t} := (0, 1, 1, 0)$ and $d = 1$. All U-splines corresponding to $\mathbf{t}$ are a multiple of $B(x|0, 1, 1)$, while the sorted sequence $\mathbf{s} := \operatorname{sort}(\mathbf{t}) = (0, 0, 1, 1)$ gives rise to $B(x|0, 0, 1)$ and $B(x|0, 1, 1)$. That is, global sorting of the knots changes the spline space. Lemma 3.3 will characterize the relation in more detail.

U-splines via Recurrence An alternative approach to B-splines is to construct higher-degree B-splines from lower degree ones. Since the recurrence relations for B-splines are based on the divided difference representation, the recurrence applies equally well to U-splines. Only the initialization differs. For a U-spline the initialization has to reflect property (5) that $U(x|p, q) = -U(x|q, p)$.

Lemma 2.2 (U-spline recurrence) *For a collocated knot sequence $t_{0:n+d}$, $x \in \mathbb{R}$ and integers i, j, k,*

$$U(x|t_{i:i+1}) = \begin{cases} 1 & \textit{if } t_i \le x < t_{i+1}, \\ -1 & \textit{if } t_{i+1} \le x < t_i, \\ 0 & \textit{otherwise.} \end{cases} \tag{8}$$

$$\begin{aligned} U(x|t_{j-1}, t_{j:k}, t_{k+1}) := &\left\{ \begin{array}{ll} \frac{x - t_{j-1}}{t_k - t_{j-1}} U(x|t_{j-1}, t_{j:k}) & \textit{if } t_k \ne t_{j-1}, \\ 0 & \textit{otherwise} \end{array} \right\} \\ &+ \left\{ \begin{array}{ll} \frac{t_{k+1} - x}{t_{k+1} - t_j} U(x|t_{j:k}, t_{k+1}) & \textit{if } t_{k+1} \ne t_j, \\ 0 & \textit{otherwise} \end{array} \right\}. \end{aligned} \tag{9}$$

Property (L) ensures that setting summands to zero is the correct limiting behavior: the denominator and the leading linear factor of U in definition (2) cancel one another and the remaining divided difference is, by (L) and (1), well-defined as a derivative of order $k-j+1$ of a polynomial of degree at most $k-j$. Irrespective of the case distinction at the recurrence start, the proof, that the recurrence (9) yields the U-splines defined in Definition 2.2, is identical to that for B-splines [Boo01, IX(13)], i.e., it is based on Leibniz' formula applied to the divided differences.

The support of $U(x|t_{i:i+k})$ is evidently $[\min(t_{i:i+k}) \mathinner{\ldotp\ldotp} \max(t_{i:i+k})]$.

3 Splines built from U-splines

To analyze the space spanned by U-splines, we need to generalize some notions that are rarely emphasized but that are crucial for the analysis of splines with non-decreasing knot sequences.

Complete knot sequences. When $\mathbf{t}$ is a non-decreasing sequence, then extending it to a biinfinite (non-decreasing) knot sequence avoids a specialized discussion of the first and last d intervals. Alternatively, one may repeat the first knot of the non-decreasing sequence k times and the last knot k times.

For unsorted knots, neither approach, making the knot sequence biinfinite or repeating knots, yields the same benefits. A biinfinite sequence may still jump back an forth and the repeated knots may lie anywhere in the original interval of support $I := [\min(t_{0:n}) \mathinner{\ldotp\ldotp} \max(t_{0:n})]$. We therefore define an augmented knot sequence that encloses I as follows.

Definition 3.1 (d-complete knot sequence) *Given the unsorted knot sequence $t_{0:n}$, $n \geq k := d+1 > 0$ and its sorted re-ordering $s_{0:n}$, let $t_{<0}$ and $t_{>n}$ be two sequences of k knots each such that $\max(t_{<0}) \leq s_0$ and $s_n \leq \min(t_{>n})$. Then $(t_{<0}, t_{0:n}, t_{>n})$ is a* complete *knot sequence for the interval $I := [s_0 \mathinner{\ldotp\ldotp} s_n]$.*

Partition of 1. Traversing a complete knot sequence in its order on the real line implies that I is traversed one more time in increasing order, from s_0 to s_n, than in decreasing order: all contributions of negative and positive characteristic functions due to reversals of traversal direction cancel one another, except for one non-decreasing sequence required to connect s_0 to s_n. By (8), for $d = 0$, this implies on I that

$$\sum_i U(x|t_{i:i+1}) = 1. \tag{10}$$

Expanding the sum of higher-degree splines in terms of lower-degree ones according to (9), as is well-defined for a complete, collocated sequence on I, we see that the resulting two sums add up to a sum of U-splines of one degree lower:

$$\sum_i U(x|t_{i:i+d+1}) = \sum_i (\frac{x-t_i}{t_{i+d}-t_i} + \frac{t_{i+d}-x}{t_{i+d}-t_i}) U(x|t_{i:i+d}) = \sum_i U(x|t_{i:i+d}).$$

Together with (10), this establishes that U-splines of a given fixed degree form a partition of 1. The partition of 1 property implies *affine invariance*,

$$A \sum_i c_i U(x|t_{i:i+k}) + a = \sum_i (Ac_i + a) U(x|t_{i:i+k}), \quad c_i, a \in \mathbb{R}^{dim},\ A \in \mathbb{R}^{dim \times dim},$$

for curves based on U-splines. However, since individual U-splines can be negative, the *local convex hull* property, $\sum_{j\in J} c_j U(x|t_{j:j+k}) = \sum_{j\in J} c_j\alpha_j(x)$ for $\sum_{j\in J}\alpha_j = 1$, $\alpha_j \geq 0$, $\#J = k$, need not hold for x in the support of all $U(x|t_{j:j+k})$. We can salvage some aspect of the property by allowing $-1 \leq \alpha_j \leq 1$ in place of $0 \leq \alpha_j \leq 1$.

Polynomial reproduction. For B-splines, Marsden's identity implies that all polynomials of degree d or less are linear combinations of B-splines of degree at least d. The proof of Marsden's identity [Boo01, IX(30)] requires only partition of 1 and valid recurrence relations (9) as implied by property (L). In particular, the proof does not depend on the knot ordering. We can therefore apply the argument of [Boo01, IX(30)] directly, starting with the signed characteristic functions (8) as the base case, to prove the following identity.

Lemma 3.1 (Marsden's identity for U-splines) *Let $t_{0:n}$ be a d-collocated knot sequence and $(t_{<0}, t_{0:n}, t_{>n})$ a d-complete knot sequence for $I := [\min t_{0:n} \,..\, \max t_{0:n}]$. For any $\tau \in \mathbb{R}$ and $x \in I$,*

$$(x-\tau)^d = \sum_i \psi_{t_{i:i+d+1}}(\tau)U(x|t_{i:i+d+1}), \qquad \psi_{t_{i:j}}(\tau) := (t_{i+1}-\tau)\cdots(t_{j-1}-\tau).$$

Example 3.1 (reproduction requires complete knot sequences) To see that Lemma 3.1 fails when the knot sequence is not complete consider the incomplete knot sequence $t_{0:4} := (0, 1, 2, 0, 1)$ and $d = 1$. Then

$$\sum_{i=0}^{2}\psi_{t_{i:i+2}}(\tau)U(x|t_{i:i+2}) = (t_1-\tau)B(x|0,1,2) + (t_2-\tau)\frac{-1}{2}B(x|0,1,2)$$
$$+ (t_3-\tau)\frac{-1}{2}B(x|0,1,2) = (1-\tau-\frac{1}{2}(2+0-2\tau))B(x|0,1,2) = 0. \tag{11}$$

On the other hand, for the completed sequence $t_{-2:6} := (-2, -1, 0, 1, 2, 0, 1, 3, 4)$ and its non-decreasing sub-sequence $\tilde{t} := (-2, -1, 0, 1, 3, 4)$, as expected,

$$\sum_{i=-2}^{4}\psi_{t_{i:i+2}}(\tau)U(x|t_{i:i+2}) = (t_{-1}-\tau)B(x|-2,-1,0) + (t_0-\tau)B(x|-1,0,1) + (t_4-\tau)B(x|0,1,3)$$
$$+ (t_5-\tau)B(x|1,3,4) = \sum \psi_{\tilde{t}_{i:i+2}}(\tau)B(x|\tilde{t}_{i:i+2}) = (\cdot-\tau), \tag{12}$$

where the first equality follows from Lemma 2.1 and (11), the second by definition, and the last from Marsden's identity for B-splines. |||

Basic interval. To establish properties of linear combinations of B-splines of degree d, requires focus on intervals where a full complement of $k := d+1$ B-splines are supported. For non-decreasing knots $\mathbf{t} := t_{0:n+d}$, all but the first d and the last d intervals are automatically in the support of exactly $k := d+1$ B-splines. Therefore, for non-decreasing knots, it makes sense to define the basic interval as $I_n := [t_d \,..\, t_n]$ and prove properties on this interval.

For establishing analogous properties of linear combinations of U-splines, the definition of I_n above does not work. For example, the knot sequence $\mathbf{t} := (0, 1, 2, 0, 1)$ of Example 3.1 and $d = 1$ yields, according to interpretation, either $I_n = [t_1 \,..\, t_3] = [0 \,..\, 1]$, or, since $t_2 = 2 > t_3$, $I_n = [0 \,..\, 2]$. In either case, the three U-splines defined on I_n before completion are by (3) each a multiple of just one B-spline (cf. (11)), i.e. do not span the space of piecewise polynomials of degree $d = 1$ on I_n. To obtain the necessary empty set as basic interval in this setting, we have to explicitly remove all support intervals of U-splines that are added by completion, namely $B(-1, 0, 1)$, $B(x|0, 1, 3)$ and $B(x|1, 3, 4)$. Indeed all of I_n is in the support of $B(-1, 0, 1)$ and $B(x|0, 1, 3)$.

Definition 3.2 (basic interval) *For $n \geq k := d + 1 > 0$, let $(t_{-k:n+d+k})$ be a completion of the unsorted, collocated knot sequence $\mathbf{t} := t_{0:n+d}$. Let $s_{d:n} := \mathrm{sort}(t_{d:n})$. Then*

$$I_b := [s_d \, .. \, s_n] \backslash ([\min(t_{-k:d}) \, .. \, \max(t_{-k:d})] \cup [\min(t_{n:n+d+k}) \, .. \, \max(t_{n:n+d+k})]).$$

is the basic interval *of* $\mathbf{t}$.

Lemma 3.2 (B-splines per basic interval) *Let $n \geq k := d+1 > 0$ and $\mathbf{t} := t_{0:n+d}$ an unsorted, collocated knot sequence with basic interval I_b. Then every interval $[s_{j-1} \, .. \, s_j] \subseteq I_b$, $j \in \{d + 1, \dots, n\}$, is in the support of k linearly independent B-splines with knots in $\mathbf{t}$.*

Proof For each interval $[s_{j-1} \, .. \, s_j] \subseteq I_b$, we form a set T_j of k knot subsequences by selecting for $\ell = 1, \dots, k$, the subsequence $t_{i:i+k}$ of $\mathbf{t}$ with least index i that contains exactly ℓ knots greater or equal to s_j. For $\ell = 1$, such a subsequence exists since, by assumption, $t_{\overline{j}}$, the first occurrence of s_j, is in I_b and there exist k elements less than s_j and listed before $t_{\overline{j}}$ in $\mathbf{t}$: if $t_b \geq s_j$ for some b, $0 \leq b \leq d$, then $[s_{j-1} \, .. \, s_j] \subseteq [\min(t_{-k:d}) \, .. \, \max(t_{-k:d})]$ contradicting the definition of I_b. For $\ell > 1$, such a subsequence exists since k elements of $t_{\overline{j}:n+d}$ are greater or equal to s_j: if not then there exists a $t_{\overline{n}} \in t_{n:n+d}$ such that $t_{\overline{n}} < s_j$, implying $t_{\overline{n}} \leq s_{j-1}$ and hence $[s_{j-1} \, .. \, s_j] \subseteq [\min(t_{n:n+d+k}) \, .. \, \max(t_{n:n+d+k})]$ contradicting the definition of I_b.

Subsequences for different ℓ in T_j have distinct, non-constant sorted re-orderings, possibly just differing by the multiplicity of knots. Therefore the B-splines obtained by (3) from the U-splines with knot sequences in T_j are linearly independent. |||

By the same argument as for B-splines [Boo01, IX(34)], Marsden's identity implies for the basic interval that for any polynomial p of degree d (or less) and any $\tau \in \mathbb{R}$,

$$p = \sum_i (\lambda_{i:i+d+1} p) U(\cdot | t_{i:i+d+1}) \tag{13}$$

$$\text{with } \lambda_{i:i+d+1} : p \mapsto \frac{1}{d!} \sum_{\nu=0}^{d} (-D)^\nu \psi_{i:i+d+1}(\tau) D^{d-\nu} p(\tau). \tag{14}$$

We note that $\psi_{i:i+d+1}$ and hence $\lambda_{i:i+d+1}$ does not depend on the ordering of the indices and that $\lambda_{i:i+d+1}$ depends linearly on $\psi_{i:i+d+1}$.

Knot averages. For any linear polynomial ℓ and $d > 0$, (13) implies that

$$\ell(x) = \sum_i \ell(t^*_{i:i+d+1}) U(x | t_{i:i+d+1}), \quad t^*_{i:i+d+1} := \frac{t_{i+1} + \cdots + t_{i+d}}{d} \tag{15}$$

since $D^k \ell = 0$ for $k > 1$ and the linear polynomial $(D^{d-1} \psi_{t_{i:i+d+1}})(\tau) = (-)^d (\tau d - \sum t_i)$ vanishes at the **Greville site** $\tau = t^*_{i:i+d+1}$.

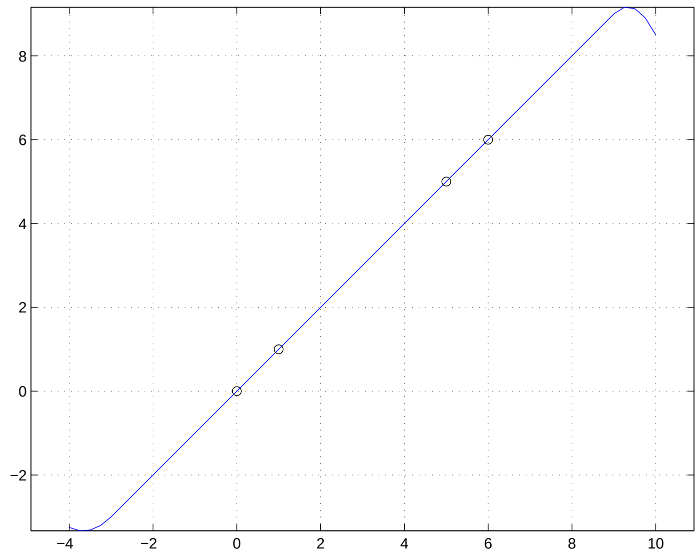

Figure 1: Graph of the C^1 quadratic spline function with unsorted knots $t_{0:10} := (-7, -5, -3, 1, -1, 3, 7, 5, 9, 11, 13)$ that interpolates values $(0, 1, 5, 6)$ at the Greville abscissae $(0, 1, 5, 6)$ by the choice of coefficients $c := (-4, -1, 0, 1, 5, 6, 7, 10)$.

Example 3.2 (linear representation of linear data) In Fig. 1, $n = 4$ points are marked on a straight line. Correspondingly, we might like the graph of a low-degree interpolating function f to have a linear segment. To linearly trace out this linear segment by the graph of a spline function f of degree $d > 1$, Equation (15) suggests matching the points by the spline's control points $(t^*_{i:i+d+1}, c_i)$. That is, we want to find knots $t_{0:n+d} := (t_0, t_1, \ldots, t_{n+d})$ so that for given $t^*_{i:i+d+1}$

$$\frac{t_{i+1} + \cdots + t_{i+d}}{d} = t^*_{i:i+d+1}, \quad i = 1, \ldots, n. \tag{16}$$

Since there are $d + 1$ more knots than Greville abscissae, the system (16) of n equations is underconstrained and, by its structure, solvable.

But if we additionally require that the knots be non-decreasing, $t_j \leq t_{j+1}$, then there may not exist a solution. In Fig. 1 the challenge is to fit a quadratic spline to the values $y^* := (0, 1, 5, 6)$ at its Greville abscissae $t^* = y^*$. That is, the challenge is to find knots t_j so that $(t_i + t_{i+1})/2 = t^*_i$:

$$t_1 + t_2 = 0,\ t_2 + t_3 = 2,\ t_3 + t_4 = 10,\ t_4 + t_5 = 12. \tag{17}$$

If we assume non-decreasing knots then $-t_1 = t_2 \geq 0$ and since $t_3 = 2-t_2$ we have $t_4 = 10-t_3 \geq 8+t_2 \geq 8$ but also $t_5 = 12 - t_4 \leq 4 - t_2 \leq 4$. But $t_4 > t_5$ contradicts the assumption that the knot sequence is non-decreasing. Reparameterization $t_i \mapsto \alpha t_i + \beta$ does not change this outcome. So there is no choice of non-decreasing knots that yields the hoped-for 'linear precision'. Fig. 1 shows a quadratic spline interpolant, made possible by unsorted knots. The interpolant is linear for $x \in [0\,..\,6]$ and, as expected, the graph is not linear outside this interval. |||

Differentiation. The (proof of the) differentiation formula

$$D(\sum_i c_{i:i+d+1} U(x|t_{i:i+d+1})) = d \sum_i \frac{c_{i:i+d+1} - c_{i-1:i+d}}{t_{i+d} - t_{i-1}} U(x|t_{i:i+d}) \tag{18}$$

does not depend on sorted knots, but does require a collocated knot sequence for the limit as $t_{i+d} \to t_{i-1}$ to be well-defined. As an example of differentiation, the spline defined in Fig. 1 has two U-splines whose knot sequences $t_{j:j+3}$ cover $x = 3$. As expected for the example, the derivative at $x = 3$ computes to

$$D\big(U(\cdot|1,-1,3,7) + 5U(\cdot|-1,3,7,5)\big)(3) = 2\frac{5-1}{7-(-1)}U(3|-1,3,7) = 1.$$

Sorting and spline space. For a non-decreasing knot sequence $\mathbf{t} := t_{0:n+d}$, we denote by $\Pi_{d,\mathbf{t}}$ the space of piecewise polynomials of degree d partitioned by $\mathbf{t}$ and such that, at each knot of $\mathbf{t}$, the polynomial pieces are connected with continuous derivatives at least d minus the consecutive multiplicity of that knot. We can retain this definition for unsorted, collocated knots $\mathbf{t}$ noting that the multiplicity of any knot and hence the removal of smoothness constraints between polynomial pieces is at most the maximum consecutive multiplicity in the sequence $\mathbf{t}$. By construction, either by Definition 2.2 or Lemma 2.2, the space of U-splines over $\mathbf{t}$ is a subspace of $\Pi_{d,\mathbf{t}}$. The next Lemma addresses globally sorting $\mathbf{t}$.

Lemma 3.3 (U-splines as a sub-space of a B-spline space) *Let* $\mathbf{s} := \mathrm{sort}(\mathbf{t})$ *be a sorted, non-decreasing reordering of* $\mathbf{t} := t_{0:n+d}$*. A linear combination of U-splines over the knot sequence* $\mathbf{t}$ *can be represented as a linear combination of B-splines over the knot sequence* $\mathbf{s}$*.*

Proof Since $\mathbf{s}$ contains all knots of $\mathbf{t}$, and since sorting can only increase the multiplicity of any knot in a knot subsequence of length $d+1$, the space of U-splines over $\mathbf{t}$ is a sub-space of the less constrained space $\Pi_{d,\mathbf{s}}$. By the Curry-Schoenberg theorem [Boo01, IX(44)], on the basic interval, $\Pi_{d,\mathbf{s}}$ equals the B-spline space over $\mathbf{s}$. |||

While Lemma 3.2 hints that U-splines may span the piecewise polynomial space corresponding to a modified $\mathrm{sort}(\mathbf{t})$ where the multiplicity of any knot is at most the maximum multiplicity in $\mathbf{t}$, the part of the Curry-Schoenberg theorem that states that the B-splines form a basis can not hold for U-splines. The following example illustrates that U-splines, as opposed to B-splines, can be locally linearly dependent.

Example 3.3 (local linear dependence) Let $d = 0$. Then $t = (-1, 1, 0, 2)$ is a collocated knot sequence complete for the interval $[0\,..\,1]$ but

$$f(x) := U(x|-1,1) + 2U(x|1,0) + U(x|0,2) \tag{19}$$

vanishes on the interval $[0\,..\,1]$. |||

The example can be daisy-chained as $\sum_i U(x|i,i+2) + 2U(x|i+1,i)$ to show linear dependence on any subset of $\mathbb{R}$. (The simpler example $\sum_i U(x|0,1) + U(x|1,0) = 0$ does not cover much ground.)

Local linear dependence also implies that U-splines need not satisfy the inequality that establishes good *condition* of a basis.

4 The de Boor recurrence and unsorted knots

A spline represented by a linear combination of U-splines can be evaluated by evaluating each U-spline separately. For example, we can use Lemma 2.1 to convert each U-spline to B-spline form, and then sum the terms. However, this is not very efficient. Famously, splines were made practical by providing a stable evaluation algorithm [Boo71, Boo72, Cox72] in terms of the coefficients of their representation in *B-form*,

$$\sum_i c_{i:i+d+1} B(x|t_{i:i+d+1}). \tag{20}$$

Reversing the construction of higher-degree B-splines from lower ones [Boo01, Ch X(26)] or applying Ramshaw's blossoming approach [Ram89], yields the following recurrence relation for the B-spline coefficients.

$$\text{For } i < k \text{ and } x \in [t_i \mathinner{..} t_k], \qquad c_{t_{i:k}} := \frac{t_k - x}{t_k - t_i} c_{t_{i-1:k}} + \frac{x - t_i}{t_k - t_i} c_{t_{i:k+1}}. \tag{21}$$

Recurrence (21), the de Boor recurrence, is equally well-defined and has non-zero denominators for an unsorted knot sequence whenever the knot sequence is collocated. The only change in the case of unsorted sequences is in the final step. In the final step, potentially several contributions $c_{t_{i:i+1}}$ from intervals $[t_i \mathinner{..} t_{i+1}]$ covering x are summed with signs according to (8), i.e., negative if $t_{i+1} < t_i$. Recurrence (21) is efficient for evaluation if we apply it only to coefficients whose associated U-spline is non-zero at the point x of evaluation. For sorted knots, at the recursion level associated with polynomials of degree d, exactly $k := d + 1$ consecutive B-spline support intervals cover x; and the recurrence only forms convex combinations $\frac{t_k - x}{t_k - t_i}$, $\frac{x - t_i}{t_k - t_i} \in [0 \mathinner{..} 1]$. For U-splines, the lack of knot ordering allows more than k intervals to cover and combinations are only affine: at the final recurrence level of constant U-splines, all coefficients need to be summed whose interval covers the evaluation point; and where $x < t_i < t_k$, the fraction $\frac{x - t_i}{t_k - t_i}$ in (21) is negative.

Example 4.1 (recurrence for U-spline coefficients) Consider the quadratic spline
$f(x) := 12U(x|3, 4, 1, 5)$, all of whose coefficients are zero except for $c_{3,4,1,5} = 12$. To evaluate f at $x = 2$ by recurrence, we compute

$$\begin{aligned}
c_{3,4,1} &= \frac{2-3}{1-3} 12 = 6, \quad c_{4,1,5} = \frac{5-2}{5-4} 12 = 36, \\
c_{3,4} &= \frac{2-3}{4-3} c_{3,4,1} = -6, \quad c_{4,1} = \frac{1-2}{1-4} c_{3,4,1} + \frac{2-4}{1-4} c_{4,1,5} = 2 + 24 = 26, \\
c_{1,5} &= \frac{5-2}{5-1} c_{4,1,5} = 27.
\end{aligned}$$

Since x lies outside the interval $[3, 4]$, inside the decreasing interval $[4, 1]$ and inside the increasing interval $[1, 5]$, the value of $f(2) = 27 - 26 = 1$. Alternatively, by Lemma 2.1, $12U(x = 2|3, 4, 1, 5) = \frac{5-3}{5-1} 12B(2|1, 3, 4, 5) = 6/6 = 1$. |||

The next example shows that, as assumed throughout, we need to remove all (coefficients of) U-splines with equal first and last knot at the outset, before evaluation via recursion (21).

Example 4.2 (removal of zero-valued U-splines is necessary) First, we apply the recurrence (21), which only looks at subsequences, to the spline $f := U(\cdot|0, 2, 2, 2, 0)$ with $c_{0,2,2,2,0} = 1$ and all other coefficients zero. Evaluating f by recurrence at $x = 1$ yields

$$
\begin{aligned}
c_{0,2,2,2} &= \frac{1-0}{2-0}, \qquad c_{2,2,2,0} = \frac{0-1}{0-2},\\
c_{0,2,2} &= \frac{1}{2}\frac{1-0}{2-0}, \quad c_{2,2,0} = \frac{0-1}{0-2},\\
c_{0,2} &= \frac{1}{4}, \qquad c_{2,0} = -\frac{1}{2}.
\end{aligned}
$$

The summation of coefficients associated to intervals covering $x = 1$ yields $\frac{3}{4}$. However the correct value according to (6) is $f(1) = U(x|0, 2, 2, 2, 0) = 0$. |||

Each level of the recurrence corresponds to **knot insertion**. Let $\check{\mathfrak{t}}$ be the knot sequence after inserting an extra knot into the sequence $\mathfrak{t}$. The relation [Boo01, XI(21)], between functionals $\lambda_{i:i+d+1}$ before and after knot insertion, does not depend on ordering and implies for $\sum_j c_{j:j+d+1}U(\cdot|t_{j:j+d+1}) = \sum_j \check{c}_{j:j+d+1}U(\cdot|\check{t}_{j:j+d+1})$ that [Boo01, XI(23)]

$$
\begin{aligned}
\check{c}_{j:j+d+1} &= (1 - \check{\omega}_{jd}(x))c_{j-1:j+d} + \check{\omega}_{jd}(x)c_{j:j+d+1}, \qquad (22)\\
\check{\omega}_{jd}(x) &:= \begin{cases} 0, & \text{for } x \le t_j;\\ \frac{x-t_j}{t_d-t_j}, & \text{for } t_j < x < t_d;\\ 1, & \text{for } t_{j+d} \le x. \end{cases}
\end{aligned}
$$

But since the weights of knot insertion are not restricted to $[0 \,..\, 1]$ as they are in the sorted case, the standard observation [Boo01, XI(28)] concerning variation diminuition does not hold for U-splines. For example, the spline $f(x) := 2U(x|1, 2, 0) + 3U(x|2, 0, 3) = -B(x|0, 1, 2) + B(x|0, 2, 3)$ evaluates to $(f(0), f(1), f(2), f(3)) = (0, -1/2, 1, 0)$, i.e. changes sign, even though the coefficient sequence $(2, 3)$ does not. Interestingly, Property xii, called variation diminuition in [Boo01, XI(29)], holds for the example: $f(1)f(2) < 0$, $2U(1|1, 2, 0)f(1) = 1/2 > 0$ and $3U(2|2, 0, 3)f(2) = 1 > 0$.

5 Why require knot sequences to be collocated ?

At the outset, we stipulated that sequences have to be collocated. This was necessary to avoid division by zero in the relations of Definition 2, Lemma 2.2 and (21). But divided differences need not be defined by recursion (1). What if we dropped the assumption that the sequence be collocated? The example below shows that this requires at the very least some rethinking of key algorithms such as de Boor's recurrence.

Example 5.1 (convex recurrence requires collocated knots) We consider the spline f of degree $d = 4$ that is a multiple of a U-spline with knots $(0, 0, 2, 2, 0, 2)$. The knots are not 4-collocated. Let $c_{0,0,2,2,0,2} = 8$ and all other coefficients zero. For the recursion, whenever the denominators in (21) are zero because the first and last knot agree, we choose an affine combination with as of yet undetermined weights $k_i \in \mathbb{R}$ and

$k_i' := 1 - k_i$ in lieu of $\frac{t_j - x}{t_j - t_i}$ and $\frac{x - t_i}{t_j - t_i}$. Evaluation at $x = 1$ yields the following recurrence:

$$\begin{aligned}
c_{0,0,2,2,0} &= 8k_1, \quad c_{0,2,2,0,2} = \frac{2-1}{2-0} 8 = 4, \\
c_{0,0,2,2} &= 8k_1 \frac{1-0}{2-0} = 4k_1, \quad c_{0,2,2,0} = 8k_1 k_2' + 4k_2, \quad c_{2,2,0,2} = 4k_3', \\
c_{0,0,2} &= 4k_1 \frac{1-0}{2-0} = 2k_1, \\
c_{0,2,2} &= \frac{2-1}{2-0}(4k_1 + 8k_1 k_2') + \frac{1-0}{2-0} 4k_2 = 2k_1 + 4k_1 k_2' + 2k_2, \\
c_{2,2,0} &= \frac{0-1}{0-2}(8k_1 k_2' + 4k_2) + \frac{1-2}{0-2} 4k_3' = 4k_1 k_2' + 2k_2 + 2k_3', \quad c_{2,0,2} = 4k_3' k_4' \\
c_{0,2} &= 2k_1 \frac{2-1}{2-0} + \frac{1-0}{2-0}(2k_1 + 4k_1 k_2' + 2k_2) + \frac{2-1}{2-0} 4k_3' k_4' \\
&= k_1 + k_1 + 2k_1 k_2' + k_2 + 2k_3' k_4', \\
c_{2,0} &= \frac{0-1}{0-2}(4k_1 k_2' + 2k_2 + 2k_3') + \frac{1-2}{0-2} 4k_3' k_4' = 2k_1 k_2' + k_2 + k_3' + 2k_3' k_4'.
\end{aligned}$$

For convex weights $0 \leq k_i \leq 1$, the recurrence yields the value $c_{0,2} - c_{2,0} = 2k_1 - k_3' < 3$. But, by Lemma 2.1,

$$f(1) = 8U(x = 1|0, 0, 2, 2, 0, 2) = 8U(1|0, 0, 0, 2, 2, 2) = 3.$$

That is, the recurrence can not yield $f(1)$ if all averages are convex. |||

6 Summary and Conclusion

A new notion, collocated knots, proved to be both necessary and sufficient to define splines for unsorted knot sequences. We saw, by a concrete example, that, if the knots are not collocated, no two-term recurrence with convex weights for stable evaluation is readily available. This points to collocated knot sequences as a maximal practical generalization of non-decreasing knot sequences.

U-splines were constructed both via divided difference tables and via recurrence. Given a collocated knot sequence, the paper proved that any individual U-spline can be represented as a multiple of a B-spline. Establishing properties of the space spanned by linear combinations of U-splines forced a fresh and detailed look at the notions by which Carl de Boor put B-splines on their solid foundation. For example, the deceptively simple definition of the basic interval for linear combinations of B-splines has to explicitly state the basic interval's role of providing a spanning set.

From the outset, U-splines lacked positivity and local linear independence. It is therefore remarkable that linear combinations of U-splines retain so many useful properties familiar from B-splines and non-decreasing knot sequences.

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