Linear Envelopes for Uniform B–spline Curves

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Abstract. We derive an efficiently computable, tight bound on the distance between a uniform spline and its B–Spline control polygon in terms of the second differences of the control points. The bound yields a piecewise linear envelope enclosing the spline and its control polygon. For quadratic and cubic splines, the envelope has minimal possible width at the break points and for all degrees the maximal width shrinks by a factor of 4 under uniform refinement. We extend the construction to tight envelopes for parametric curves.

§1. Motivation and Overview

The central feature that allows reasoning about nonlinear piecewise polynomials is the fact that a spline is closely outlined by its B-spline control polygon. The efficiency of many applications depends crucially on a good estimate of the distance separating spline and control polygon. For rendering, a refined control polygon is rendered instead of the curve itself. For curve-intersection, an efficient and robust technique is to recursively refine and intersect control polygons [2]. Assessing the exactness of these operations requires a uniform, linear bound on the distance of the curve and its (refined B-spline) control polygon. The efficiency is improved if the effect of the refinement can be predicted rather than just measured. Of the two classical bounding contructs, axis-aligned min-max coefficient boxes and the convex hull, the first yields only a loose envelope and neither yields *a priori* estimates.

This paper introduces quantitative bounds that can be computed more efficiently than convex hulls and yield a simple piecewise linear envelope enclosing spline and control polygon (see Figure 1) whose maximal width contracts to 1/4th when the knot spacing is halved. The computation of the envelope of a degree d spline consists of computing the second differences of its control points and looking up or calculating $d-1$ constants, the values of a fixed set of splines. The sum of the constants, $(d+1)/24$, provides a second, even

Saint-Malo Proceedings
XXX, XXX, and Larry L. Schumaker (eds.), pp. 1–8.

ISBN 1-xxxxx-xxx-x.

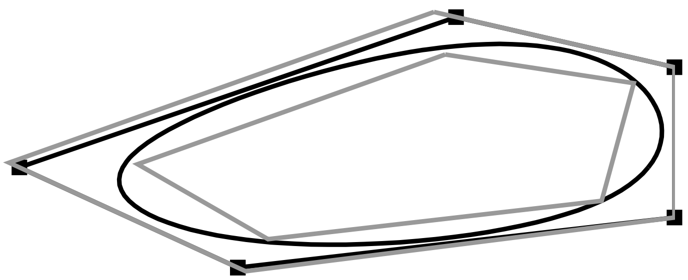

Fig. 1. A cubic curve *(black)* and its control points *(black squares)*. The envelope *(grey)* is constructed with the bound from Theorem 2.

simpler, but generally much coarser bound (Figure 2). Both bounds are piecewise linear with breaks at the corners of the control polygon and are *sharp for quadratic and cubic splines* in the sense that at every corner of the control polygon the distance between the spline and polygon is matched exactly.

This paper derives these bounds for *functions*, and establishes the convergence of the bound under uniform refinement; the bounds are then applied to curves to obtain *localized* envelopes.

§2. Notation

A scalar-valued piecewise polynomial p of degree d is a uniform B–spline, if it can be represented as

$$p = \sum_{k \in \mathbb{Z}} b^k N^k, \quad b^k \in \mathbb{R}, \quad N^k = N(\cdot - k),$$

where N is the B-spline of degree d supported on the interval $[0, d+1)$ and with the uniform knot sequence $\mathbb{Z}$ (c.f. [2]). For simplicity, we assume that both the control point sequence and the knot sequence are biinfinite.

The control polygon ℓ of p is the piecewise linear interpolant of the control points b^k at the Greville abscissae

$$t_k^* = k + (d+1)/2.$$

Over the interval $[t_k^*, t_{k+1}^*]$, the control polygon is $\ell(t) = \mathsf{L}_k(t\,; b^k, b^{k+1})$, where we denote the line segment from (t_k^*, a_1) to (t_{k+1}^*, a_2) by

$$\mathsf{L}_k(t\,; a_1, a_2) = a_1(t_{k+1}^* - t) + a_2(t - t_k^*).$$

The linear interpolant of a function f over this interval will be abbreviated as $\mathsf{L}_k\,(f) = \mathsf{L}_k(\cdot\,; f(t_k^*), f(t_{k+1}^*))$. The (centered) second differences of b are defined as

$$\Delta_2 b^i = b^{i-1} - 2b^i + b^{i+1}.$$

The first and the last basis function that are supported on $[t_k^*, t_{k+1}^*]$ are $N^{\underline{k}}$ and $N^{\overline{k}}$ with

$$\underline{k} = k + 1 - \lfloor d/2 \rfloor, \qquad \overline{k} = k - 1 + \lfloor d/2 \rfloor.$$

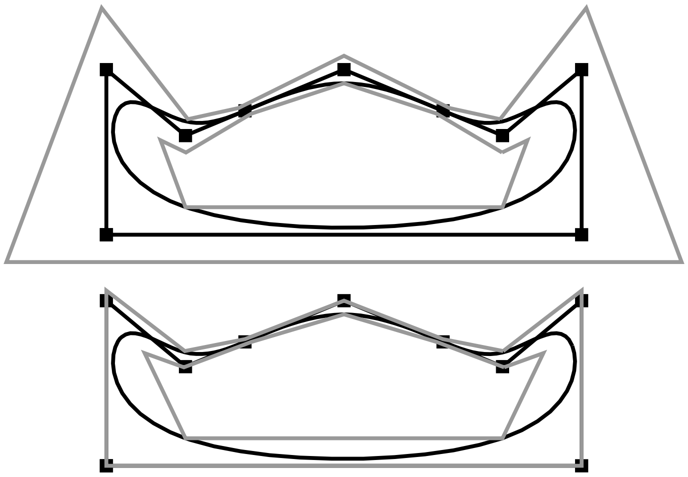

Fig. 2. A cubic curve *(black)* and its control points *(black squares)*. On top the envelope *(grey)* is constructed with the bound from Theorem 3, on the bottom from the tighter bounds of Theorem 2.

§3. Uniform B–splines Bounds

The key observation for deriving the bounds is that the difference between a uniform B–spline p and its control polygon can be factored into two parts: the second differences of the control polygon, and splines β_{ki} which are independent of p.

Theorem 1. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is given by*

$$p - \ell = \sum \Delta_2 b^i \, \beta_{ki}, \qquad \beta_{ki} = \begin{cases} \sum_{j=-\infty}^{i} (i-j) N^j & i \le k \\ \sum_{j=i}^{\infty} (j-i) N^j & i > k. \end{cases}$$

The functions β_{ki} are non-negative and convex on the interval $[t_k^*, t_{k+1}^*]$, and $\beta_{ki}(t_k^*) > 0$ if and only if $i \in [\underline{k}, \overline{k}]$.

Proof: We write $p - \ell$ over $[t_k^*, t_{k+1}^*]$ as

$$\sum_{i=\underline{k}}^{\overline{k}} b^i \, \alpha_{ki} = \sum_{i=\underline{k}}^{\overline{k}} b^i \left(N^i(t) - \mathtt{L}_k(t\,; \delta_{ik}, \delta_{i,k+1}) \right),$$

where $\delta_{ik} = 1$ if $i = k$ and 0 otherwise. We show that $\alpha_{ki} = \Delta_2 \beta_{ki}$: the partition of unity $\sum_i N^i = 1$ implies that $\sum_i \alpha_{ki} = 0$ and the linear precision of B–Splines, $\sum_i t_i^* N^i(t) = t$ implies on the interval $[t_k^*, t_{k+1}^*]$ that $\sum_i i\alpha_{ki} = 0$. Hence, for any i, $\sum_j (j-i)\alpha_{kj} = 0$. For $i > k$,

$$\beta_{ki} = \sum_{j=i}^{\overline{k}} (j-i) N^j - \sum_{j=\underline{k}}^{\overline{k}} (j-i)\alpha_{kj} = \sum_{j=\underline{k}}^{i} (i-j)\alpha_{kj},$$

so that $\beta_{ki} = \sum_{j=\underline{k}}^{i}(i-j)\alpha_{kj}$ for any i. It is now straightforward to verify that $\Delta_2\beta_{ki} = \alpha_{ki}$, and summation by parts yields

$$p - \ell_k = \sum_{i=\underline{k}}^{\overline{k}} b^i\,\alpha_{ki} = \sum_{i=\underline{k}}^{\overline{k}} b^i\,\Delta_2\beta_{ki} = \sum_{i=\underline{k}}^{\overline{k}} \Delta_2 b^i\,\beta_{ki}.$$

The functions β_{ki} are non-negative since their B–spline coefficients are non-negative. The convexity of the β_{ki} over $[t_k^*, t_{k+1}^*]$ follows from the convexity of their B–spline control polygons: for $i > k$, the part of the control polygon of β_{ki} that influences β_{ki} over $[t_k^*, t_{k+1}^*]$ lies on the function $\max\{\cdot - t_i^*, 0\}$, while for $i \le k$ it lies on $\max\{t_i^* - \cdot, 0\}$. In both cases, the control polygon of β_{ki}, and hence β_{ki}, is non-negative and convex. □

Theorem 1 immediately gives us a piecewise linear envelope on $p - \ell$:

Theorem 2. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is bounded by*

$$\mathtt{L}_k\left(\sum \Delta_2^- b^i\,\beta_{ki}\right) \le p - \ell \le \mathtt{L}_k\left(\sum \Delta_2^+ b^i\,\beta_{ki}\right),$$

where $\Delta_2^+ b^i = \max\{\Delta_2 b^i, 0\}$ and $\Delta_2^- b^i = \min\{\Delta_2 b^i, 0\}$.

Proof: We have from Theorem 1

$$p - \ell = \sum_i \Delta_2 b^i\,\beta_{ki} = \sum_i \Delta_2^+ b^i\,\beta_{ki} + \sum_i \Delta_2^- b^i\,\beta_{ki}.$$

The positivity of the β_{ki} implies that the first sum on the right–hand side is positive and the second is negative, and therefore

$$\sum_i \Delta_2^- b^i\,\beta_{ki} \le p - \ell \le \sum_i \Delta_2^+ b^i\,\beta_{ki}.$$

Since the β_{ki} are convex over $[t_k^*, t_{k+1}^*]$, they can be bounded linearly to yield the bound of Theorem 2. □

An even simpler envelope can be derived by bounding the sum of the β_{ki} at t_k^* by the constant $(d+1)/24$.

Theorem 3. *Over the interval $[t_k^*, t_{k+1}^*]$, the difference between a uniform B–spline p and its control polygon ℓ is bounded by*

$$|p - \ell| \le \frac{d+1}{24}\,\mathtt{L}_k(\,\cdot\,; \|\Delta_2 b\|_k, \|\Delta_2 b\|_{k+1}),$$

where $\|\Delta_2 b\|_k = \max\{|\Delta_2 b^i| : i \in [\underline{k}, \overline{k}]\}$. If $d = 2$ or $d = 3$, the relation holds with equality at the t_k^*.

Proof: By Theorem 1 and the convexity of the β_{ki} over $[t_k^*, t_{k+1}^*]$, we have

$$\begin{aligned} |p - \ell| &\le \sum_i |\Delta_2 b^i|\,\beta_{ki} \le \mathtt{L}_k\left(\sum_i |\Delta_2 b^i|\,\beta_{ki}\right) \\ &\le \mathtt{L}_k(\,\cdot\,; \|\Delta_2 b\|_k \sum_i \beta_{ki}(t_k^*), \|\Delta_2 b\|_{k+1} \sum_i \beta_{ki}(t_{k+1}^*)). \end{aligned} \tag{1}$$

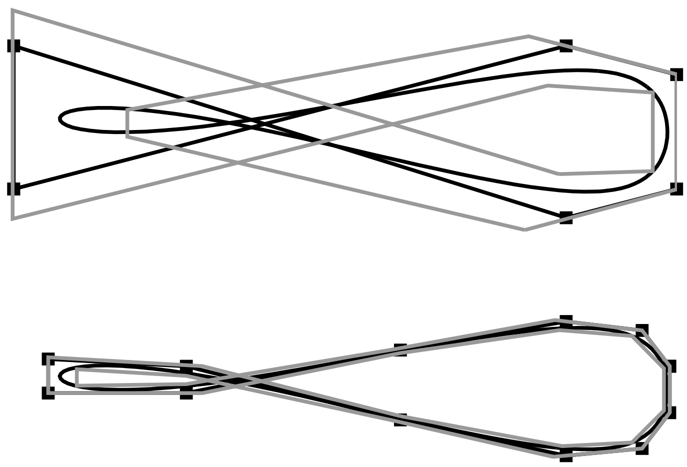

Fig. 3. A self–intersecting quartic curve *(black)* and its control points *(black squares)*. The envelopes *(grey)* are constructed with the bound from Theorem 2. The envelope converges rapidly to the curve as the comparison of the original envelope, top, and the envelope after one step of uniform refinement, bottom, shows.

The theorem follows if we can show that $\sum_i \beta_{ki} \leq (d+1)/24$:

$$\sum_i \beta_{ki} = \sum_{i=\underline{k}}^{\overline{k}} \sum_{j=\underline{k}}^{i} (i-j)\,\alpha_{kj} = \sum_{j=\underline{k}}^{\overline{k}} \sum_{i=j}^{\overline{k}} (i-j)\,\alpha_{kj} = \sum_j \alpha_{kj} \sum_{i=0}^{\overline{k}-j} i$$
$$= \sum_j \binom{j-\overline{k}}{2} \alpha_{kj} = \sum_j \binom{j}{2} \alpha_{kj} = \sum_j \binom{j-k}{2} N^j =: z.$$

Regardless of the degree of p, z is the quadratic polynomial

$$z(t) = \frac{t^2}{2} - \frac{t_k^* + t_{k+1}^*}{2} t + \frac{1}{2}\left(\frac{t_k^* + t_{k+1}^*}{2}\right)^2 + \frac{d-2}{24}.$$

Since z is a positive and convex function, z attains its maximum over $[t_k^*, t_{k+1}^*]$ at one of the endpoints of the interval. Its values there are

$$z(t_k^*) = z(t_{k+1}^*) = \frac{d+1}{24},$$

and hence $z(t) = \sum_i \beta_{ki} \leq (d+1)/24$ for all $t \in [t_k^*, t_{k+1}^*]$.

The number of β_{ki} that are nonzero at t_k^* is $d-1$ for d even and $d-2$ for d odd, i.e. only β_{kk} is nonzero at t_k^* if p is quadratic or cubic. But then all inequalities of equation (1) become equalities as claimed. $\square$

Computing the bounds.

To compute the bounds for quadratics or cubics, no B–spline evaluation is required, since only $\beta_{kk}(t_k^*) = (d+1)/24$ is nonzero. For $d > 3$, it suffices to look up tabulated values $\beta_{ki}(t_k^*)$ for $-\lfloor d/2 \rfloor < i < \lfloor d/2 \rfloor$. Forming the inner products of Theorem 2 and Theorem 3 at t_k^* is straightforward.

§4. Uniform Refinement

An important operation on B–splines is the refinement of the knot sequence or knot insertion. Knot insertion changes the representation of the piecewise polynomial p over the original knot sequence to one over a larger knot sequence, and reduces the distance between spline and control polygon (c.f. Figure 3).

After halving the distance between knots, the new control points $\hat{b}^k$ in $p(t) = \sum_k b^k N^k(t) = \sum_k \hat{b}^k N^k(2t)$ are given by

$$\hat{b}^{2i} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j} b^{i-j}, \qquad \hat{b}^{2i+1} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j+1} b^{i-j}. \tag{2}$$

Theorem 4. *The second differences $\Delta_2 \hat{b}^i$ of the refined control polygon are bounded by the second differences $\Delta_2 b^i$ of the original control polygon*

$$\max_i |\Delta_2 \hat{b}^i| = \frac{1}{4} \max_i |\Delta_2 b^i|.$$

Proof: The second derivative p'' of p is given by

$$p''(t) = \sum_k \Delta_2 b^{k-1} N^k_{d-2}(t) = \sum_k \Delta_2 \hat{b}^{k-1} N^k_{d-2}(2t)$$

which means that the $\Delta_2 \hat{b}^i$ can be obtained from the $\Delta_2 b^i$ via (2) as

$$2^d \Delta_2 \hat{b}^{2i} = \sum_j \binom{d-1}{2j-1} \Delta_2 b^{i-j}, \quad 2^d \Delta_2 \hat{b}^{2i+1} = \sum_j \binom{d-1}{2j} \Delta_2 b^{i-j}.$$

The proof follows from $\sum_j \binom{d-1}{j} = 2^{d-1}$ and $\sum_j \binom{d-1}{2j} = \sum_j \binom{d-1}{2j-1} = 2^{d-2}$. □

Theorem 4 yields the following *a priori* estimate on the number of subdivisions σ needed to bring spline and control polygon within a given distance ε:

$$\sigma(p, \varepsilon) = \lceil \log_4 \frac{(d+1)\|\Delta_2 b\|}{24\varepsilon}.\rceil$$

Examples: For quadratic B–Splines, uniform refinement is called Chaikin's algorithm and

$$\hat{b}^{2i} = 2^{-2}(3b^{i-1} + b^i) \qquad \hat{b}^{2i+1} = 2^{-2}(b^{i-1} + 3b^i).$$

This yields

$$\Delta_2 \hat{b}^{2i} = \Delta_2 \hat{b}^{2i-1} = \frac{1}{4} \Delta_2 b^{i-1},$$

i.e. every second difference is guaranteed to decrease by a factor of four. Similarly, for cubic B–Splines we have

$$\hat{b}^{2i} = 2^{-3}(b^{i-2} + 6b^{i-1} + b^i) \qquad \hat{b}^{2i+1} = 2^{-3}(4b^{i-1} + 4b^i).$$

and

$$\Delta_2 \hat{b}^{2i} = \frac{1}{4} \Delta_2 b^{i-1} \quad \text{and} \quad \Delta_2 \hat{b}^{2i+1} = \frac{1}{4} \frac{\Delta_2 b^{i-1} + \Delta_2 b^i}{2}.$$

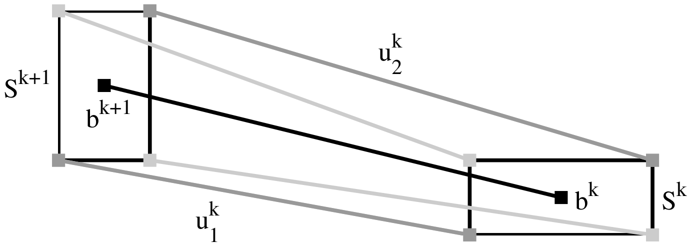

Fig. 4. Constructing the envelope of a curve from the bounding rectangles S^k and S^{k+1}: only the outer line segments $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$ are part of the convex hull of $S^k \cup S^{k+1}$ and the envelope.

§5. Curve Envelopes

A parametric curve $\boldsymbol{p}$ is in uniform B–spline form if $\boldsymbol{p} = \sum_j \boldsymbol{b}^j N^j$, where the $\boldsymbol{b}^j \in \mathbb{R}^n$ are the control points of $\boldsymbol{p}$ and the uniform B–spline basis N^j is defined as in Section 2. The curve $\boldsymbol{p}$ is closed if the control point sequence $(\boldsymbol{b}^j)$ is periodic.

The functional bounds are applied componentwise to parametric curves. Then each control point and the curve point corresponding to its Greville abscissa lie in a box whose width in the ith component is the bound in the ith component. It is now convenient to restate the bounds from Theorems 2 and 3 more abstractly as

$$\underline{e}(t) \leq p(t) - \ell(t) \leq \overline{e}(t), \qquad \text{for } t \in [t_k^*, t_{k+1}^*]. \tag{3}$$

For curves $\boldsymbol{p}$, the bound in the i–th component is denoted by $\underline{e}_i \leq \boldsymbol{p}_i - \ell_i \leq \overline{e}_i$.

By (3), $\boldsymbol{p}(t_k^*)$ is located in the axis–aligned box S^k,

$$S^k = \{\boldsymbol{x} \mid \underline{e}_i(t_k^*) \leq \boldsymbol{x}_i - \boldsymbol{b}_i^k \leq \overline{e}_i(t_k^*) \quad \text{for all } i = 1, \ldots, n\}.$$

Each point of the curve segment $\boldsymbol{p}(t)$, $t \in [t_k^*, t_{k+1}^*]$, lies in a box $S(t)$, that by the linearity of $\underline{e}$ and $\overline{e}$ is a convex combination of S^k and S^{k+1}:

$$S(t) = \mathtt{L}_k(t\,; S^k, S^{k+1}).$$

The curve segment is therefore contained in the union of all $S(t)$, $t \in [t_k^*, t_{k+1}^*]$, which is the convex hull H^k of the corners of S^k and S^{k+1}. To be specific, we discuss the case of planar curves.

Enveloping planar curves

Let $\boldsymbol{v}_i^k$, $i = 1, \ldots, 4$, be the line segments connecting corresponding corners of S^k and S^{k+1}; that means $\boldsymbol{v}_1^k$ connects the lower left corner of S^k to the lower left corner of S^{k+1}, $\boldsymbol{v}_2^k$ connects the lower right corner of S^k to the lower right corner of S^{k+1} etc. as in Figure 4.

H^k consists of parts of the boundaries of S^k and S^{k+1} and exactly two additional line segments $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$ chosen from the $\boldsymbol{v}_i^k$. Since $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$ are part of the convex hull H^k, they do not intersect the interiors of S^k and S^{k+1}. We do not need to actually compute intersections of the $\boldsymbol{v}_i^k$ and S^k, S^{k+1} to select $\boldsymbol{u}_1^k$ and $\boldsymbol{u}_2^k$: since S^k and S^{k+1} are axis–aligned it suffices to look at the signs of the slopes of the $\boldsymbol{v}_i^k$. The $\boldsymbol{u}_i^k$ are separated by the line from $\boldsymbol{b}^k$ to $\boldsymbol{b}^{k+1}$; we call the one lying to the left of this line $\boldsymbol{u}_1^k$ and the one lying to the right of this line $\boldsymbol{u}_2^k$.

The sets $U_i = \{\boldsymbol{u}_i^k\}$ are not yet polylines: consecutive line segments $\boldsymbol{u}_i^k$ and $\boldsymbol{u}_i^{k+1}$ may intersect or not touch at all. But note that the line extending $\boldsymbol{u}_i^k$ always intersects the one extending $\boldsymbol{u}_i^{k-1}$. We obtain a proper polyline W_i with exactly one line segment for each control point of $\boldsymbol{p}$ by taking this intersection as starting point and the intersection with the line through $\boldsymbol{u}_i^{k+1}$ as the end point of W_i. The polylines W_1 and W_2 then form a *local* envelope of $\boldsymbol{p}$: the curve–piece $\boldsymbol{p}([t_k^*, t_{k+1}^*])$ lies in the quadrangle spanned by the k–th pieces of W_1 and W_2.

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