Generalized spline subdivision

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- Polynomial Heritage
- Computing Moments
- Shape and Eigenvalues
Polynomial heritage
of generalized spline subdivision

• Doo-Sabin

Catmull-Clark
Polynomial heritage
of generalized spline subdivision

- Increasing regions are regular: points and faces have standard valence
Polynomial heritage
of generalized spline subdivision

• Doo-Sabin bi-2 B-spline
• Catmull-Clark bi-3 B-spline
• Midedge Zwart-Powell $C^1$ box-spline
• Loop $C^2$ box-spline

box-spline = generalization of B-spline to shift-invariant partitions
book: [de Boor, Hollig, Riemenschneider 94]
Polynomial heritage
of generalized spline subdivision

- Subdivision of the Zwart-Powell $C^1$ quadratic box-spline

![Diagram of polynomial heritage](image)
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of generalized spline subdivision

Mid-edge Rule
(“simplest rule”)  

Zwart-Powell subdivision
= 2 steps of Midedge subdivision

regular: 4-valence, quadrilaterals

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Polynomial heritage
of generalized spline subdivision

- Increasing regions are regular (polynomial)
- Union of surface layers at an extraordinary point
Polynomial heritage of generalized spline subdivision

• Uses:
  
  Representation as Bezier patches

  Evaluation at non-binary points

  Fast moment computation
Generalized spline subdivision

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• Polynomial Heritage
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Moments
of objects enclosed by generalized subdivision surfaces

- Challenge: Exponential increase in the number of facets!

Volume
Center of mass

Inertia Frame

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Theory: Gauss’ Divergence Theorem:

The integral of the divergence over the volume

\[ \int_{V} \nabla \cdot f \, dV \]

equals the integral of the normal component over the surface S

\[ \int_{S} f \cdot n/|n| \, dS \]

\[ \int_{V} \nabla \cdot f \, dV = \int_{S} f \cdot n/|n| \, dS = \int_{U} f \cdot n \, dU \]

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Theory: Change of variables

The area of the surface element $S$ equals the integral of the Jacobian $|n|$ of the surface parametrization $(x,y,z)$ over the domain $U$.

\[
\int_{U} |n| \, dU = \int_{S} dS
\]

\[
\int_{V} \nabla \cdot f \, dV = \int_{S} f \cdot n/|n| \, dS = \int_{U} f \cdot n \, dU
\]
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of objects enclosed by generalized subdivision surfaces

For example, \( f = [0,0,z] \) \( n = x_u y_v - x_v y_u \)

\( f \cdot n = z (x_u y_v - x_v y_u) \) is piecewise polynomial in regular regions

Volume = \( \int_V 1 \ dV = \int_U z [x_u y_v - x_v y_u] \ du \ dv \)
= \( \sum_{\text{patch } p} \int_{U_p} z^p (x_u^p y_v^p - x_v^p y_u^p) \ du \ dv \)

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“Volume” patch \( p = \int_{U_p} z^p (x^p_u y^p_v - x^p_v y^p_u) \, du \, dv \)

Schema for bi-3 Bezier patch
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Volume patch $p = \int_{U_p} z^p (x^p_u y^p_v - x^p_v y^p_u) \, du \, dv$
Moments of objects enclosed by generalized subdivision surfaces

Volume patch $p = \int z^p (x^p_u y^p_v - x^p_v y^p_u) \, du \, dv$
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Volume patch $p = \int_{U_p} z^p (x_u^p y_v^p - x_v^p y_u^p) \, du \, dv$
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Work: at each subdivision step linear
for each extraordinary point
add volume contribution of 3n patches

Doo-Sabin

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\[ V_i = \sum_{p \text{ in layer } i_{UP}} \int f^{p} (x_{u_{v}}^{p} y_{v}^{p} - x_{v_{u}}^{p} y_{u}^{p}) \, du \, dv \]

Volume \( = \sum_{i=0}^{m} V_i + W_m \)
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- Error estimation: bounding boxes
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- Geometric decay of error volume \(1, \frac{1}{8}, \frac{1}{64}, \ldots\)
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- Computing geometry given a fixed volume

Bisection
Moments of objects enclosed by generalized subdivision surfaces

- Higher moments and the inertia frame

$$\int \int \int_V V dV \quad dV \quad \text{center of mass}$$

$$\int \int \int_V xy dV \quad dV \quad \text{inertia tensor:}$$

$$\int \int \int_V x dV, \int \int \int_V y dV, \int \int \int_V z dV$$

$$\Rightarrow \Rightarrow \Rightarrow \Rightarrow$$

$$\Rightarrow \Rightarrow \Rightarrow$$
Moments
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• Higher moments and the inertia frame
Moments of objects enclosed by generalized subdivision surfaces

- Physics-based animation

Center of mass support

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Moments
of objects enclosed by generalized subdivision surfaces

• Simple registration, comparison

matching frames = computing a 3x3 matrix $Q$:

$IP Q = IS$
Moments of objects enclosed by generalized subdivision surfaces

Solution: Moments efficiently and exactly computed via Gauss' theorem and polynomial heritage.

- Volume
- Center of mass
- Inertia Frame
Shape and eigenvalues

- Union of surface layers at an extraordinary point
- Control points transformed by the subdivision matrix
Shape and eigenvalues

\[ B_{m+1} = A \cdot B_m = A^{m+1} B_0 \]

\[ A \cdot v_i = \lambda_i v_i \]

\( \lambda_i \) is the eigenvalue to the eigenvector \( v_i \)

\[ B_0 = \sum \alpha_i v_i \]  
**eigenvector expansion**

\[ B_{m+1} = A^{m+1} B_0 = \sum \alpha_i A^{m+1} v_i \]

\[ = \sum \alpha_i \lambda_i^{m+1} v_i \]
Shape and eigenvalues

- If all $\lambda < 1$, then collapse
- If some $\lambda > 1$, then unbounded growth
- Good sequence: $1, \ell, \ell, \ldots$ where $|\ell| < 1$
- Eigenvectors of $\ell$ determine the tangent plane

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Shape and eigenvalues

- Fast contraction of 3-sided facets
  \[ \ell = \frac{1 + \cos(\frac{2\pi}{3})}{2} = 0.25 \]

- Slow contraction of large facets
  \[ \ell = \frac{1 + \cos(\frac{2\pi}{16})}{2} = 0.962... \]
Shape and eigenvalues

- adjust subdominant eigenvalues
  (modified midedge subdivision)

\[ \iff \lambda = 0.5 \]
Shape and eigenvalues
Generalized spline subdivision

Summary

- Polynomial Heritage
  - regular regions
- Computing Moments
  - Gauss’ theorem
- Shape and Eigenvalues
  - subdominant values

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