Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients

D. Lutterkort J. Peters* U. Reif†

November 1, 1998

Abstract Given a polynomial p of degree n we want to find a best L_2-approximation over the unit interval from polynomials of degree $m < n$. This problem is shown to be equivalent to the problem of finding the best Euclidean approximation of the vector of Bernstein-Bézier coefficients of p from the vector of degree-raised Bernstein-Bézier coefficients of polynomials of degree m.

1 Motivation

Optimal degree reduction to exchange, convert or reduce data, or compare geometric entities is an important task in CAGD. Since the piecewise linear control structure of the commonly used Bernstein-Bézier form efficiently captures geometric properties it is tempting to find the optimal lower degree approximant by just comparing control points. This is in general a flawed approach since the L_2-norm $\|\cdot\|_{\mathrm{L}}$ of the polynomials and the Euclidean norm $\|\cdot\|_{\mathrm{E}}$ of the coefficients are not similar. For a simple example, consider the univariate linear polynomials $p(t) = 6(1-t)$ and $q(t) = 4(1-t) + 3t$. Then

$$\|p\|_{\mathrm{L}} = \frac{6}{\sqrt{3}} < \frac{\sqrt{37}}{\sqrt{3}} = \|q\|_{\mathrm{L}} \;, \quad \text{but} \quad \|p\|_{\mathrm{E}} = 6 > 5 = \|q\|_{\mathrm{E}} \;.$$

The result proven in this paper is therefore not obvious: to find a best L_2-approximation over the unit interval from polynomials of degree m to a given polynomial p of degree $n > m$ is equivalent to finding the best Euclidean approximation of the vector of Bernstein-Bézier coefficients of p from vectors of Bernstein-Bézier coefficients of polynomials of degree m raised to degree n.

Bypassing the many interesting identities encountered along the way, we present the result succinctly in the next pages — but not before pointing out *prior work*. Lachance [5] and Eck [2], [3] analyze Chebyshev economization,

*NSF National Young Investigator Award 9457806-CCR

†supported by the Leibniz Foundation

Brunnett et al. [1] focus on separability of degree reduction into the different spatial components and the geometry of the control polygon. Endpoint constrained L_2-approximation coupled with subdivision is discussed in [4] which also contains a summary of earlier literature on economization.

2 Characterization of degree-raised polynomials

The linear space of polynomials of degree less than or equal to n is denoted by $\mathbb{P}_n$. We shall use two different bases of $\mathbb{P}_n$, namely the Bernstein-Bézier (BB) basis and the Lagrange basis with respect to the points $0, \ldots, n$. The corresponding row vectors of polynomials are

$$B^n := [B_0^n, \ldots, B_n^n], \quad \text{where } B_i^n(t) := \binom{n}{i}(1-t)^{n-i}t^i$$
$$Q^n := [Q_0^n, \ldots, Q_n^n], \quad \text{where } Q_i^n(t) := \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t-j}{i-j}.$$

With $b \in \mathbb{R}^{n+1}$ a column vector of coefficients, we write polynomials in BB form and Lagrange form as $B^n b$ and $Q^n b$, respectively. The latter form is used to relate a discrete polynomial dependence of the coefficients on the vector index to a continuous polynomial. For example, if the coefficients $b(i) = i^2 - i$ depend quadratically on the index i, then $Q^n(t)b = t^2 - t$ is the corresponding quadratic polynomial. The following lemma is well-known.

Lemma 2.1 *A polynomial $B^n b$ is of degree $\leq m$ if and only if the vector of coefficients is a polynomial of degree $\leq m$ in its index, i.e.*

$$B^n b \in \mathbb{P}_m \;\Leftrightarrow\; Q^n b \in \mathbb{P}_m.$$

Proof We define the column vector of alternating binomial coefficients by

$$v_k(i) := (-1)^i \binom{k}{i}, \quad i = 0, \ldots, n \tag{1}$$

with the understanding that $\binom{k}{i} = 0$ if $i > k$. First, when expanding the BB form into monomial form we obtain

$$B^n(t)b = \sum_{k=0}^{n} (b^{\mathrm{t}} v_k)(-1)^k \binom{n}{k} t^k.$$

Hence, $B^n b \in \mathbb{P}_m$ if and only if $b^{\mathrm{t}} v_k = \sum_i b(i) v_k(i) = 0$ for $m < k \leq n$. Second, the divided difference of $Q^n b$ at the points $0, \ldots, k$ is

$$[0, \ldots, k](Q^n b) = \frac{b^{\mathrm{t}} v_k}{k!}.$$

¿From the Newton form

$$Q^n(t)b = \sum_{k=0}^{n} \frac{b^{\mathrm{t}} v_k}{k!} \prod_{j=1}^{k} (t-j)$$

we see that also $Q^n b \in \mathbb{P}_m$ if and only if $b^{\mathrm{t}} v_k = 0$ for $m < k \leq n$, what concludes the proof. $\bowtie$

3 Equivalence of orthogonal complements

Like any approximation problem in a Hilbert space, degree reduction is closely related to determining the orthogonal complement of the approximation space with respect to the embedding space. The following result states an unexpected coincidence:

Theorem 3.1 *The orthogonal complements of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the L_2-inner product*

$$\langle f, g \rangle_{\mathrm{L}} := \int_0^1 f(t) g(t)\, dt \tag{2}$$

and the Euclidean inner product of the BB coefficients

$$\langle B^n b, B^n c \rangle_{\mathrm{E}} := \sum_{i=0}^{n} b_i c_i \tag{3}$$

are equal.

Proof Denote the orthogonal complement of $\mathbb{P}_m$ in $\mathbb{P}_n$ with respect to the Euclidean inner product by $\mathbb{P}_{m,n}$, and let $B^n w_{m+1}, \ldots, B^n w_n$ be some basis of this space. By equality of dimensions it suffices to show that $\mathbb{P}_{m,n}$ is contained in the orthogonal complement with respect to the L_2-inner product, i.e. the polynomials $B^n w_k$ have to be L_2-orthogonal to all polynomials in $\mathbb{P}_m$,

$$\left\langle B^n w_j, t^i \right\rangle_{\mathrm{L}} = 0\,, \quad 0 \leq i \leq m < j \leq n\,.$$

Defining the column vector p_i by

$$p_i(k) := \int_0^1 B_k^n(t) t^i\, dt\,, \quad k = 0, \ldots, n$$

we rewrite $\left\langle B^n w_j, t^i \right\rangle_{\mathrm{L}} = \langle B^n w_j, B^n p_i \rangle_{\mathrm{E}}$. By definition, the latter expression vanishes if and only if $B^n p_i \in \mathbb{P}_m$, and by Lemma 2.1 this is equivalent to $Q^n p_i \in \mathbb{P}_m$. In other words, we have to show that $p_i(k)$ is polynomial in k of degree $\leq m$ for all $i = 0, \ldots, m$. Using the formula $\int_0^1 B_k^d(t)\, dt = 1/(d+1)$, this follows easily from

$$p_i(k) = \frac{\binom{n}{k}}{\binom{n+i}{k+i}} \int_0^1 B_{k+i}^{n+i}(t)\, dt = \frac{n!}{(n+i+1)!} \prod_{\ell=1}^{i} (k+\ell)\,.$$

$\bowtie$

¿From the proof of Lemma 2.1 we can see that a possible choice for the basis $B^n w_{m+1}, \ldots, B^n w_n$ is provided by the coefficient vectors $v_{m+1}, \ldots, v_n$. In particular, we see for $m = n-1$ that

$$B^n v_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i^n$$

is L_2-orthogonal to all polynomials of degree $< n$. Hence, up to scaling, $B^n v_n$ is just the Legendre polynomial of degree n on the unit interval in BB form.

4 Consequences

The implications of Theorem 3.1 to degree reduction are straightforward.

Corollary 4.1 *Given a polynomial $B^n b$ of degree n, the approximation problem*

$$\min_{p \in \mathbb{P}_m} \|B^n b - p\|$$

has the same minimizer for the norm induced either by the L_2-inner product (2) *or the Euclidean inner product* (3)*.*

Proof The polynomial $B^n b$ can be decomposed uniquely according to

$$B^n b = p + q\,, \quad p \in \mathbb{P}_m\,,\ q \in \mathbb{P}_{m,n}$$

and, by orthogonality, p is the wanted solution for both norms. $\bowtie$

The following corollary affirms that the degree reduction process factors, e.g. k-fold degree reduction by one yields the same best approximand as a single reduction by k degrees. This is of interest, for example, when seeking an approximand of least degree that still lies within a prescribed tolerance.

Corollary 4.2 *Denote by $\mathcal{P}_{m,n}$ the linear operator mapping polynomials $B^n b \in \mathbb{P}_n$ to their best L_2 or Euclidean approximant $p \in \mathbb{P}_m$. Then*

$$\mathcal{P}_{m,n} = \mathcal{P}_{m,\ell}\mathcal{P}_{\ell,n}\,, \quad m \leq \ell \leq n\,.$$

The factorization of degree reduction is well-known in the L_2-case, non-trivial to prove directly in the discrete Euclidean case, and in general false in other norms, e.g. for Chebyshev approximation.

5 Practical considerations and an example

In practice, one is often interested in the BB form $p = B^m c$ of the best degree reduction to the polynomial $B^n b$. In order to compare coefficients, p has to be represented in terms of B^n, i.e. $p = B^n \tilde{c}$. The degree raising matrix $A_{n,m}$ for mapping the BB coefficients c to $\tilde{c}$ has dimension $(n+1) \times (m+1)$ and can be decomposed into elementary degree-raising steps [5] as $A_{n,m} = A_{n,n-1} A_{n-1,n-2} \cdots A_{m+1,m}$, where

$$A_{k,k-1}(i,j) = \begin{cases} i/k & \text{if } j = i-1 \\ 1 - i/k & \text{if } j = i \\ 0 & \text{else}. \end{cases}$$

Then, with $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^{n+1}$, degree reduction amounts to solving the least squares problem

$$\min_{c \in \mathbb{R}^{m+1}} \|b - A_{m,n} c\| .$$

The solution is given by the pseudo inverse $P_{m,n}$ of the degree raising matrix,

$$c = P_{n,m} b := (A^{\mathrm{t}}_{m,n} A_{m,n})^{-1} A^{\mathrm{t}}_{m,n} b .$$

¿From Corollary 4.2 it follows that $P_{m,n}$ can be factored corresponding to a sequence of elementary degree reduction steps,

$$P_{m,n} = P_{m,m+1} P_{m+1,m+2} \cdots P_{n-1,n} .$$

Hence, in order to get easy access to arbitrary degree reduction matrices, it suffices to precompute the matrices $P_{k,k+1}$, the first few of which are

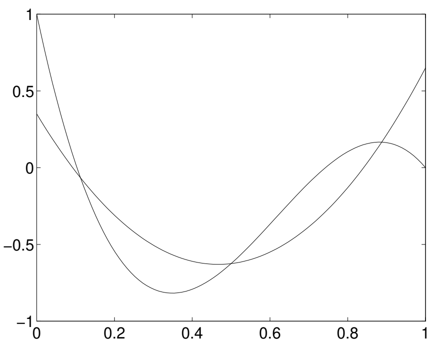
$$P_{0,1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad P_{2,3} = \frac{1}{20} \begin{bmatrix} 19 & 3 & -3 & 1 \\ -5 & 15 & 15 & -5 \\ 1 & -3 & 3 & 19 \end{bmatrix}$$

$$P_{1,2} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ -1 & 2 & 5 \end{bmatrix}, \quad P_{3,4} = \frac{1}{210} \begin{bmatrix} 207 & 12 & -18 & 12 & -3 \\ -53 & 212 & 102 & -68 & 17 \\ 17 & -68 & 102 & 212 & -53 \\ -3 & 12 & -18 & 12 & 207 \end{bmatrix}.$$

We obtain for instance

$$P_{1,3} = P_{1,2} P_{2,3} = \frac{1}{10} \begin{bmatrix} 7 & 4 & 1 & -2 \\ -2 & 1 & 4 & 7 \end{bmatrix}.$$

As an example, consider the cubic polynomial $B^3 b^3$ with BB coefficients $b^3 = [1, -3, 1, 0]^{\mathrm{t}}$. The best approximating quadratic $B^2 b^2$ has coefficients $b^2 = [7/20, -35/20, 13/20]^{\mathrm{t}}$, while the best approximating linear polynomial $B^1 b^1$ to either $B^2 b^2$ or $B^3 b^3$ has coefficients $b^1 = [-4/10, -1/10]^{\mathrm{t}}$. The result is illustrated in Figure 1.



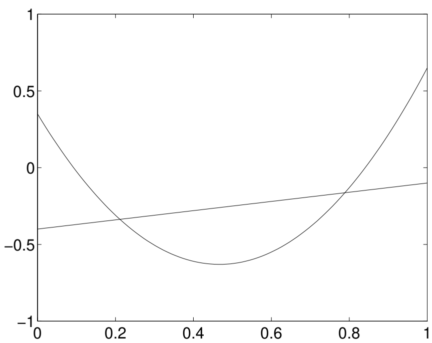


Figure 1: L_2 degree reduction of a cubic to degree 2 (left) and degree 1 (right).

References

[1] Brunnett, G., Schreiber, T., and Braun, J. The geometry of optimal degree reduction of Bézier curves. *Computer Aided Geometric Design*, 13 (1996), 773–788.

[2] Eck, M. Degree reduction of Bézier curves. *Computer Aided Geometric Design 10*, 3 (Aug. 1993), 237–252.

[3] Eck, M. Degree reduction of Bézier surfaces. In *Design and application of curves and surfaces (Edinburgh, 1992)*. Oxford Univ. Press, New York, 1994, pp. 135–154.

[4] Eck, M. Least squares degree reduction. *Computer Aided Design*, 27 (1995), 845–851.

[5] Lachance, M. Chebyshev economization for parametric surfaces. *Computer Aided Geometric Design*, 5 (1988), 195–205.