A sharp degree bound on G^2-refinable multi-sided surfaces

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Vilnius University

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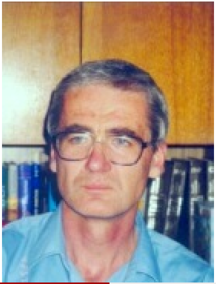
University of Florida

SPM 2020



Motivation

Design: needs multi-sided surface "caps"

Design: needs good shape

Engineering Analysis
needs **flexibility increasing refinability**

Key result for G^2 (curvature continuous) surfaces:

- bi-5 surfaces are not flexibly G^2-refinable
- bi-6 surfaces are flexibly G^2-refinable

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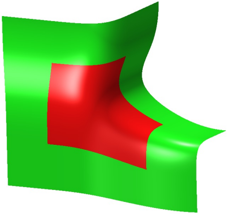
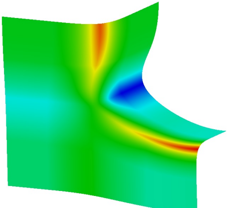
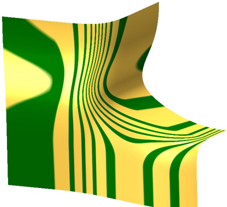




mean curvature



highlight lines

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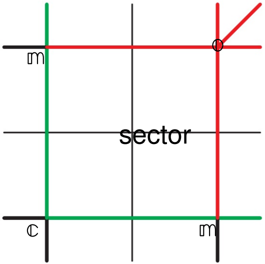

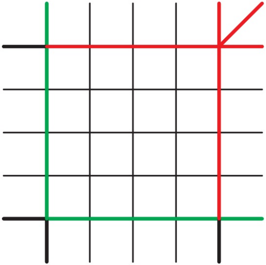

= increase degrees of freedom both along boundaries and in the interior.

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Outline

1. Technical Toolkit
2. Lower bound: Bi-5 caps are not flexibly G^2-refinable
3. Upper bound: Bi-6 caps are flexibly G^2-refinable

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2. Lower bound: Bi-5 caps are not flexibly G^2-refinable

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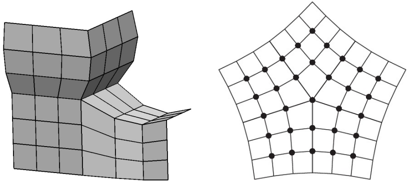
Setup: Multi-sided surfaces in bi-cubic B-spline complex



extended CC-net

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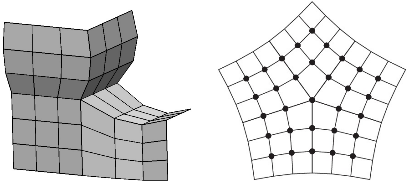

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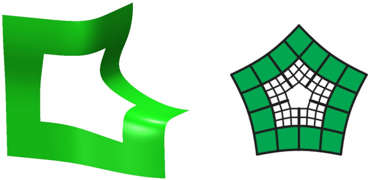

bicubic ring + tensor-border

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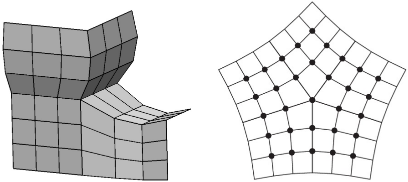

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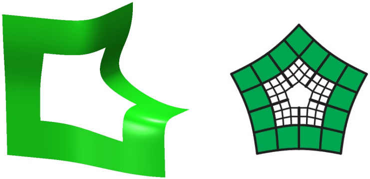

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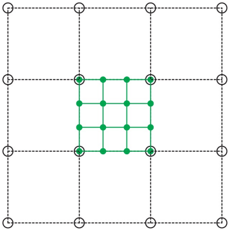
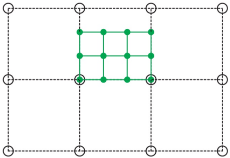


bicubic patch: B-to-BB form conversion



tensor-border

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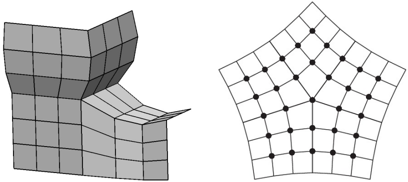

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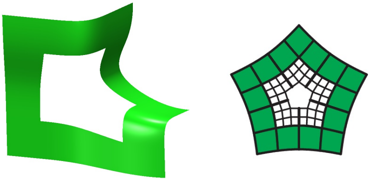

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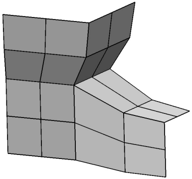

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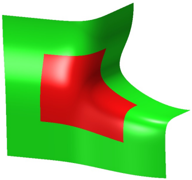
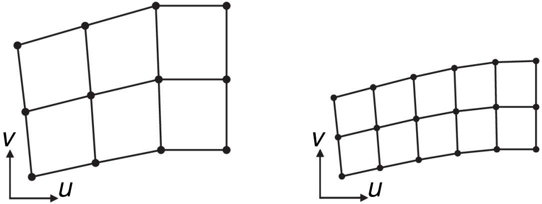


cap

Geometric continuity, reparameterizations

$$\tilde{\mathbf{t}}(u,v) := \mathbf{t} \circ \rho(u,v)$$



$$\mathbf{t}(u,v) := \mathbf{t}(u,0) + \partial_v \mathbf{t}(u,0)v + \tfrac{1}{2}\partial_v^2 \mathbf{t}(u,0)v^2$$
$$\rho(u,v) := (u + b(u)v + \tfrac{1}{2}e(u)v^2,\;\; a(u)v + \tfrac{1}{2}d(u)v^2)$$

for $k = 0, 1, 2,\quad \partial^k \tilde{\mathbf{f}} = \partial^k(\mathbf{f} \circ \rho)$

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G^2 constraints between
two surface pieces $\tilde{\mathbf{f}}, \mathbf{f} : (u, v) \in \mathbb{R}^2 \to \mathbb{R}^3$ along common edge $(u, 0)$:
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G^1 constraints: $\quad \partial_v \tilde{\mathbf{f}} = a\,\partial_v \mathbf{f} + b\,\partial_u \mathbf{f}$

G^2 constraints: $\quad \partial^2_{vv} \tilde{\mathbf{f}} = a^2\,\partial^2_{vv}\mathbf{f} + 2a\,b\,\partial_u\partial_v \mathbf{f} + b^2\,\partial^2_{uu}\mathbf{f} + d\,\partial_v \mathbf{f} + e\,\partial_u \mathbf{f}$

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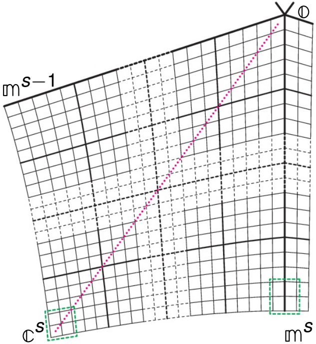

Hypothetical bi-5 many-piece sector

Reasonable G^2 constructions are

- diagonally symmetric (invariant under reversal of indices)
- unbiased (invariant under relabeling)
- sectors are internally C^k (as opposed to G^k)

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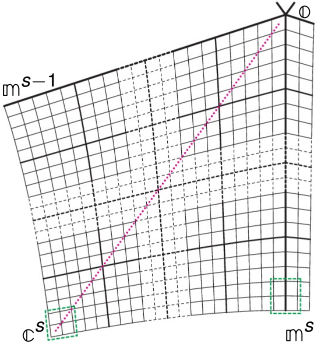

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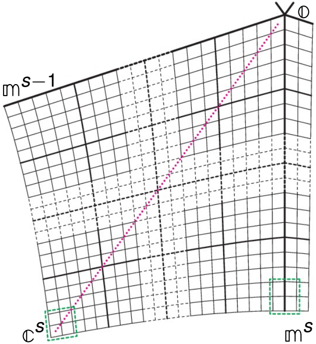

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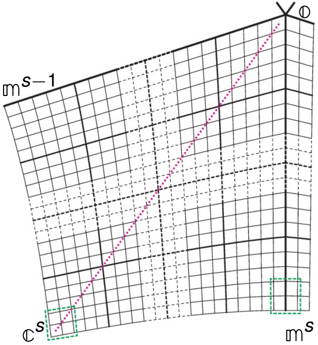

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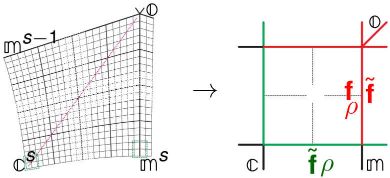
Flexible G^2-refinement



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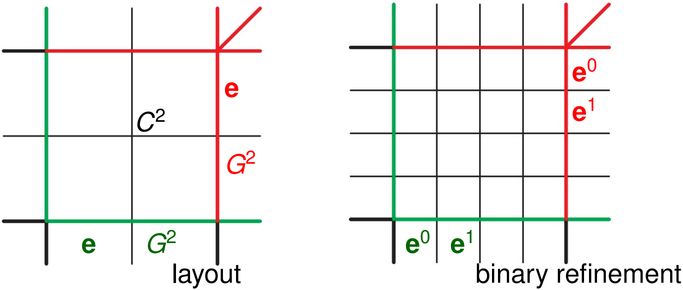

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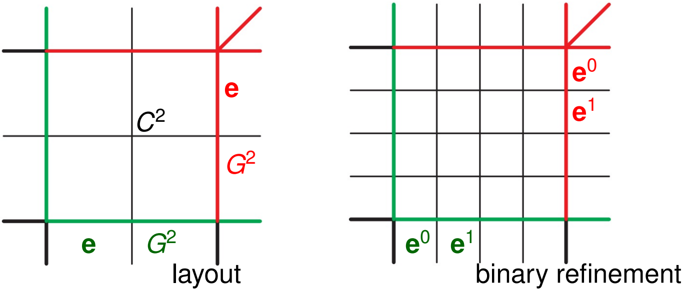

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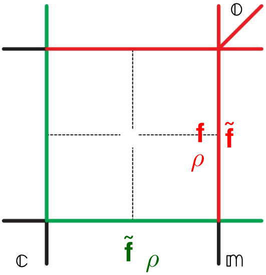
Bi-5 splines are not flexibly G^2-refinable: Technical Lemmas



across $\mathbb{c}$–$\mathbb{m}$: a, b, d, e must be polynomial

degree $a, b, d, e \le 1, 2, 2, 3$.

$a \equiv 1$, $b \equiv d \equiv e \equiv 0$. (internal C^2, refinability)

$\Rightarrow \rho$ is identity

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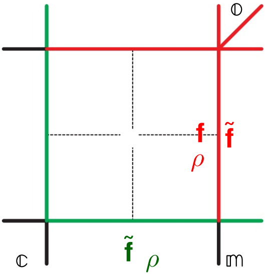

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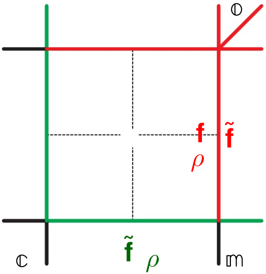

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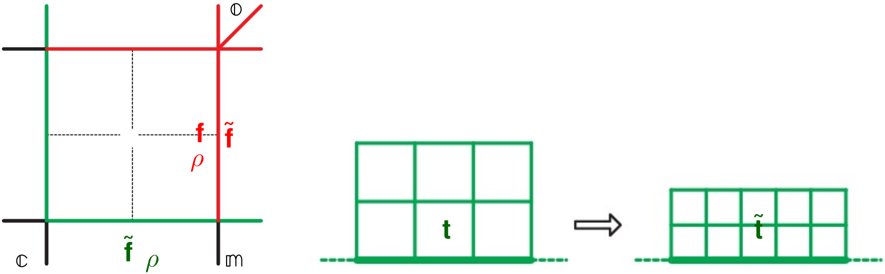

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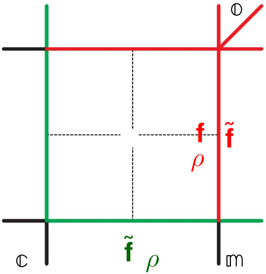

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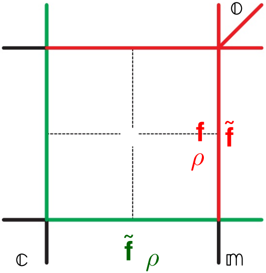

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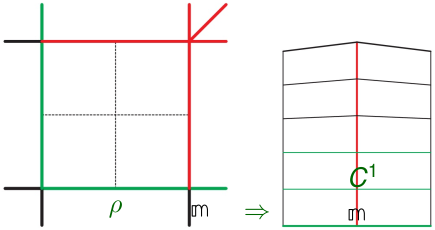
Lower bound: Bi-5 is not flexibly G^2-refinable: Proof



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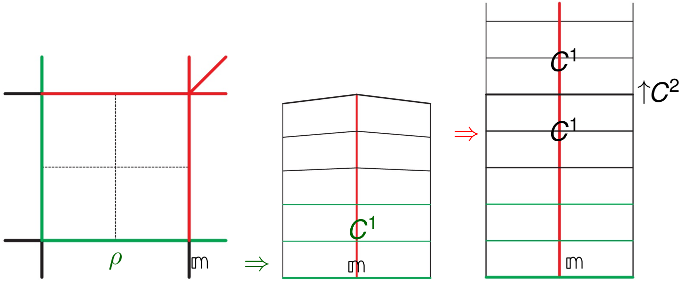

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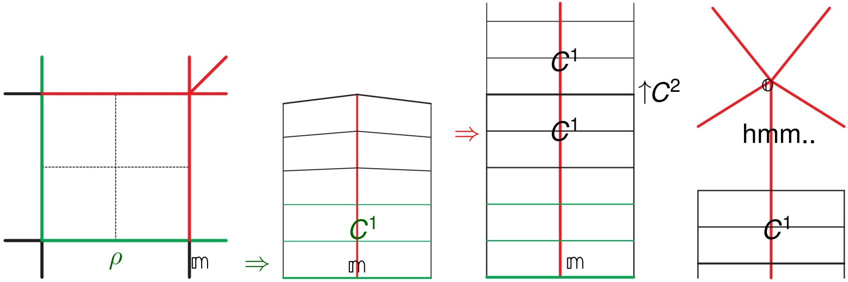

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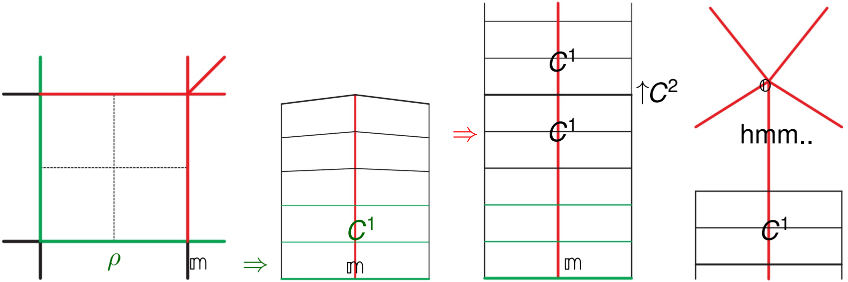
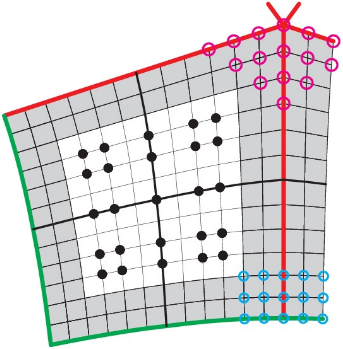


$\rho(u, v)$ = identity $\qquad$ $\rho(u, v)$ = (u,-v) $\qquad$ conflict at eop!

Least upper bound Bi-6 flexibly G^2-refinable surface



layout

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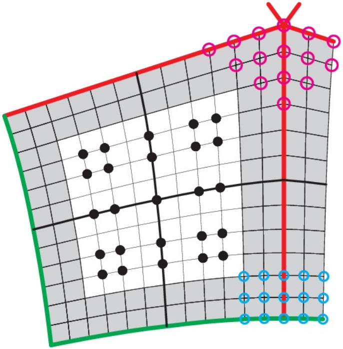

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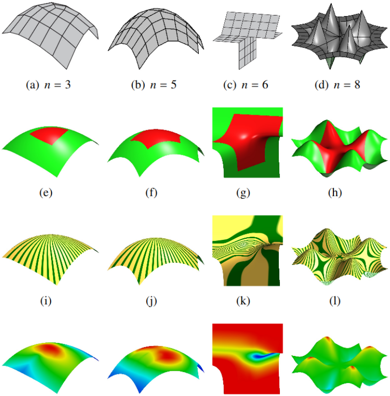


(a) $n = 3$ (b) $n = 5$ (c) $n = 6$ (d) $n = 8$

(e) (f) (g) (h)

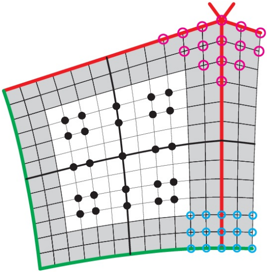
(i) (j) (k) (l)

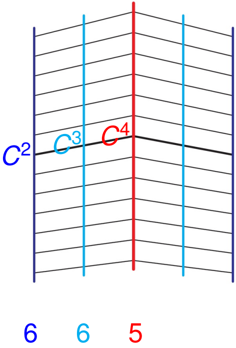
(m) min (n) Gaussian (o) max (p) mean

multi-sided surface caps

G^2-refinability between sectors

G^2 bi-6 degree$(b(u))$ = 2; $d(u) := 0$, $e(u) := b(u)b'(u)$





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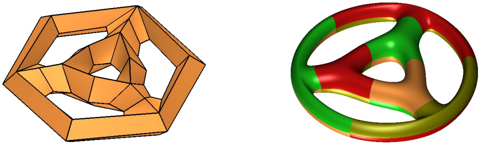

Thank you