A sharp degree bound on $G^2$-refinable multi-sided surfaces

Kęstutis Karčiauskas
Vilnius University

Jörg Peters
University of Florida

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Motivation

Design: needs multi-sided surface "caps"

Design: needs good shape

Engineering Analysis needs flexibility increasing refinability

Key result for $G^2$ (curvature continuous) surfaces:

- bi-5 surfaces are not flexibly $G^2$-refinable
- bi-6 surfaces are flexibly $G^2$-refinable

= a sharp degree bound on $G^2$-refinable multi-sided surfaces
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Engineering Analysis (e.g. solving a P.D.E. on surface):
needs **flexibility increasing refinability**

= increase degrees of freedom both along boundaries and in the interior.

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Outline

1. Technical Toolkit

2. Lower bound: Bi-5 caps are not flexibly $G^2$-refinable

3. Upper bound: Bi-6 caps are flexibly $G^2$-refinable
1 Technical Toolkit

2 Lower bound: Bi-5 caps are not flexibly $G^2$-refinable

3 Upper bound: Bi-6 caps are flexibly $G^2$-refinable
Setup: Multi-sided surfaces in bi-cubic B-spline complex
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extended CC-net

bicubic ring + tensor-border
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bicubic patch: B-to-BB form conversion

tensor-border
Setup: Multi-sided surfaces in bi-cubic B-spline complex

- extended CC-net
- bicubic ring + tensor-border
- CC-net
- cap
Geometric continuity, reparameterizations

\[ \tilde{t}(u, v) := t \circ \rho(u, v) \]

\[ t(u, v) := t(u, 0) + \partial_v t(u, 0) v + \frac{1}{2} \partial_v^2 t(u, 0) v^2 \]

\[ \rho(u, v) := (u + b(u)v + \frac{1}{2} e(u)v^2, a(u)v + \frac{1}{2} d(u)v^2) \]

for \( k = 0, 1, 2, \quad \partial^k \tilde{f} = \partial^k (f \circ \rho) \)
Geometric continuity, reparameterizations

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\( G^2 \) constraints between two surface pieces \( \tilde{f}, f : (u, v) \in \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) along common edge \((u, 0)\):
for \( k = 0, 1, 2 \), \( \partial^k \tilde{f} = \partial^k (f \circ \rho) \)
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\( G^1 \) constraints: \( \partial_v \tilde{f} = a \partial_v f + b \partial_u f \)

\( G^2 \) constraints: \( \partial_{vv} \tilde{f} = a^2 \partial_{vv} f + 2ab \partial_u \partial_v f + b^2 \partial_{uu} f + d \partial_v f + e \partial_u f \)
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Reasonable \( G^2 \) constructions are

- diagonally symmetric (invariant under reversal of indices)
- unbiased (invariant under relabeling)
- sectors are internally \( C^k \) (as opposed to \( G^k \))

Hypothetical bi-5 many-piece sector
Geometric continuity, reparameterizations

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Flexible $G^2$-refinement

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3. Upper bound: Bi-6 caps are flexibly $G^2$-refinable
Bi-5 splines are not flexibly $G^2$-refinable: Technical Lemmas

across $\mathbb{C} \rightarrow \mathbb{M}$: $a, b, d, e$ must be polynomial degree $a, b, d, e \leq 1, 2, 2, 3$.

$a \equiv 1$, $b \equiv d \equiv e \equiv 0$. (internal $C^2$, refinability)

$\Rightarrow \rho$ is identity

across $\mathbb{M} \rightarrow \mathbb{C}$: $b$ polynomial and $a \equiv -1$
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Lower bound: Bi-5 caps are not flexibly $G^2$-refinable: Proof

\[ \rho(u, v) = \text{identity} \]
Lower bound: Bi-5 is not flexibly $G^2$-refinable: Proof

$$\rho(m) = \text{identity} \quad \rho(u, v) = (u, -v)$$
Lower bound: Bi-5 caps are not flexibly $G^2$-refinable: Proof

\[ \rho(u, v) = \text{identity} \quad \quad \rho(u, v) = (u, -v) \]
Lower bound: **Bi-5 is not flexibly $G^2$-refinable: Proof**

\[ \rho(m, m) = \text{identity} \quad \quad \quad \quad \rho(u, v) = (u,-v) \quad \quad \quad \quad \text{conflict at eop!} \]
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3 Upper bound: Bi-6 caps are flexibly $G^2$-refinable
Least upper bound Bi-6 flexibly $G^2$-refinable surface

layout
Upper bound: Bi-6 caps are flexibly $G^2$-refinable

Least upper bound Bi-6 flexibly $G^2$-refinable surface

layout

multi-sided surface caps
$G^2$-refinability between sectors

$G^2$ bi-6 \[ \text{degree}(b(u)) = 2; \quad d(u) := 0, \quad e(u) := b(u)b'(u) \]
Conclusion

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Thank you