A comparison of classical, discrete differential and isogeometric methods at irregular points

Thien Nguyen Kęstutis Karčiauskas Jörg Peters

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Geometrically smooth (G^k surface) construction yield smooth (C^k) iso-geometric elements

Topics not covered ...

- *Box-Splines on Crystallographic Lattices* • SIAM Annual Meeting 2012 MS89-4.
- *Refinability of splines derived from regular tessellations* (hex splines are not refinable) • BIRS 2013 Algebraic and Geom. Design
- *Correct resolution rendering of trimmed spline surfaces* • same accuracy as ray casting, but much faster!
- Solving Poisson's equation with Box Splines

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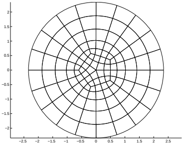
Overview



Poisson's equation on

- • classical finite elements,
  • discrete differential approach and
  • four iso-geometric constructions
- IgA: singular polar parameterization; $O(h^3)$ convergence
- IgA using C^1 functions on complex domains based on G^1 constructions; • $O(h^3)$ L^2 convergence, $O(h^2)$ L^∞ convergence.

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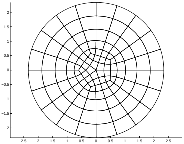

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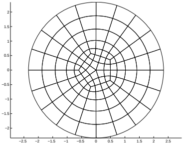

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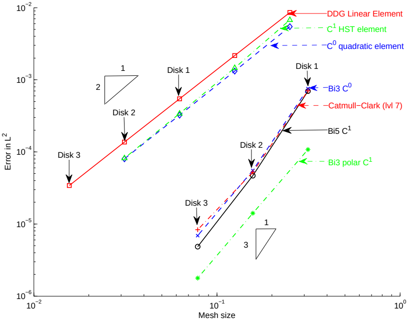
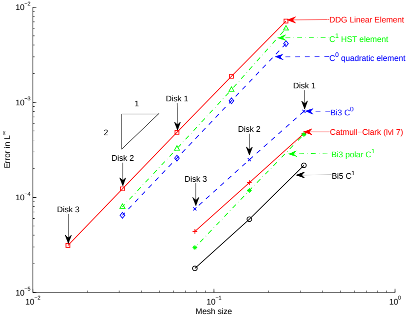


L^2 error

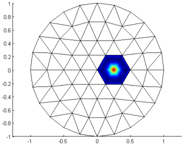


L^∞ error

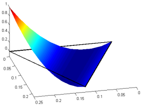
C^0 quadratic triangular elements

Linear Strain Triangle (LST) or Veubeke triangle:
$0 \le u, v \le 1, \; 0 \le u + v \le 1$

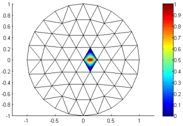
$$b^{\triangle}(u,v) := \sum_{i+j+k=2} c_{ijk} \frac{2!}{i!j!k!}(1-u-v)^i u^j v^k, \; i,j,k \in \mathbb{N}_0.$$



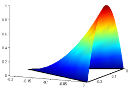
Nodal basis function



BB-piece

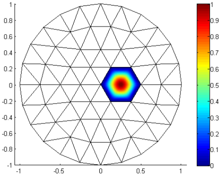


Mid-edge basis function

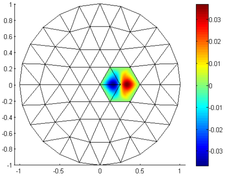


BB-piece

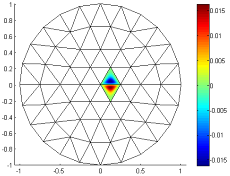
C^1 Hsieh-Clough-Tocher Elements



$b_{3i}^{\triangle}$: nodal basis function



$b_{3i+1}^{\triangle}$: x-derivative basis function

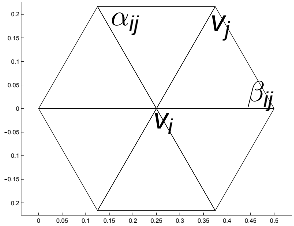


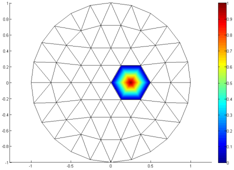
$b_{3N+k}^{\triangle}$: mid-edge normal derivative function

The discrete differential geometry approach

cotan operator [Pinkall+Polthier,Desbrun,...]

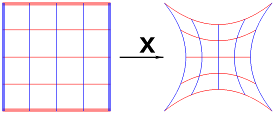
$$\Delta_M f(v_i) := \frac{3}{\mathcal{A}(v)} \sum_{j \in \mathcal{N}_1(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} [f(v_j) - f(v_i)]$$



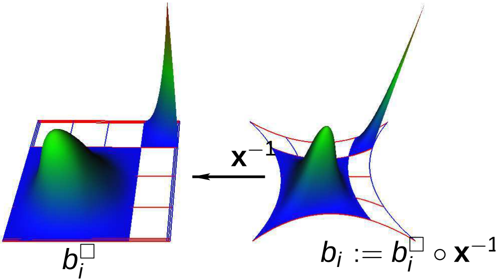


Top view of the linear 'hat' function

The Iso-geometric approach



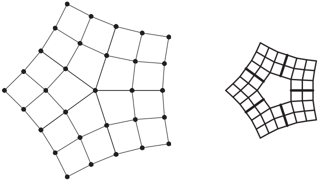
T to physical domain Ω



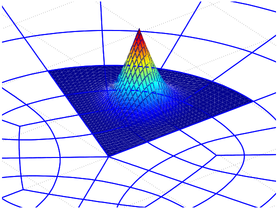
Basis function $b_i^{\square}$ on T, basis on Ω

- IgA = iso-parametric analysis using splines both to describe the domain and the approximate PDE solution.

C^0 bi-3 element

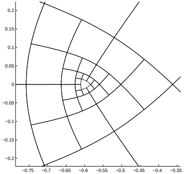


control net and C^2 extension in BB-form

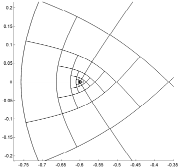


C^0 bi-3 basis function

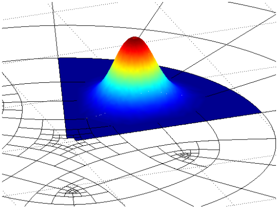
Subdivision (Catmull-Clark) elements



Level 3



Level 7

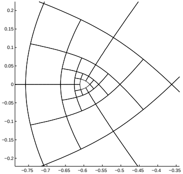


A Catmull-Clark subdivision function

[Barendrecht 2014]

Quadrature: [Halstead, Kass, DeRose 1993]

Subdivision (Catmull-Clark) elements



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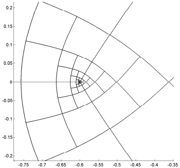

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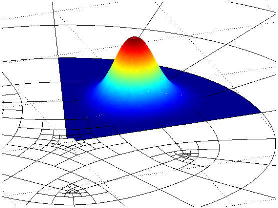

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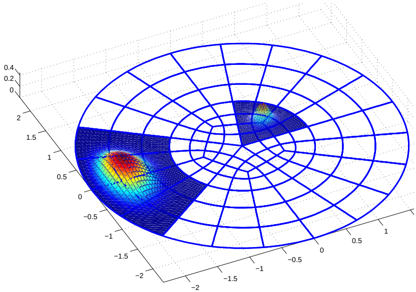
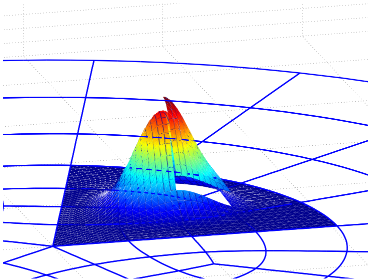
G^1 bi-3/bi-5 elements [KP 2013]



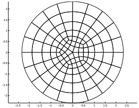
Two G^1 bi-3/bi-5 basis functions (G^2 at eop)



G^1 bi-3/bi-5 basis function — one BB-patch lifted up

Matched G-continuity yields C-continuity

Ω := physical domain parameterized piecewise by n maps



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$\to$ arXiv 1406.4229 (math.NA)

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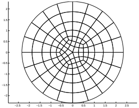

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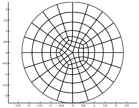

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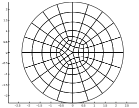

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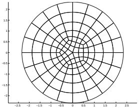

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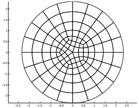

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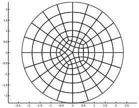

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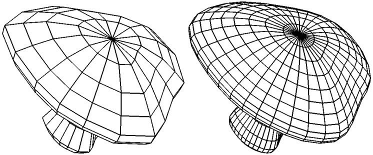
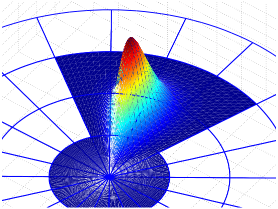
$\to$ arXiv 1406.4229 (math.NA)

Polar elements for polar configurations



Modeling with C^1 polar functions



A C^1 polar basis function

Solving Poisson's equation

Poisson's equation: find u such that

$$-\Delta u = f, \qquad u(\partial\Omega) = 0.$$

DDG: directly as $-\Delta_M u = f$.
Other methods: solve **weak form.** Find $u \in H_0^1$ such that for all $v \in H_0^1$

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\,\Omega = \int_\Omega f v \, \mathrm{d}\,\Omega.$$

We seek an approximate solution in terms of functions $b_i : \Omega \to \mathbb{R}$ by determining the coefficients $c_i \in \mathbb{R}$ in

$$u_h := \sum_1^N c_i b_i. \tag{1}$$

Solving Poisson's equation

Using **Galerkin's** method, we set $v := b_i$ and obtain the constraints

$$\int_\Omega \nabla\Big(\sum_1^N c_j b_j\Big) \cdot \nabla b_i \, \mathrm{d}\,\Omega = \int_\Omega f b_i \, \mathrm{d}\,\Omega.$$

This yields a system of linear equations

$$K\mathbf{c} = \mathbf{f}, \qquad \text{where } K_{ij} := \int_\Omega \nabla b_i \cdot \nabla b_j \, \mathrm{d}\,\Omega, \quad \text{and} \quad \mathbf{f}_i := \int_\Omega f b_i \, \mathrm{d}\,\Omega$$

The vector of coefficients $\mathbf{c} := [c_1, \cdots, c_n]^{\mathrm{t}}$ is to be determined.

Solving Poisson's equation

For IgA: physical domain $\Omega := \cup \mathbf{x}_\alpha(T)$,

$$K_{ij} = \int_{\mathbf{x}(T)} \nabla(b_i^\square \circ \mathbf{x}^{-1}) \cdot \nabla(b_j^\square \circ \mathbf{x}^{-1})\, \mathrm{d}\,\Omega = \ldots$$

$$= \int_T (\nabla b_i^\square)^t [J^{-1}]^t J^{-1} (\nabla b_j^\square) |\det J|\, \mathrm{d}\, T$$

J =transpose of Jacobian of $\mathbf{x} : (s,t) \in T \to [x(s,v)\ y(s,v)]^t$.

$$J := \begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix}, \qquad \det J = x_s y_t - x_t y_s, \qquad J^{-1} = \frac{1}{\det J} \begin{bmatrix} y_t & -y_s \\ -x_t & x_s \end{bmatrix}$$

$$[J^{-1}]^t J^{-1} |\det J| = \frac{1}{|\det J|} \begin{bmatrix} x_t^2 + y_t^2 & -x_s x_t - y_s y_t \\ -x_s x_t - y_s y_t & x_s^2 + y_s^2 \end{bmatrix}.$$

Similarly, for the right hand side term,

$$\int_\Omega f b_i\, \mathrm{d}\,\Omega = \int_T (f \circ \mathbf{x}) b_i^\square |\det J|\, \mathrm{d}\, T.$$

Solving Poisson's equation

For IgA: physical domain $\Omega := \cup \mathbf{x}_\alpha(T)$,

$$K_{ij} = \int_{\mathbf{x}(T)} \nabla(b_i^\square \circ \mathbf{x}^{-1}) \cdot \nabla(b_j^\square \circ \mathbf{x}^{-1})\, \mathrm{d}\,\Omega = \dots$$

$$= \int_T (\nabla b_i^\square)^{\mathrm{t}} [J^{-1}]^{\mathrm{t}} J^{-1} (\nabla b_j^\square) |\det J|\, \mathrm{d}\, T$$

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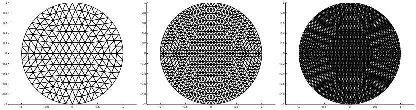
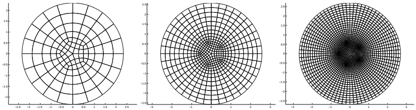
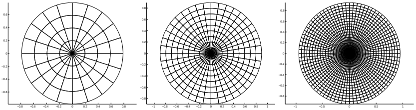
Numerical results and comparison



C^0 quadratic, HCT, DDG elements: 384, $\times 4$, $\times 16$

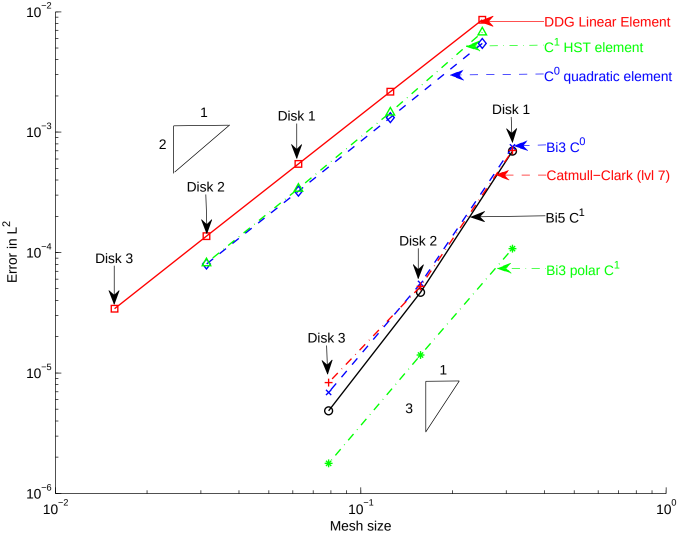


bi-3 C^0, CC, G^1 bi-3/bi-5 elements: 120, $\times 4$, $\times 16$

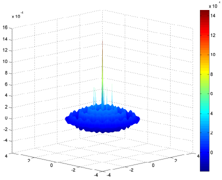


Polar C^1 elements: 100, $\times 4$, $\times 16$

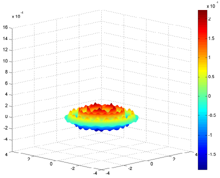
L_2 error


Error in L^2
Mesh size
10^-2
10^-3
10^-4
10^-5
10^-6
10^-1
10^0
DDG Linear Element
C^1 HST element
C^0 quadratic element
Bi3 C^0
Catmull−Clark (lvl 7)
Bi5 C^1
Bi3 polar C^1
Disk 1
Disk 2
Disk 3
Disk 1
Disk 2
Disk 3
1
2
1
3

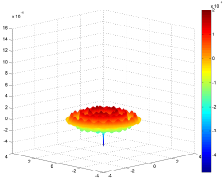
Poisson's equation: exact - computed.



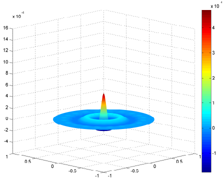
C^0, bi-3



G^1 bi-3/bi-5



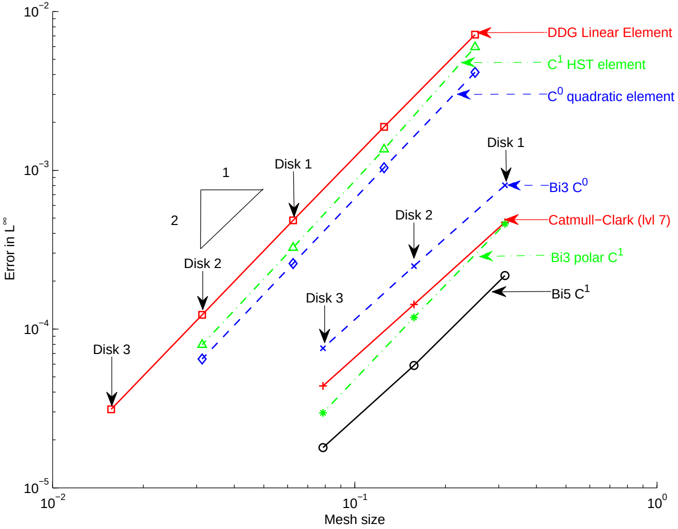
Catmull-Clark



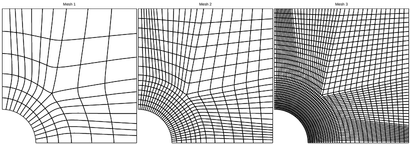
C^1 polar, bi-3

When $f := 1$ the exact solution is $u := (1 - x^2 - y^2)/4$.

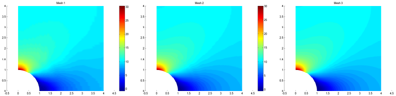
L^∞ error



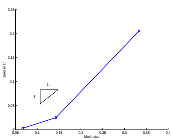
G^1 bi-3/bi-5 elements: elastic plate with circular hole



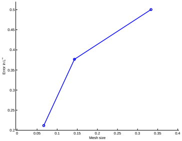
h-refinement



Contour plots of σ_{xx}



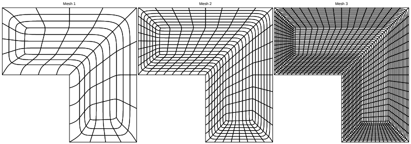
L^2-error



L^∞-error

exact solution:
$\sigma_{xx}(r,\theta) = T - T\frac{R^2}{r^2}(3/2\cos(2\theta) + \cos(4\theta)) + T\frac{3R^4}{2r^4}\cos(4\theta)$ where $r(x,y) := \sqrt{x^2+y^2}$ and $\theta(x,y) := \text{atan}(y/x)$.

G^1 bi-3/bi-5 elements: Poisson's equation on L-shape



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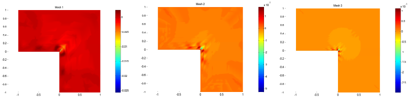
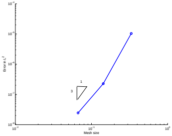
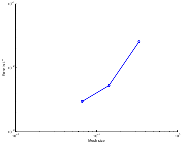


Exact - Computed.



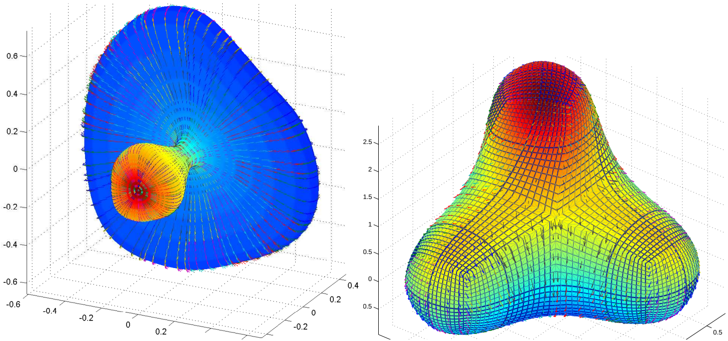
L^2-error



L^∞-error

exact solution: $u(x, y) = r^{\frac{2}{3}} \sin(2a/3 + \pi/3)$, where
$r(x, y) := \sqrt{x^2 + y^2}$ and $a(x, y) := \mathrm{atan}(x/y)$.

Heat equation on surfaces



Summary

- *G* construction yields *C* isogeometric element
- G^1 construction: $O(h^3)$ L^2 convergence, $O(h^2)$ L^∞ convergence.
- useful for surfaces, bihormonic equations
- Supported by NSF CCF 1117695

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G^k construction yields C^k isogeometric element

Faa di Bruno's law (chain and product rules) yields a complex combination of derivatives, but only terms up to kth order.

By geometric continuity, derivatives up to kth order evaluated along e_i^{-1} agree.

(e_i^{-1} is the pre-image of the edge e common to $\mathbf{x}_i(\square)$ and $\mathbf{x}_j(\square)$).

$$\begin{aligned}\partial^k(u_i \circ \mathbf{x}_i^{-1})(e) &= \partial^k(u_j \circ \rho \circ (\mathbf{x}_j \circ \rho)^{-1})(e) = \partial^k(u_j \circ \rho \circ \rho^{-1} \circ (\mathbf{x}_j)^{-1})(e) \\ &= \partial^k(u_j \circ \mathbf{x}_j^{-1})(e)\end{aligned}$$

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G^1 construction yields C^1 isogeometric element

$$\rho : e_i^{-1} \to e_j^{-1} \qquad \partial \mathbf{x}_i(e_i^{-1}) = \partial(\mathbf{x}_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial(u_j \circ \rho)(e_i^{-1})$$

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&= \partial_\perp u_j \circ x_j^{-1}(e)
\end{aligned}$$

G^1 construction yields C^1 isogeometric element

$$\rho : e_i^{-1} \to e_j^{-1} \qquad \partial \mathbf{x}_i(e_i^{-1}) = \partial(\mathbf{x}_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial(u_j \circ \rho)(e_i^{-1})$$

$$x^{-1} \circ x = id$$
$$\partial x^{-1}(x) \cdot \partial x = I \quad \Longrightarrow \quad (\partial x)^{-1} = \partial x^{-1}$$

$$\partial_\perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial_\perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_\perp x_i(e_i^{-1}))^{-1}$$
$$= \partial(u_j \circ \rho)(e_i^{-1})(\partial_\perp(x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial\rho(e_i^{-1})(\partial_\perp x_j(e_j^{-1}) \partial\rho(e_i^{-1}$$
$$= \partial u_j(e_j^{-1}) \partial\rho(e_i^{-1})(\partial\rho(e_i^{-1}))^{-1}(\partial_\perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_\perp x_j(e_j^{-1}))^{-1}$$
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$$\begin{aligned} x^{-1} \circ x &= id \\ \partial x^{-1}(x) \cdot \partial x &= I \quad \Longrightarrow \quad (\partial x)^{-1} = \partial x^{-1} \end{aligned}$$

$$\begin{aligned} &\partial_\perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial_\perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_\perp x_i(e_i^{-1}))^{-1} \\ &= \partial(u_j \circ \rho)(e_i^{-1})(\partial_\perp(x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial\rho(e_i^{-1})(\partial_\perp x_j(e_j^{-1}) \partial\rho(e_i^{-1} \\ &= \partial u_j(e_j^{-1}) \partial\rho(e_i^{-1})(\partial\rho(e_i^{-1}))^{-1}(\partial_\perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_\perp x_j(e_j^{-1}))^{-1} \\ &= \partial_\perp u_j \circ x_j^{-1}(e) \end{aligned}$$

G^1 construction yields C^1 isogeometric element

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$$\partial x^{-1}(x) \cdot \partial x = I \quad \Longrightarrow \quad (\partial x)^{-1} = \partial x^{-1}$$

$$\begin{aligned}
&\partial_\perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial_\perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_\perp x_i(e_i^{-1}))^{-1} \\
&= \partial(u_j \circ \rho)(e_i^{-1})(\partial_\perp(x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial_\perp x_j(e_j^{-1}) \partial \rho(e_i^{-1} \\
&= \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial_\perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_\perp x_j(e_j^{-1}))^{-1} \\
&= \partial_\perp u_j \circ x_j^{-1}(e)
\end{aligned}$$

L_2 error – more players