A comparison of classical, discrete differential and isogeometric methods at irregular points

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Geometrically smooth ($G^k$ surface) construction yield smooth ($C^k$) iso-geometric elements
Topics not covered ...

- **Box-Splines on Crystallographic Lattices** • SIAM Annual Meeting 2012 MS89-4.
- Refinability of splines derived from regular tessellations (hex splines are not refinable) • BIRS 2013 Algebraic and Geom. Design
- Correct resolution rendering of trimmed spline surfaces • same accuracy as ray casting, but much faster!
- Solving Poisson’s equation with Box Splines
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Poisson’s equation on

- classical finite elements,
- discrete differential approach and
- four iso-geometric constructions

- IgA: singular polar parameterization; $O(h^3)$ convergence
- IgA using $C^1$ functions on complex domains based on $G^1$ constructions; $O(h^3)$ $L^2$ convergence, $O(h^2)$ $L^\infty$ convergence.
Overview

Poisson’s equation on

- classical finite elements,
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IgA: singular polar parameterization; $O(h^3)$ convergence

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IgA using $C^1$ functions on complex domains based on $G^1$ constructions; • $O(h^3)$ $L^2$ convergence, $O(h^2)$ $L^\infty$ convergence.
$L^2$ error  $L^\infty$ error
$C^0$ quadratic triangular elements

Linear Strain Triangle (LST) or Veubeke triangle:
$0 \leq u, v \leq 1$, $0 \leq u + v \leq 1$

$$b^\Delta(u, v) := \sum_{i+j+k=2} c_{ijk} \frac{2!}{i! j! k!} (1 - u - v)^i u^j v^k, \quad i, j, k \in \mathbb{N}_0.$$
$C^1$ Hsieh-Clough-Tocher Elements

$b_{3i}^\triangle$: nodal basis function

$b_{3i+1}^\triangle$: $x$-derivative basis function

$b_{3N+k}^\triangle$: mid-edge normal derivative function
The discrete differential geometry approach

cotan operator [Pinkall+Polthier,Desbrun,...]

\[ \Delta_M f(v_i) := \frac{3}{A(v)} \sum_{j \in N_1(i)} \left( \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} \right) [f(v_j) - f(v_i)] \]
The Iso-geometric approach

- IgA = iso-parametric analysis using splines both to describe the domain and the approximate PDE solution.
The Iso-parametric (iso-geometric) approach and finite elements

$C^0$ bi-3 element

control net and $C^2$ extension in BB-form

$C^0$ bi-3 basis function
The Iso-parametric (iso-geometric) approach and finite elements

Subdivision (Catmull-Clark) elements

Level 3

Level 7

A Catmull-Clark subdivision function

[Barendrecht 2014]

Quadrature: [Halstead, Kass, DeRose 1993]
Subdivision (Catmull-Clark) elements

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A Catmull-Clark subdivision function

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The Iso-parametric (iso-geometric) approach and finite elements

$G^1$ bi-3/bi-5 elements [KP 2013]

Two $G^1$ bi-3/bi-5 basis functions ($G^2$ at eop)

$G^1$ bi-3/bi-5 basis function — one BB-patch lifted up
Matched $G$-continuity yields $C$-continuity

$\Omega := \text{physical domain parameterized piecewise by } n \text{ maps}$

$x_i : T \rightarrow \mathbb{R}^d$, $T := [0..1]^2$, $d \in \{2, 3\}$

$(s, t) \mapsto x_i(s, t) =: (x_i(s, t), y_i(s, t))$.

$x_i$ and $x_j$ join $G^k$ along $E := x_i(s, 0) = x_j(\rho(s, 0)) = x_j(0, t)$

$\partial^k x_i(s, 0) = \partial^k x_j(\rho(s, 0))$ \quad $\rho := \mathbb{R}^2 \rightarrow \mathbb{R}^2, \partial^k = k - \text{jet}$

If also $\partial^1 u_i = \partial^1 (u_j \circ \rho)$ then $C^1$ continuity of $u \circ x^{-1}$ across $E \partial_\perp \cdot (e)$:

$\partial_\perp (u_i \circ x_j^{-1}) = \partial u_i \partial_\perp x_j^{-1} = \partial u_j \partial \rho (\partial \rho)^{-1} \partial_\perp x_j^{-1} = \partial_\perp (u_j \circ x_j^{-1})$.

$\implies$ Every $G$ construction yields a $C$ iso-geometric construction.

\text{arXiv 1406.4229 (math.NA)}
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\Rightarrow \quad \text{Every } G \text{ construction yields a } C \text{ iso-geometric construction.}
\Rightarrow \quad \text{arXiv 1406.4229 (math.NA)}$$
The Iso-parametric (iso-geometric) approach and finite elements

Matched $G$-continuity yields $C$-continuity

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Polar elements for polar configurations

Modeling with $C^1$ polar functions

A $C^1$ polar basis function
Solving Poisson’s equation

Poisson’s equation: find $u$ such that

$$-\Delta u = f, \quad u(\partial \Omega) = 0.$$ 

DDG: directly as $-\Delta_M u = f$.

Other methods: solve weak form. Find $u \in H^1_0$ such that for all $v \in H^1_0$

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} fv \, d\Omega.$$

We seek an approximate solution in terms of functions $b_i : \Omega \rightarrow \mathbb{R}$ by determining the coefficients $c_i \in \mathbb{R}$ in

$$u_h := \sum_{1}^{N} c_i b_i.$$  \hspace{1cm} (1)
Solving Poisson’s equation

Using Galerkin’s method, we set \( v := b_i \) and obtain the constraints

\[
\int_{\Omega} \nabla \left( \sum_{j=1}^{N} c_j b_j \right) \cdot \nabla b_i \, d\Omega = \int_{\Omega} f b_i \, d\Omega.
\]

This yields a system of linear equations

\[
K c = f, \quad \text{where } K_{ij} := \int_{\Omega} \nabla b_i \cdot \nabla b_j \, d\Omega, \quad \text{and} \quad f_i := \int_{\Omega} f b_i \, d\Omega
\]

The vector of coefficients \( c := [c_1, \cdots, c_n]^t \) is to be determined.
Solving Poisson’s equation

For IgA: physical domain $\Omega := \cup x(\alpha, T)$,

$$K_{ij} = \int_{x(T)} \nabla (b_i \Box \circ x^{-1}) \cdot \nabla (b_j \Box \circ x^{-1}) \, d\Omega = \ldots$$

$$= \int_T (\nabla b_i \Box)^t [J^{-1}]^t J^{-1} (\nabla b_j \Box) | \det J | \, d\, T$$

$J = \text{transpose of Jacobian of } x : (s, t) \in T \rightarrow [x(s, v) \ y(s, v)]^t$.

$$J := \begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix}, \quad \det J = x_s y_t - x_t y_s, \quad J^{-1} = \frac{1}{\det J} \begin{bmatrix} y_t & -y_s \\ -x_t & x_s \end{bmatrix}$$

$$[J^{-1}]^t J^{-1} | \det J | = \frac{1}{|\det J|} \begin{bmatrix} x_t^2 + y_t^2 & -x_s x_t - y_s y_t \\ -x_s x_t - y_s y_t & x_s^2 + y_s^2 \end{bmatrix}.$$  

Similarly, for the right hand side term,

$$\int_{\Omega} f b_i \, d\Omega = \int_T (f \circ x) b_i \Box | \det J | \, d\, T.$$
Solving Poisson’s equation

For IgA: physical domain \( \Omega := \bigcup x_{\alpha}(T) \),

\[
K_{ij} = \int_{x(T)} \nabla(b_i \circ x^{-1}) \cdot \nabla(b_j \circ x^{-1}) \, d\Omega = \ldots
\]

\[
= \int_T (\nabla b_i)\,^t \left[J^{-1}\right]^t J^{-1} (\nabla b_j) \mid \det J \mid d\, T
\]

\( J^\top \) = transpose of Jacobian of \( x: (s, t) \in T \rightarrow [x(s, v) \, y(s, v)]^\top. \)

\[
J := \begin{bmatrix}
x_s & y_s \\
x_t & y_t
\end{bmatrix}, \quad \det J = x_s y_t - x_t y_s, \quad J^{-1} = \frac{1}{\det J} \begin{bmatrix}
y_t & -y_s \\
-x_t & x_s
\end{bmatrix}
\]

\[
\left[J^{-1}\right]^t J^{-1} \mid \det J\mid = \frac{1}{\mid \det J\mid} \begin{bmatrix}
x_t^2 + y_t^2 & -x_s x_t - y_s y_t \\
-x_s x_t - y_s y_t & x_s^2 + y_s^2
\end{bmatrix}.
\]

Similarly, for the right hand side term,

\[
\int_{\Omega} f b_i \, d\Omega = \int_T (f \circ x) b_i \, \mid \det J\mid \, d\, T.
\]
Numerical results and comparison

$C^0$ quadratic, HCT, DDG elements: 384, $\times 4$, $\times 16$

bi-3 $C^0$, CC, $G^1$ bi-3/bi-5 elements: 120, $\times 4$, $\times 16$

Polar $C^1$ elements: 100, $\times 4$, $\times 16$
Numerical results and comparison

$L_2$ error

![Graph showing mesh size vs. error in $L_2$ for different elements and datasets.]

- DDG Linear Element
- $C^1$ HST element
- $C^0$ quadratic element
- Catmull–Clark (lvl 7)
- Bi3 $C^0$
- Bi5 $C^1$
- Bi3 polar $C^1$

T Nguyen, K. Karčiauskas, J. Peters (UF, VU) HCT, DDG, IgA irregular $O(h^3)$ convergence
Poisson’s equation: exact - computed.

When \( f := 1 \) the exact solution is \( u := \frac{(1 - x^2 - y^2)}{4} \).
Numerical results and comparison

$L^\infty$ error

![Graph showing $L^\infty$ error vs mesh size for different methods: Catmull–Clark (lvl 7), Bi3 polar $C^1$, Bi5 $C^1$, Bi3 C$^0$, quadratic element, DDG Linear Element, C$^1$ HST element, C$^0$ quadratic element. The graph illustrates the convergence behavior for three disks labeled as Disk 1, Disk 2, and Disk 3.](image-url)
$G^1$ bi-3/bi-5 elements: elastic plate with circular hole

$h$-refinement

Contour plots of $\sigma_{xx}$

$L^2$-error  

$L^\infty$-error

exact solution:

$$\sigma_{xx}(r, \theta) = T - \frac{T}{r^2} \left( \frac{3}{2} \cos(2\theta) + \cos(4\theta) \right) + \frac{T}{2r^4} \frac{3R^4}{2r^4} \cos(4\theta)$$

where

$$r(x, y) := \sqrt{x^2 + y^2}$$

and

$$\theta(x, y) := \text{atan}(y/x).$$
$G^1$ bi-3/bi-5 elements: Poisson’s equation on L-shape

$h$-refinement

Exact - Computed.

$L^2$-error

$L^\infty$-error

exact solution: $u(x, y) = r^{2/3} \sin(2a/3 + \pi/3)$, where $r(x, y) := \sqrt{x^2 + y^2}$ and $a(x, y) := \text{atan}(x/y)$. 

T Nguyen, K. Karčiauskas, J. Peters (UF, VU, HCT, DDG, IgA irregular $O(h^3)$ convergence
Heat equation on surfaces

(T Nguyen, K. Karčiauskas, J. Peters)
Summary

- $G$ construction yields $C$ isogeometric element
- $G^1$ construction: $O(h^3)$ $L^2$ convergence, $O(h^2)$ $L^\infty$ convergence.
- useful for surfaces, biharmonic equations
- Supported by NSF CCF 1117695
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$G^k$ construction yields $C^k$ isogeometric element

Faa di Bruno’s law (chain and product rules) yields a complex combination of derivatives, but only terms up to $k$th order. By geometric continuity, derivatives up to $k$th order evaluated along $e_i^{-1}$ agree. ($e_i^{-1}$ is the pre-image of the edge $e$ common to $x_i(□)$ and $x_j(□)$).

$$\partial^k (u_i \circ x_i^{-1})(e) = \partial^k (u_j \circ \rho \circ (x_j \circ \rho)^{-1})(e) = \partial^k (u_j \circ \rho \circ \rho^{-1} \circ (x_j)^{-1})(e) = \partial^k (u_j \circ x_j^{-1})(e)$$
$G^k$ construction yields $C^k$ isogeometric element

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\[
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$G^k$ construction yields $C^k$ isogeometric element

Faa di Bruno’s law (chain and product rules) yields a complex combination of derivatives, but only terms up to $k$th order.
By geometric continuity, derivatives up to $k$th order evaluated along $e_i^{-1}$ agree.
$(e_i^{-1}$ is the pre-image of the edge $e$ common to $x_i(□)$ and $x_j(□)$).

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\partial^k(u_i \circ x_i^{-1})(e) = \partial^k(u_j \circ \rho \circ (x_j \circ \rho)^{-1})(e) = \partial^k(u_j \circ \rho \circ \rho^{-1} \circ (x_j)^{-1})(e) = \partial^k(u_j \circ x_j^{-1})(e)
\]
\[
G^1 \text{ construction yields } C^1 \text{ isogeometric element}
\]

\[
\rho : e_i^{-1} \rightarrow e_j^{-1} \quad \partial x_i(e_i^{-1}) = \partial (x_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial (u_j \circ \rho)(e_i^{-1})
\]

\[
x^{-1} \circ x = id
\]

\[
\partial x^{-1}(x) \cdot \partial x = I \quad \implies \quad (\partial x)^{-1} = \partial x^{-1}
\]

\[
\partial \perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial \perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial \perp x_i(e_i^{-1}))^{-1}
\]

\[
= \partial (u_j \circ \rho)(e_i^{-1})(\partial \perp (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \perp x_j(e_j^{-1}))^{-1} \partial \rho(e_i^{-1})
\]

\[
= \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial \perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial \perp x_j(e_j^{-1}))^{-1}
\]

\[
= \partial \perp u_j \circ x_j^{-1}(e)
\]
$G^1$ construction yields $C^1$ isogeometric element

$$
\rho : e_i^{-1} \to e_j^{-1} \quad \partial x_i(e_i^{-1}) = \partial(x_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial(u_j \circ \rho)(e_i^{-1})
$$

$$
x^{-1} \circ x = id
$$

$$
\partial x^{-1}(x) \cdot \partial x = I \quad \implies \quad (\partial x)^{-1} = \partial x^{-1}
$$

$$
\partial \perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1})\partial \perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial \perp x_i(e_i^{-1}))^{-1} \\
= \partial(u_j \circ \rho)(e_i^{-1})(\partial \perp (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial \perp x_j(e_j^{-1}))\partial \rho(e_i^{-1}) \\
= \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial \perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial \perp x_j(e_j^{-1}))^{-1} \\
= \partial \perp u_j \circ x_j^{-1}(e)
$$
$G^1$ construction yields $C^1$ isogeometric element

$$\rho : e_i^{-1} \rightarrow e_j^{-1} \quad \partial x_i(e_i^{-1}) = \partial(x_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial(u_j \circ \rho)(e_i^{-1})$$

$$x^{-1} \circ x = id \quad \partial x^{-1}(x) \cdot \partial x = I \quad \Rightarrow \quad (\partial x)^{-1} = \partial x^{-1}$$

$$\partial \perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial \perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial \perp x_i(e_i^{-1}))^{-1}$$

$$= \partial(u_j \circ \rho)(e_i^{-1})(\partial \perp (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \perp x_j(e_j^{-1}))^{-1} \partial \rho(e_i^{-1})$$

$$= \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial \perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial \perp x_j(e_j^{-1}))^{-1}$$

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\[
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\partial_{\perp} u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1})\partial_{\perp} x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_{\perp} x_i(e_i^{-1}))^{-1}
\]

\[
= \partial(u_j \circ \rho)(e_i^{-1})(\partial_{\perp} (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial_{\perp} x_j(e_j^{-1})\partial \rho(e_i^{-1})^{-1}
\]

\[
= \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1}
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$$= \partial_{\perp} u_j \circ x_j^{-1}(e)$$
\( G^1 \) construction yields \( C^1 \) isogeometric element

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\[ x^{-1} \circ x = id \]
\[ \partial x^{-1}(x) \cdot \partial x = I \quad \implies (\partial x)^{-1} = \partial x^{-1} \]

\[ \partial_{\perp} u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1})\partial_{\perp} x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_{\perp} x_i(e_i^{-1}))^{-1} \]
\[ = \partial (u_j \circ \rho)(e_i^{-1})(\partial_{\perp} (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1} \]
\[ = \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1} \]
\[ = \partial_{\perp} u_j \circ x_j^{-1}(e) \]
$G^1$ construction yields $C^1$ isogeometric element

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\[ \partial x_i(e_i^{-1}) = \partial (x_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial (u_j \circ \rho)(e_i^{-1}) \]

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\[ \partial_\perp u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1}) \partial_\perp x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_\perp x_i(e_i^{-1}))^{-1} \]

\[ = \partial (u_j \circ \rho)(e_i^{-1})(\partial_\perp (x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1}) \partial_\perp \rho(e_i^{-1})(\partial_\perp x_j(e_j^{-1}))^{-1} \]

\[ = \partial u_j(e_j^{-1}) \partial_\perp \rho(e_i^{-1})(\partial_\rho(e_i^{-1}))^{-1}(\partial_\perp x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_\perp x_j(e_j^{-1}))^{-1} \]

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$\rho : e_i^{-1} \rightarrow e_j^{-1}$

$\partial x_i(e_i^{-1}) = \partial(x_j \circ \rho)(e_i^{-1})$

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$\partial x^{-1}(x) \cdot \partial x = I \quad \implies (\partial x)^{-1} = \partial x^{-1}$

$\partial_{\perp} u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1})\partial_{\perp} x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_{\perp} x_i(e_i^{-1}))^{-1}$

$= \partial(u_j \circ \rho)(e_i^{-1})(\partial_{\perp}(x_j \circ \rho)(e_i^{-1}))^{-1} = \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial_{\perp} x_j(e_j^{-1}))\partial \rho(e_i^{-1})$

$= \partial u_j(e_j^{-1})\partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1}(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1}$

$= \partial_{\perp} u_j \circ x_j^{-1}(e)$
$G^1$ construction yields $C^1$ isogeometric element

$$\rho : e_i^{-1} \rightarrow e_j^{-1} \quad \partial x_i(e_i^{-1}) = \partial(x_j \circ \rho)(e_i^{-1}) \quad \partial u_i(e_i^{-1}) = \partial(u_j \circ \rho)(e_i^{-1})$$

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$$\partial_{\perp} u_i \circ x_i^{-1}(e) = \partial u_i(e_i^{-1})\partial_{\perp} x_i^{-1}(e) = \partial u_i(e_i^{-1})(\partial_{\perp} x_i(e_i^{-1}))^{-1} = \partial(u_j \circ \rho)(e_i^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})\partial \rho(e_j^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial u_j(e_j^{-1})(\partial_{\perp} x_j(e_j^{-1}))^{-1} = \partial_{\perp} u_j \circ x_j^{-1}(e)$$
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\[ = \partial_\perp u_j \circ x_j^{-1}(e) \]
$L_2$ error – more players

![Graph showing error in $L_2$ for different mesh sizes and methods.](image)