On G^1 stitched bi-cubic Bézier patches with arbitrary topology

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Abstract

Lower bounds, mandating a minimal number and degree of polynomial pieces, represent a major achievement in the theory of geometrically smooth (G^1) constructions. On one hand, they establish a floor when searching for optimal constructions, on the other they can be used to flag complex constructions for potential flaws. In particular, quadrilateral meshes of arbitrary topology can not in general be converted to G^1-connected Bézier patches of bi-degree 3 with one piece per quad or use just linear reparameterizations. This note illustrates how lower bounds indicate otherwise difficult-to-find flaws in a complex new surface construction.

1 Introduction

For many applications, for example artistic rendering and sculpting, a few steps of refinement and averaging provide a pleasing rounding of the original polyhedral shape. The simplicity of subdivision, in particular, when it has small and local stencils (refinement rules) is appealing and Catmull-Clark subdivision [CC78] in particular is a staple of geometric modeling environments when creating computer graphics assets. However Catmull-Clark surfaces have also been shown to inherently have shape deficiencies, such as pinching of highlight lines, that can be traced back to its simple stencil-based rules [KPR04, KP17].

The algorithm of [ASC17] proposes an approach to obtaining 'C^2 continuous Bi-Cubic Bézier patches that are guaranteed to be stitched with G^1 continuity regardless of the underlying mesh topology'. This approach consists of applying not Catmull-Clark but *Doo-Sabin subdivision* to an initial polyhedral input mesh. The approach then derives quadrilateral facets and Bézier control points from the refined mesh and constructs n bi-cubic patches for each n-sided facet.

Beyond demonstrating aesthetic rounding, [ASC17] emphasizes that the result is a 'smooth surface with G^1 continuity' [1]. If true, this would be remarkable since this contradicts or circumvents the restrictions on bi-cubic G^1 spline complexes that were derived in [PF09, Section 3]. Moreover, if [ASC17] were correct then the special

[1] G^1 is typeset as G_1 in several places in [ASC17].

constructions [PS15, SP16], that were published earlier in the same conference series, would be superfluous.

Below we show that, while the surfaces generated by the approach of [ASC17] often appear to be smooth, in general, as predicted by the lower bounds on the required number and degree of the polynomial pieces, they are not smooth.

Overview. Section 2 summarizes the algorithm in [ASC17] and the lower bound result of [PF09] as it applies to bi-cubic G^1 constructions. Section 3 exhibits an explicit, minimal counterexample to the claim that the approach in [ASC17] always generates G^1 surfaces. Section 4 succinctly surveys the state of the art when constructing both formally smooth and near-smooth bi-cubic surfaces.

2 A quick review of G^1 continuity, the construction of [ASC17] and a lower bound theorem

The construction of [ASC17] applies two steps[2] of Doo-Sabin subdivision to an initial polyhedral input mesh $\mathcal{M}$ and then places the corners of bicubic patches at the Doo-Sabin limit points of the facets obtained in the initial subdivision (Fig 5 of [ASC17]). That is, every vertex and every face of $\mathcal{M}$ has a corner of a bi-cubic patch associated with it. This layout looks more general, and therefore more challenging than the one in [HBC08] which used 2×2 bi-cubics to per quadrilateral face of the input, but could not guarantee G^1 continuity, since it violated the lower bound on bi-cubic surfaces.

For the construction of [ASC17], denote by $\mathbf{v}$ and $\mathbf{w}$ the limit points associated with adjacent facets of $\mathcal{M}$ (see Fig. 1). Since $\mathcal{M}$ is unrestricted, $\mathbf{v}$ and $\mathbf{w}$ and their tangent planes can be freely adjusted – as is desirable for flexible modeling. The construction in [ASC17] is therefore G^1 *vertex-localized* in the sense that the Taylor expansion at $\mathbf{v}$ is not tightly linked to that at $\mathbf{w}$. It also does not matter (and it should not) whether $\mathbf{v}$ is listed first or $\mathbf{w}$. That is, the construction uses *unbiased* G^1 *constraints* along the boundary $\mathbf{p}(u,0) = \mathbf{q}(u,0)$ between the two patches $\mathbf{p}, \mathbf{q} : (u,v) \to \mathbb{R}^3$: for $u \in [0..1]$,

$$\partial_2\mathbf{p}(u,0) + \partial_2\mathbf{q}(u,0) = \alpha(u)\partial_1\mathbf{p}(u,0). \qquad (1)$$

Therefore, as indicated in Fig 8 of [ASC17], the four bicubic patches meeting at the midpoint $\mathbf{m}$ of the edge $\mathbf{v}, \mathbf{w}$ (see Fig. 1) join C^1, the following theorem applies.

Theorem 1 ([PF09]: lower bound = two double knots per edge needed)
A vertex-localized unbiased G^1 bi-cubic spline surface construction without forced linear boundary segments, requires in general at least two internal double knots per edge.

In other words, Theorem 1 states that to satisfy G^1 constraints along $\mathbf{v}$ and $\mathbf{w}$ (and not have straight line segments embedded in the surface) unrestricted input data require at least three polynomial boundary segments. [ASC17] connects $\mathbf{v}$ and $\mathbf{w}$ with two polynomial boundary segments. One might hope that the specific initialization via

[2] In [ASC17] there is some ambiguity as to whether two or three steps of Doo-Sabin subdivision should be applied, but for this note the outcome is the same.

Doo-Sabin or more refinement steps might side-step the conditions for Theorem 1 to apply. But even then a more subtle constraint (Lemma 4 of [PF09] underlying the proof of Theorem 1) holds: that α needs to be quadratic somewhere. The next section therefore looks more closely at the construction of [ASC17] expecting to find an error in the claim of G^1 continuity.

Below the bicubic tensor-product polynomial surface patches $\mathbf{p}$, $\mathbf{q}$ of bi-degree 3 are expressed in Bernstein-Bézier (BB) form, e.g.

$$\mathbf{p}(u,v) := \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{p}_{ij} B_i^3(u) B_j^3(v), \quad (u,v) \in \square := [0..1]^2,$$

where $B_k^3(t) := \binom{3}{k}(1-t)^{3-k}t^k$ is the kth Bernstein-Bézier (BB) polynomials of degree 3 and $\mathbf{p}_{ij} \in \mathbb{R}^3$ are the BB-coefficients [Far02, PBP02].

3 A Counterexample: an input mesh where [ASC17] does not yield a G^1 output

Since the algorithm of [ASC17] applies initially multiple steps of Doo-Sabin subdivision, finding a simple explicit counterexample seems a formidable challenge. The refinement means that visual inspection does not easily reveal flaws and that any flaws that one observes under zoom could be due to rounding. However, the lower bound encourages a detailed search and analysis.

Reducing the input data to the simplest configuration, a regular tetrahedron $\mathcal{M}$ with vertices

$$A := \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad B := \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad D := \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \tag{2}$$

turned out to prove that the construction of [ASC17] as stated can not, in general, generate G^1 surfaces. This was not the first configuration tested and is surprising since this input is combinatorially symmetric and admits a bi-cubic solution with the layout – just not the one of [ASC17].)

Consider Fig. 1. Let $\mathbf{m}$ be the intersection point of the curves connecting the limit points associated with A and B and the curves connecting $\mathbf{v}$, the center of the face B, A, D, to the center of A, C, D. We analyze G^1 continuity along the edge from $\mathbf{v}$ to $\mathbf{m}$. To compute with integers throughout, we scale $\mathcal{M}$ by $2^2 3^2 \cdot 5 \cdot 7$. Following the algorithm of [ASC17] up to the claim 'Our calculation of the control points guarantees G_1 continuity', the mesh points and BB-coefficients can then be computed as integers. Three rows of BB-coefficients determine the G^1 continuity constraints (1) between the resulting two adjacent bi-cubic patches $\mathbf{p}$ and $\mathbf{q}$. We focus on on the BB-coefficients of $\mathbf{p}_{ij}$ and $\mathbf{q}_{ij}$ for $i = 0,1,2,3$ and $j = 0,1$. Below $\sim$ indicates proportionality after scaling the BB-coefficients to the right of $\sim$ to the smallest integer values:

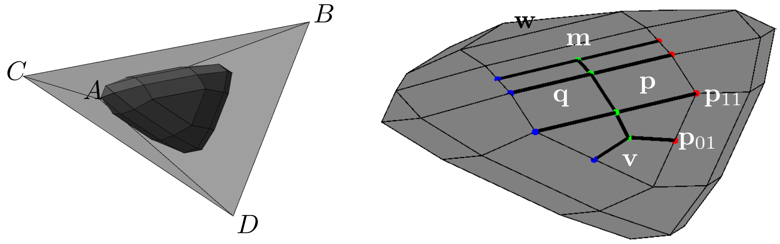

Figure 1: Counterexample: (*left*) the input mesh $\mathcal{M}$ is a regular tetrahedron. The darker quad-mesh inside is the result of applying two steps of Doo-Sabin subdivision. (*right*) The subnet of 12 Bernstein-Bézier control points of interest are sketched on the refined mesh: from the 3-valent point $\mathbf{v}$ to the 4-valent point $\mathbf{m}$, these are the BB-coefficients of (3) that determine the G^1 continuity between the two bi-cubic patches $\mathbf{p}$ and $\mathbf{q}$.

(after multiplication by 210)

$$\begin{matrix} \mathbf{p}_{i1} \\ \mathbf{p}_{i0} = \mathbf{q}_{i,3} \sim \\ \mathbf{q}_{i2} \end{matrix} \quad \begin{matrix} \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}, \\ \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}, \\ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix}. \end{matrix} \tag{3}$$

Then the coefficients of the derivatives across and along the common edge are (after multiplication by 630)

$$\begin{matrix} \partial_2 \mathbf{p} \\ \partial_1 \mathbf{p} = \partial_1 \mathbf{q} \sim \\ \partial_2 \mathbf{q} \end{matrix} \quad \begin{matrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \end{matrix} \tag{4}$$

Taking the dot-product of (1) with $\partial_1 \mathbf{p}(u,0) \times \partial_2 \mathbf{q}(u,0)$ implies that $\det|\partial_2 \mathbf{p}(u,0), \partial_1 \mathbf{p}(u,0), \partial_2 \mathbf{q}(u,0)| = 0$. However, for the counterexample, the scaled BB-coefficients of the determinant polynomial of degree $3+2+3$ are

$$\det|\partial_2 \mathbf{p}, \partial_1 \mathbf{p}, \partial_2 \mathbf{q}| \sim [0, -70, -120, 30, -4, -5, 0, 0, 0] \neq 0. \tag{5}$$

Therefore the two patches do not join with G^1 continuity.

4 Synopsis of alternative bi-cubic constructions in the literature

To place lower bounds on the number and degree of piecewise polynomial constructions into context, we list a number of attempts to generalize bi-cubic splines to irregular layouts. Starting with Malcolm Sabin's technical report 50 years ago, publications include [Sab68], [Bez77], [Bee86], [CC78], [vW86], [Sar87], [GZ94], [Pet91], [Pet94] to list just a few. While some of these constructions achieve G^1 continuity, others, as predicted by the lower bounds, only work in geometrically restricted settings. The construction of [FP08] uses 3×3 bi-cubic patches per quad and achieves the lower bound determined by Theorem 1. [FP08] covers multi-sided caps included in bi-cubic T-spline constructions. [Pet00] requires more pieces and can have poor shape already because it caps Catmull-Clark subdivision meshes. [SP16] focuses on how to restrict input meshes to ensure that G^1 bi-cubic surfaces can be built with fewer pieces.

Car styling and many other high-end outer surfaces primarily demand a good distribution of highlight lines, in addition to smoothness. However aesthetic requirements are at present mathematically ill-defined and certainly highly non-linear. Currently the most effective approach to obtaining bi-cubic surfaces with a good highlight line distribution is to employ a guide shape (of higher polynomial degree). For example, using a guide improves the shape of bi-cubic singularly parameterized surfaces [KP16a]. Since the formal proof that a surface construction generates 'fair' surfaces is not mathematically well-posed, currently the best approach is to assess new algorithms by testing them against an obstacle course [KP] of challenging but not unreasonable local input meshes.

Exploring the world of 'approximate smoothness', the paper "Can bi-cubic surfaces be class A?" [KP15a] emphasizes the distinction between exact G^1 continuity and acceptable shape in terms of curvature distribution and highlight lines. This distinction, accompanied by mathematical estimates of the jump in normals, could also be useful in the context of [ASC17].

5 Conclusion

A number of finite bi-cubic surface G^1 constructions exist in the literature. However few, currently only one, ensure good shape for a basic obstacle course of input meshes. Testing new algorithms on the obstacle course is necessary at this time of writing since the formal characterization of 'good shape' has remained illusive and may differ in mathematics *vs.* practice, as well as among design stylists.

There are several constructions of degree higher than bi-cubic that satisfy the lower bounds, use few patches and pass the obstacle course (e.g. [KP15b, KP16b]). These constructions should be considered when least polynomial degree is not crucial. The approach of [ASC17] rounds shapes but cannot guarantee G^1 continuity.

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References

[ASC17] Ergun Akleman, Vinod Srinivasan, and Jianer Chen. Interactive modeling of smooth manifold meshes with arbitrary topology: G^1 stitched bi-cubic bzier patches. *Computer & Graphics, Special Issue on SMI 2017*, 66:64–73, 2017.

[Bee86] E. Beeker. Smoothing of shapes designed with free-form surfaces. *Computer-Aided Design*, 18(4):224–232, May 1986.

[Bez77] Pierre E. Bezier. *Essai de definition numerique des courbes et des surfaces experimentales*. Ph.d. thesis, Universite Pierre et Marie Curie, February 1977.

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10:350–355, 1978.

[Far02] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Academic Press, San Diego, 2002.

[FP08] Jianhua Fan and Jörg Peters. On smooth bicubic surfaces from quad meshes. In G. Bebis et al., editor, *ISVC (1)*, volume 5358 of *Lecture Notes in Computer Science*, pages 87–96. Springer, 2008.

[GZ94] John A. Gregory and Jianwei Zhou. Filling polygonal holes with bicubic patches. *Computer Aided Geometric Design*, 11(4):391–410, 1994.

[HBC08] Stefanie Hahmann, Georges-Pierre Bonneau, and Baptiste Caramiaux. Bicubic G1 interpolation of irregular quad meshes using a 4-split. In Falai Chen and Bert Jüttler, editors, *Advances in Geometric Modeling and Processing, 5th International Conference, GMP 2008, Hangzhou, China, April 23-25, 2008. Proceedings*, volume 4975 of *Lecture Notes in Computer Science*, pages 17–32. Springer, 2008.

[KP] Kęstutis Karčiauskas and Jörg Peters. Quad-net obstacle course. `http://www.cise.ufl.edu/research/SurfLab/shape_gallery.shtml`. Accessed: 2017-09-05.

[KP15a] Kęstutis Karčiauskas and Jörg Peters. Can bi-cubic surfaces be class A? In *SGP 2015, Symposium Geometry Processing 2015, Graz*, July 6-8 2015.

[KP15b] Kęstutis Karčiauskas and Jörg Peters. Improved shape for multi-surface blends. *Graphical Models*, 8:87–98, 2 2015.

[KP16a] Kęstutis Karčiauskas and Jörg Peters. Curvature continuous bi-4 constructions for scaffold- and sphere-like surfaces. *Computer Aided Design (SPM 2016)*, 78:48–59, June 17 2016.

[KP16b] Kęstutis Karčiauskas and Jörg Peters. Minimal bi-6 G^2 completion of bicubic spline surfaces. *Computer Aided Geometric Design*, 41:10–22, Jan 2016.

[KP17] Kęstutis Karčiauskas and Jörg Peters. Improved shape for refinable surfaces with singularly parameterized irregularities. *Computer Aided Design*, 90:191–198, Sept 2017.

[KPR04] K. Karciauskas, J. Peters, and U. Reif. Shape characterization of subdivision surfaces – case studies. *Computer-Aided Geometric Design*, 21(6):601–614, july 2004.

[PBP02] Hartmut Prautzsch, Wolfgang Boehm, and Marco Paluszny. *Bézier and B-spline techniques*. Springer Verlag, 2002.

[Pet91] J. Peters. Smooth interpolation of a mesh of curves. *Constructive Approximation*, 7:221–247, 1991.

[Pet94] J. Peters. Surfaces of arbitrary topology constructed from biquadratics and bicubics. In Nickolas S. Sapidis, editor, *Designing fair curves and surfaces: shape quality in geometric modeling and computer-aided design*, pages 277–293. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1994.

[Pet00] J. Peters. Patching Catmull-Clark meshes. In Kurt Akeley, editor, *Siggraph 2000, Computer Graphics Proceedings*, Annual Conference Series, pages 255–258. ACM Press / ACM SIGGRAPH / Addison Wesley Longman, 2000.

[PF09] J. Peters and Jianhua Fan. On the complexity of smooth spline surfaces from quad meshes. *Computer-Aided Geometric Design*, 27:96–105, 2009.

[PS15] Jörg Peters and Martin Sarov. Polynomial spline surfaces with rational linear transitions. *Computers & Graphics*, 51:43–51, Oct 2015.

[Sab68] M. Sabin. Conditions for continuity of surface normals between adjacent parametric surfaces. Technical Report VTO/MS/151, British Aircraft Corporation, 1968.

[Sar87] R. Sarraga. G^1 interpolation of generally unrestricted cubic Bézier curves. *Computer Aided Geometric Design*, 4(1-2):23–40, 1987.

[SP16] Martin Sarov and Jörg Peters. Refinable polycube G-splines. *Computers & Graphics (SMI 2016)*, 58:92–101, August 2016.

[vW86] J. van Wijk. Bicubic patches for approximating non-rectangular control-point meshes. *Computer Aided Geometric Design*, 3(1):1–13, 1986.