Refinable Polycube G-Splines

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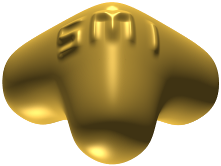
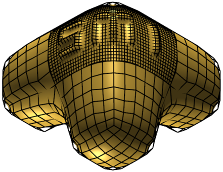
NSF CCF-0728797





Background

Quest

Surfaces with rational linear reparameterization

=

'as close as possible' to regular splines

Geometric Continuity

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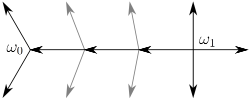

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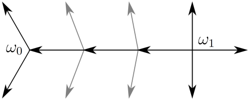

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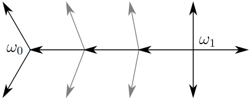

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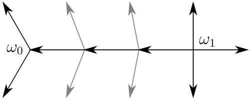

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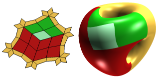


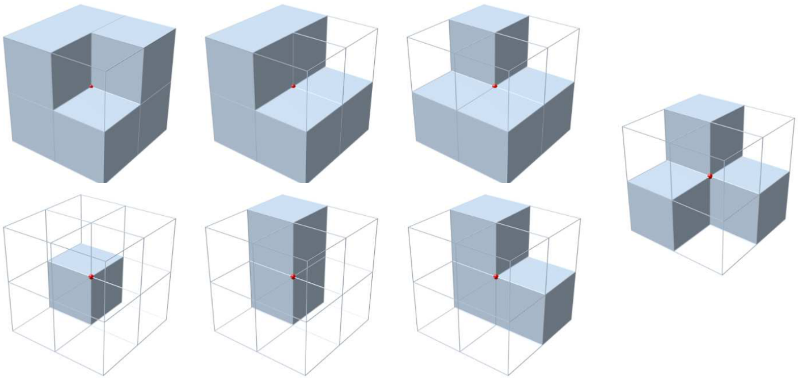
- [PS15]: valence restricted to 3, 6

Outline

1. Polycube connectivity and control handles
2. Surface Construction
3. Multi-resolution

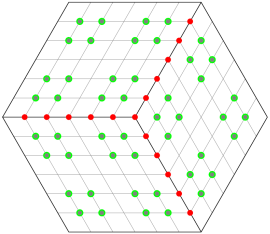
Restricted Connectivity: Polycube Configurations

$n \in \{3, 4, 5, 6\}$:



3 4 5 6

2x2 split: Control Handles $\mathbb{C}_i$



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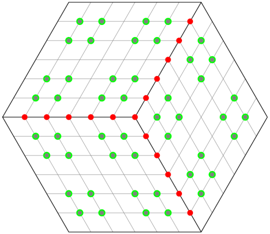

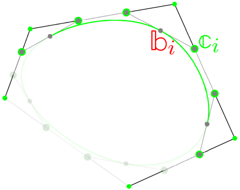

Control handles $\mathbb{c}_i$
Boundary points $\mathbb{b}_i$ (Bézier)

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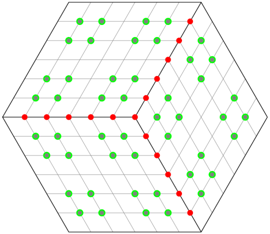

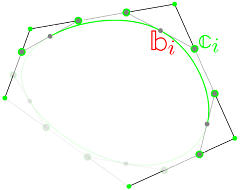

Edge Recovery: $\mathbb{c}_i \rightarrow \mathbb{b}_i$

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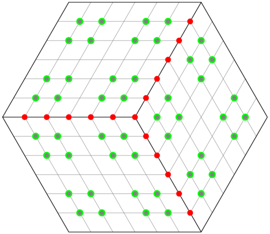

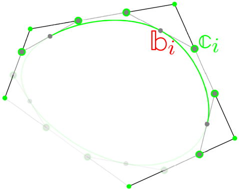

Edge Recovery: $\mathbb{c}_i \rightarrow \mathbb{b}_i$
smooth ?

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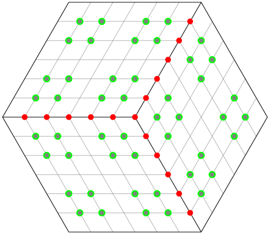
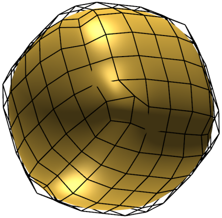
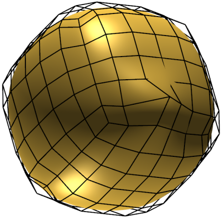


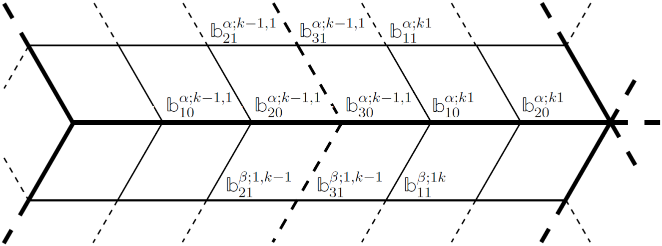




Outline

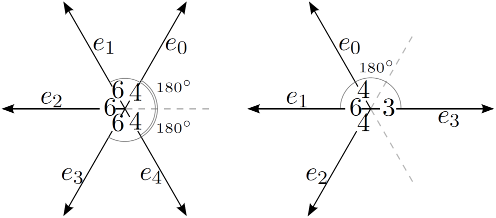
PGS: G^1 construction for vertex valences 3, 4, 5, 6

- G^1 constraints on BB-coefficients (technical)



PGS: G^1 construction for vertex valences 3, 4, 5, 6

- G^1 constraints on BB-coefficients
- Apparent valence (mimic 3–6)



$$2c_p\, \mathbb{b}_{10}^{\alpha,11} = \mathbb{b}_{10}^{\alpha+1,11} + \mathbb{b}_{10}^{\alpha-1,11}, \qquad c_p := \cos\frac{2\pi}{p}.$$

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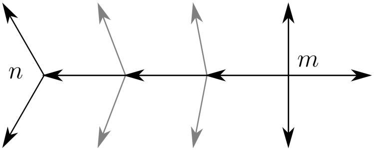
- Assign labels: $\langle n, m \rangle$ edge



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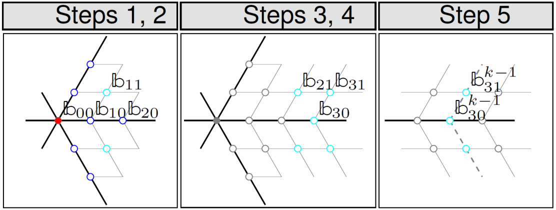
- Construction steps:



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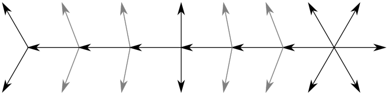
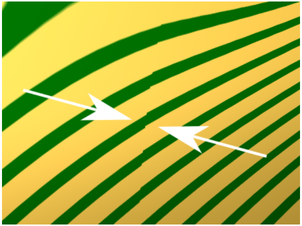
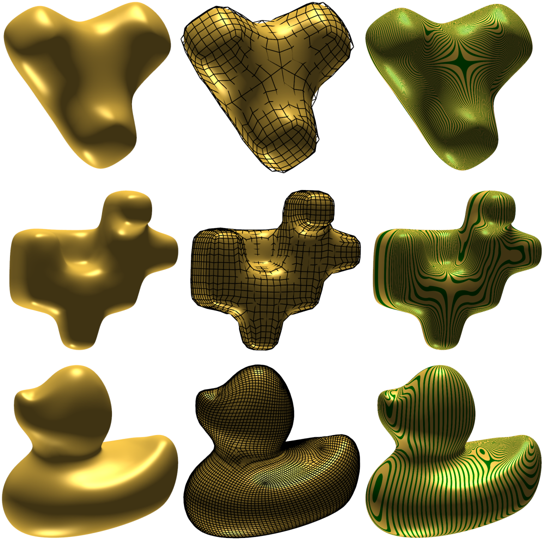




Smooth PGS surfaces



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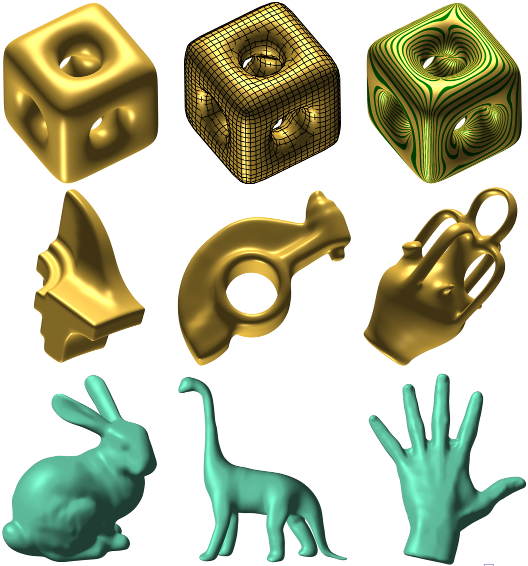
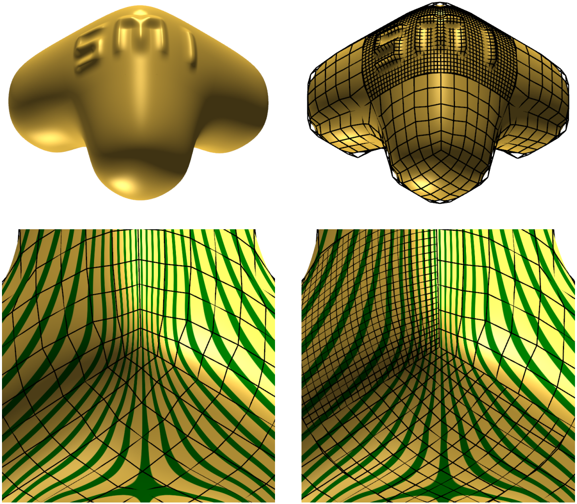


Outline

Multi-resolution of Polycube G^1-splines



Multi-resolution of Polycube G^1-splines

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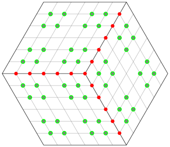

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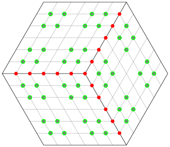

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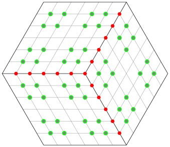

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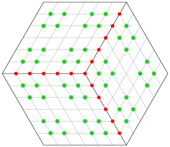

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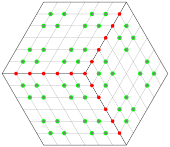

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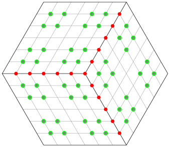

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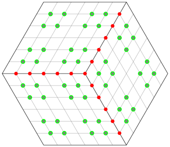

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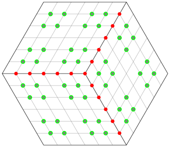

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  perturb $\mathbb{c}_i$ $\Longrightarrow$ different smooth surface

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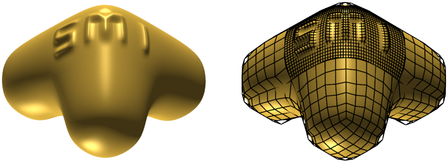

QUESTIONS ?

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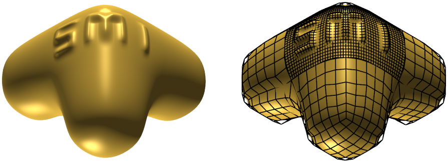


QUESTIONS ?

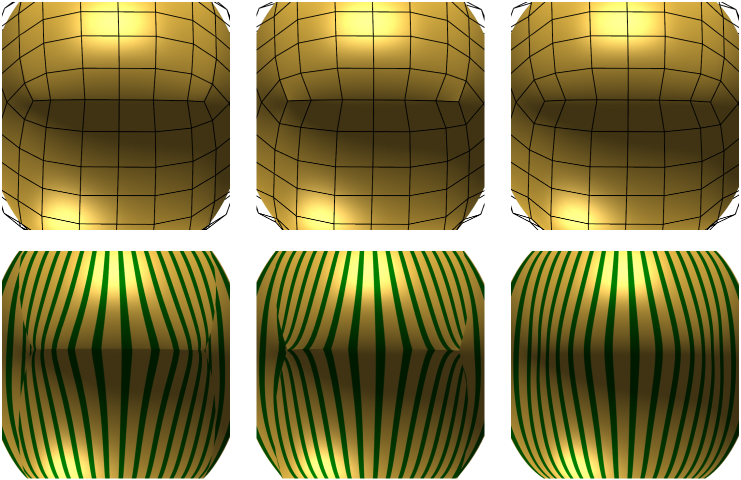
Multiresolution: algebraic challenge

- Challenge: global system, ☺: additional constraint

$$\omega(\frac{k}{2^\ell})\underbrace{(-\mathbb{b}_{10}^{\alpha;k-1,1}+2\mathbb{b}_{20}^{\alpha;k-1,1}-2\mathbb{b}_{10}^{\alpha;k1}+\mathbb{b}_{20}^{\alpha;k,1})}_{C^2\ \text{boundary}}=0.$$

☹: large support of $\mathbb{c}_i$ handles ☺: decay fast

PGS^{ER}



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- G^1 constraints on BB-coefficients:

$$\mathbb{b}(u,v) := \sum_{i=0}^{d}\sum_{j=0}^{d} \mathbb{b}_{ij} B_i^d(u) B_j^d(v),$$

$$\mathbf{p}_{01} + \mathbf{q}_{10} = \omega_0 \mathbf{p}_{10} + (2-\omega_0)\mathbf{p}_{00}, \tag{1}$$

$$\mathbf{p}_{11} + \mathbf{q}_{11} = \boldsymbol{\rho}(\omega_0,\omega_1), \tag{2}$$

$$\mathbf{p}_{21} + \mathbf{q}_{12} = \boldsymbol{\sigma}(\omega_0,\omega_1), \tag{3}$$

$$\mathbf{p}_{31} + \mathbf{q}_{13} = (2+\omega_1)\mathbf{p}_{30} - \omega_1 \mathbf{p}_{20}. \tag{4}$$

$$\boldsymbol{\rho}(\omega_0,\omega_1) := \frac{1}{3}\big(2\omega_0\mathbf{p}_{20} - \omega_1\mathbf{p}_{00} + (6-2\omega_0+\omega_1)\mathbf{p}_{10}\big)$$

$$\boldsymbol{\sigma}(\omega_0,\omega_1) := \frac{1}{3}\big(\omega_0\mathbf{p}_{30} - 2\omega_1\mathbf{p}_{10} + (6-\omega_0+2\omega_1)\mathbf{p}_{20}\big).$$

What goes wrong when $\langle n, 4\rangle$, $\langle 4, m\rangle$ with $n \neq m$?