Refinable C^1 spline elements for irregular quad layout

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Outline

1. Refinable, smooth, CAD compatible spline space incl. irregularities
2. Algorithm
3. Applications

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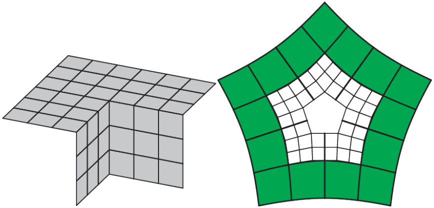
Challenge: refinable, smooth and CAD compatible



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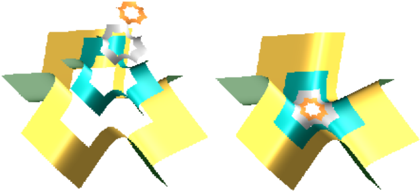
- subdivision surface:

  ☺ nested space



  ☹ infinite rings;
  ☹ industrial design infrastructure;
  ☹ integration rules;

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- subdivision surface: ☺ nested space
  ☹ infinite rings; industrial design infrastructure; integration rules;
- G^k spline complex:
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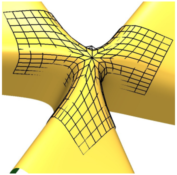
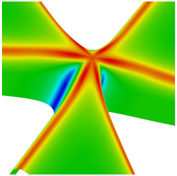

☹ refinement book keeping (non-local);
☹ or: not nested : problem for free-form surfaces!

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Challenge: combine, for multi-sided configurations,
splines with simple nested refinability.

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- singularly parameterized surface
  ☺ nested space,
  ☺ industrial design infrastructure
  (Peters 91, Neamtu 94)
  (Reif 97) ☺ proves C^1 surface (projection)

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  ☹ Surface shape is poor.

2 × 2 split construction

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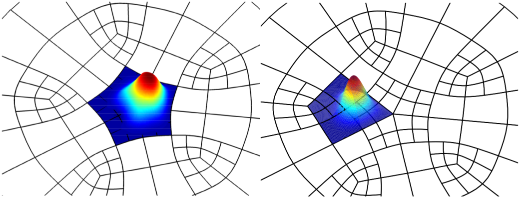
  ☺ regardless of vertex valences, each quad has 4 d.o.f.!



2×2 split construction

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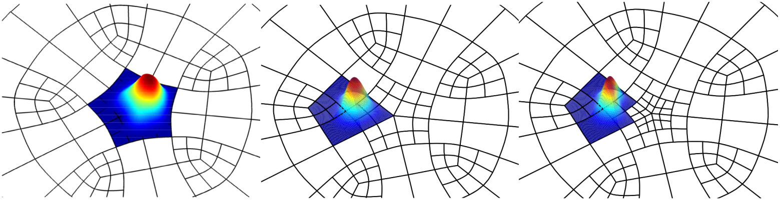
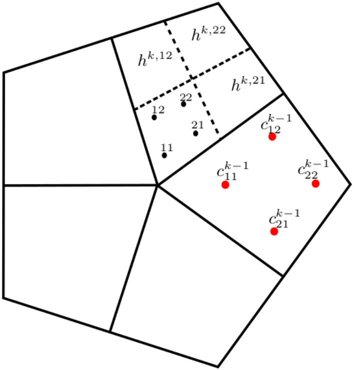
  ☺ regardless of vertex valences, each quad has 4 d.o.f.!
- C^1 bi-3 basis functions
  ☺ naturally compatible with bi-cubic PHT refinement



Outline

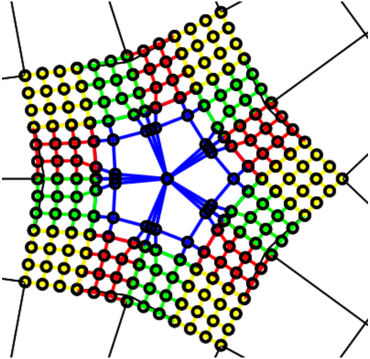
Algorithm Input



Input: B-spline-like control points c_{ij}^{ℓ}

- Recall: regular double-knot bi-3 B-spline coefficients are co-located with "inner" Bézier coefficients: $c_{11} \to \frac{1}{4} \begin{matrix} 1b_{00} & 2b_{01} \\ 2b_{10} & 4b_{11} \end{matrix}$

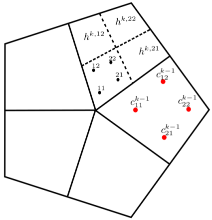
Algorithm Output



Output: Bézier points $b^{k,11}_{\alpha\beta}$ obtained by projection $\mathbf{P}$

Algorithm: **PS*c***

- Conversion to BB form: copy $a_{ij}^k := c_{ij}^k$ for $i, j \in \{1, 2\}$



make C^1 except $a_{00}^k := \sum_{k=1}^n c_{11}^k / n$.

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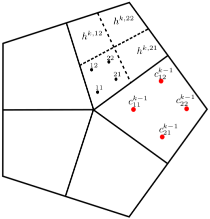

  make C^1 except $a^k_{00} := \sum_{k=1}^{n} c^k_{11}/n$.

- Subdivide $h^{k,ij} := \mathbf{S}a^k,\ i, j \in \{1, 2\}$. When $i + j > 2$ then $b^{k,ij} := h^{k,ij}$.

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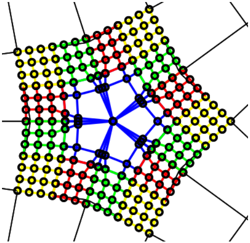
- Subpatches $h_{\alpha\beta} := h^{k,11}_{\alpha\beta}$:

$$\begin{pmatrix} b_{11} \\ b_{21} \\ b_{12} \end{pmatrix} := \mathbf{P} \begin{pmatrix} h_{11} \\ h_{21} \\ h_{12} \end{pmatrix}$$

For all edges make C^1:
$b^k_{10} := (b^k_{11} + b^{k+1}_{11}/2)$.



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- Refinable (nested spaces)

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- Linear independence of f_{ij}^k associated c_{ij}^k (proof via functionals)

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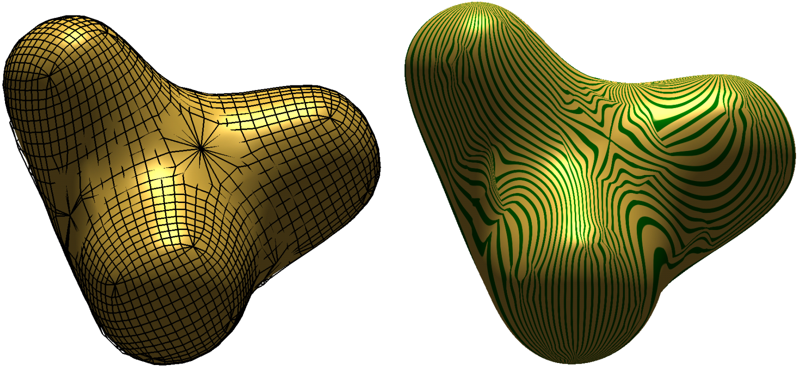
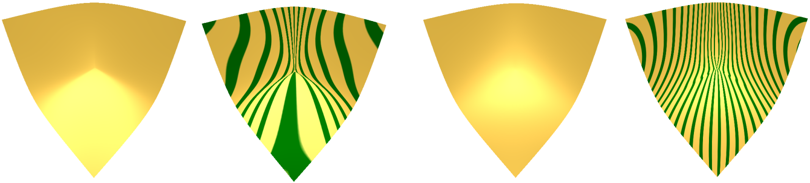
Applications: free-form surfaces





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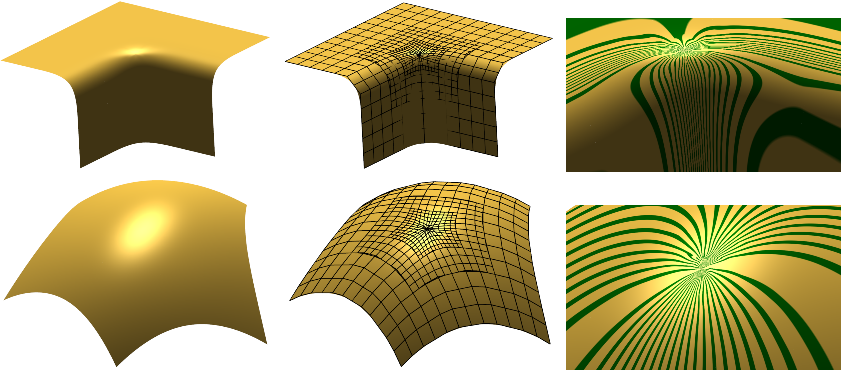

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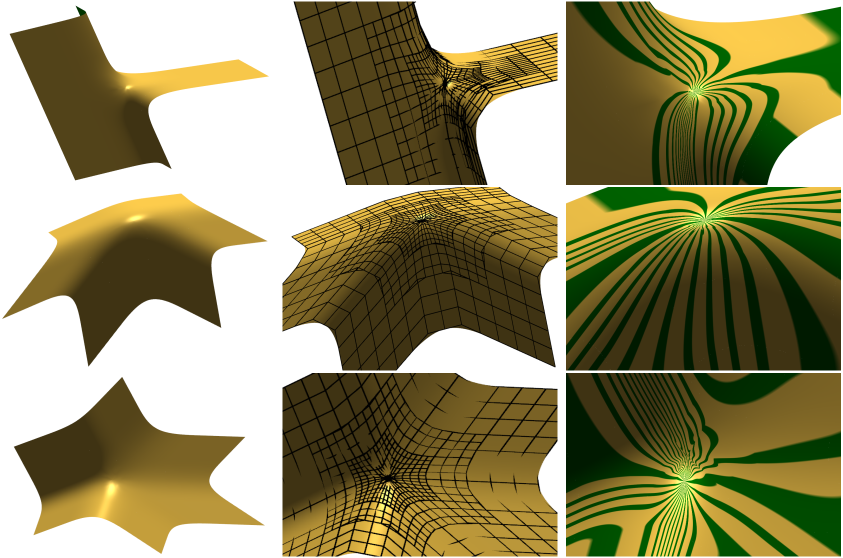
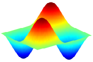
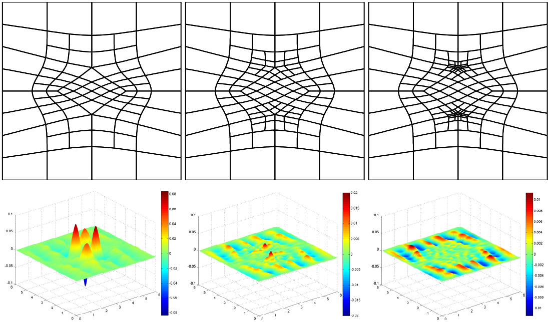


Applications: Poisson local refinement

Poisson's equation on the square $[0,6]^2$ 



. Error -0.08:0.08, -0.02:0.02, -0.01:0.01

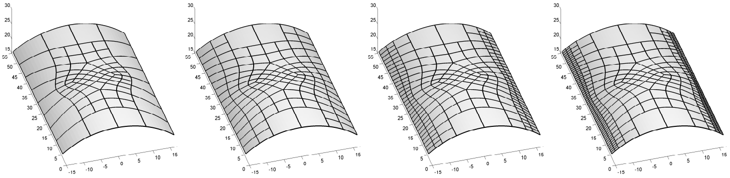
Applications: Poisson

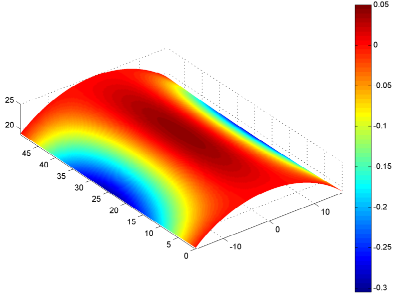
Error and approximate convergence rate (a.c.r.): close to 2^{-4}

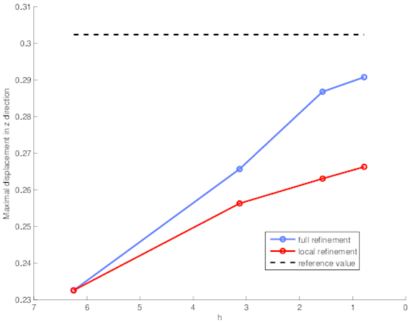
ℓ	$\|\|u-u_h\|\|_{L^2}$	a.c.r. $\|\|.\|\|_{L^2}$	$\|\|u-u_h\|\|_{L^\infty}$	a.c.r. $\|\|.\|\|_{L^\infty}$	$\|\|u-u_h\|\|_{H^1}$	a.c.r.
1	0.0625	-	0.0826	-	0.4568	
2	0.0098	6.4	0.0194	4.3	0.1448	3
3	9.7e-04	10.1	0.0015	12.9	0.0225	6
4	7.17e-5	13.5	1.3e-4	11.5	0.0033	6
5	5.29e-06	13.56	9.87e-06	13.1	4.78e-04	6

Applications: Thin shell

Scordelis-Lo roof







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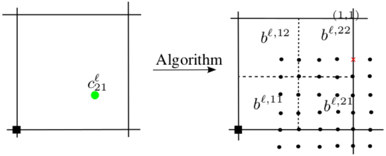
linear independence



Nonzero BB coefficients • of the basis function f^{ℓ}_{21}. The coefficient marked additionally with an $\times$ is nonzero only for f^{ℓ}_{21}. It is zero for f^{k}_{12} or f^{k}_{11}, $k = 0, \dots, n-1$ and for f^{k}_{21}, $k \neq \ell$.