Polynomial Spline Surfaces with Rational Linear Transitions

Jörg Peters and Martin Sarov

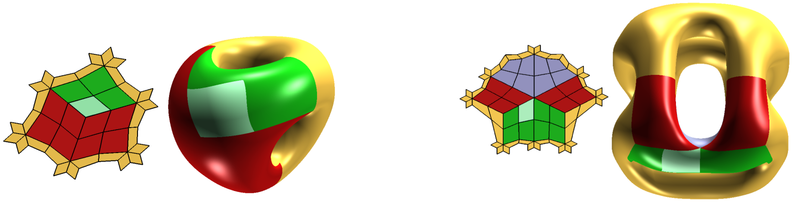
supported by: National Science Foundation grant CCF-1117695

Outline

3-6 quads

once-subdivided 3-6 quads for modelling

- smooth objects of any genus greater than one,



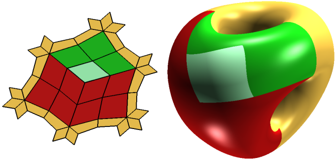
(a) 3-neighborhood, level 1 [3,6,6,6] quads, genus $g = 2$, total: 2×6 quads

(b) 6-neighborhood, level 1 [3,6,6,6] quads, genus $g = 5$, total: 8×6 quads

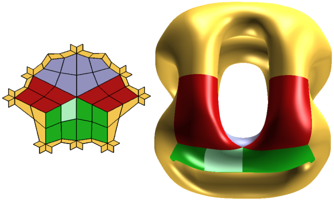
3-6 quads

once-subdivided 3-6 quads for modelling

- smooth objects of any genus greater than one,
- using one bi-cubic polynomial piece per sub-quad.

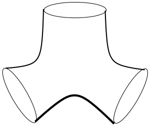


(a) 3-neighborhood, level 1 [3,6,6,6] quads, genus $g = 2$, total: 2×6 quads

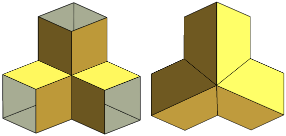


(b) 6-neighborhood, level 1 [3,6,6,6] quads, genus $g = 5$, total: 8×6 quads

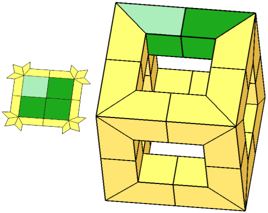
Pants



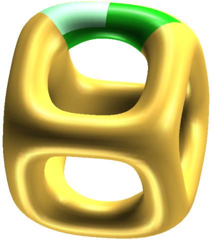
(a) (pair of) pants



(b) polyhedral pants

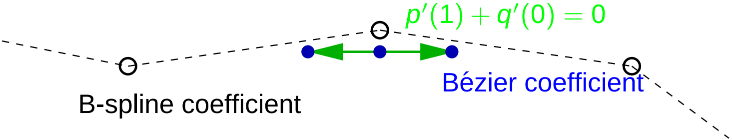


(c) Level $\ell = 1$ [6,6,6,6] quad on a genus 5 polyhedron

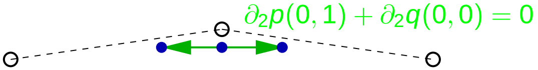


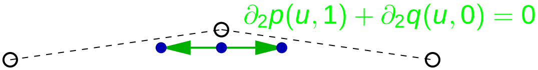
(d) spline surface

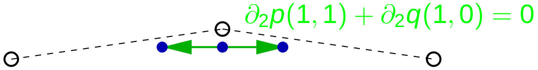
Derivative Patterns 1 (univariate C^1)



Derivative Patterns 2 (tensor-product)







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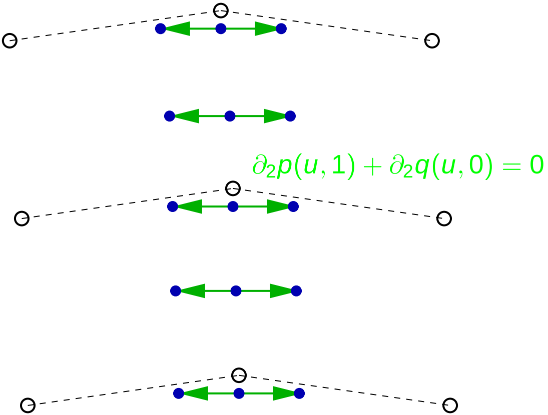
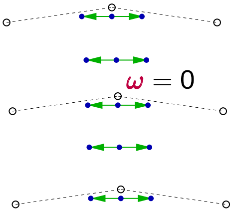


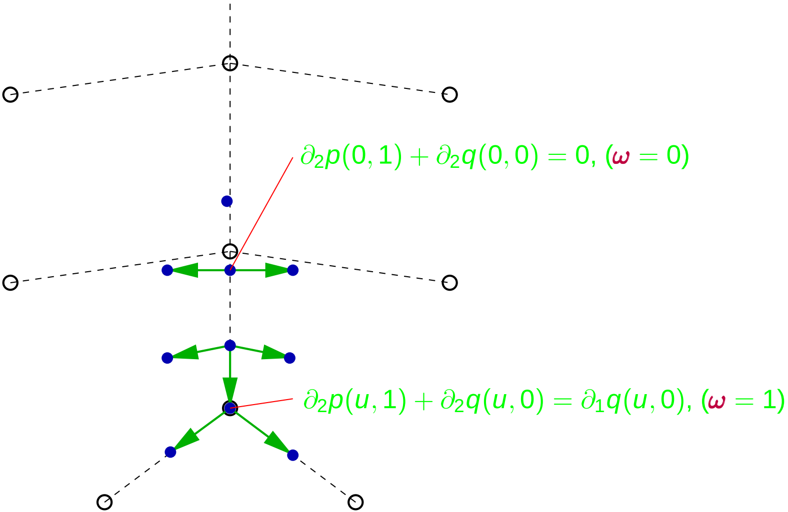
Derivative Patterns 2 (structurally symmetric)



$\rho : \mathbb{R}^2 \to \mathbb{R}^2$, G^1 surface construction is structurally symmetric if

$$\partial \mathbf{p} = \partial(\mathbf{q} \circ \rho) \quad \text{and } \partial \mathbf{q} = \partial(\mathbf{p} \circ \rho) \implies \partial_2 \mathbf{p}(t,0) + \partial_2 \mathbf{q}(0,t) = \omega(t) \partial_1 \mathbf{p}(t,0).$$

Derivative Patterns 3 (irregular point)



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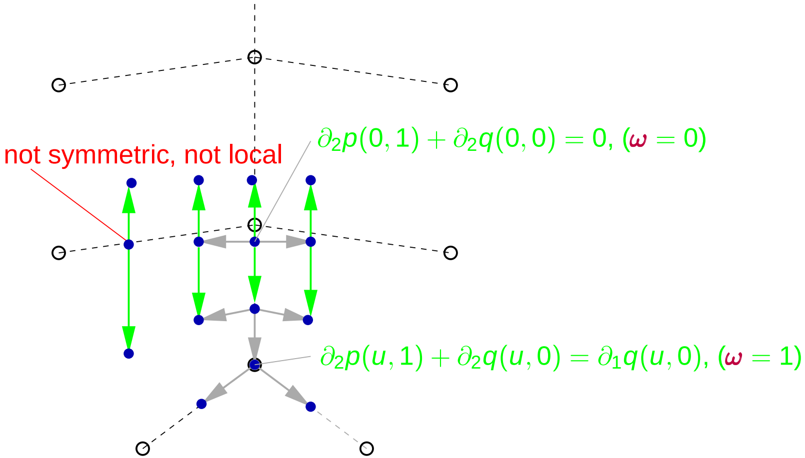
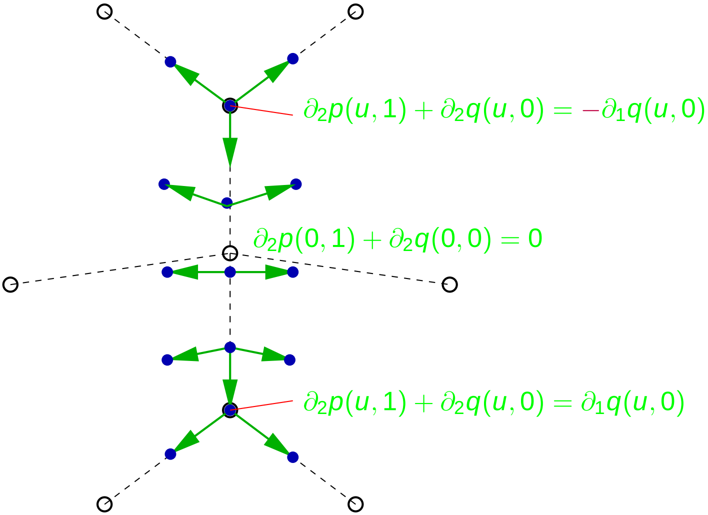


Derivative Patterns 3 (matching valence)



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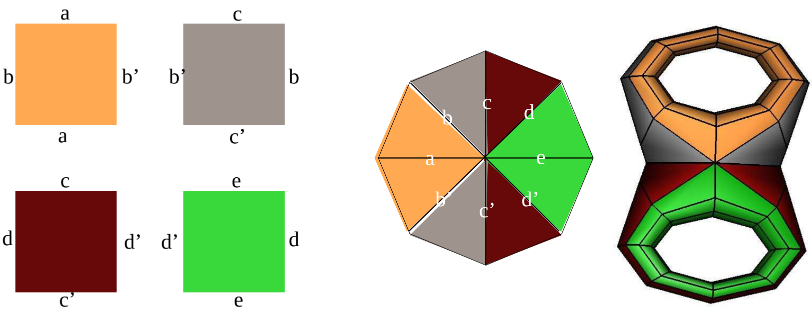

A surface has restricted patch layout = wherever two vertices are connected by a curve across which two splines match according to $\partial_2\mathbf{p}(t,0) + \partial_2\mathbf{q}(0,t) = \omega(t)\partial_1\mathbf{p}(t,0)$, with ω *linear and not constant* the vertices have equal valence (number of neighbors).

Rational Linear Reparameterization 1

Gauss-Bonnet Theorem = $\int_S K ds = 2\pi\chi(S)$: no affine atlas when $\chi \neq 0$, i.e. genus $\neq 1$.

The structurally symmetric G^k rational linear reparameterization $\boldsymbol{\rho} := r_{\mathbf{q}} \circ r_{\mathbf{p}}^{-1}$ is unique [SMI 2010]:

$$\partial^i(\mathbf{p} \circ r_{\mathbf{p}})(t, 0) = \partial^i(\mathbf{q} \circ r_{\mathbf{q}})(t, 0), \quad i = 0, \ldots, k,$$

where $\mathrm{c}_0 := \cos \frac{2\pi}{n_0}, \mathrm{s}_0 := \sin \frac{2\pi}{n_0}$,

$$r_{\mathbf{p}}(u, v) := \frac{1}{\mathrm{s}_0 + \eta v} \begin{bmatrix} -v \\ \mathrm{s} u + \mathrm{c}_0 v \end{bmatrix}, \quad r_{\mathbf{q}}(u, v) := \frac{1}{\mathrm{s}_0 - \eta v} \begin{bmatrix} \mathrm{s}_0 u - \mathrm{c}_0 v \\ v \end{bmatrix},$$

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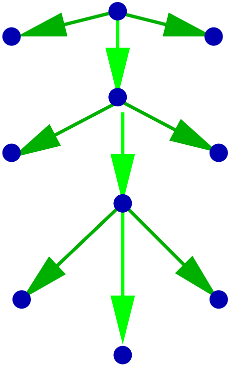
implies *linear*

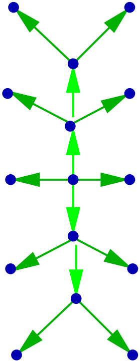
$$\boldsymbol{\omega}(t) := \partial_2 \boldsymbol{\rho}^{[2]}(t,0) = (1-t)2\mathrm{c}_0 + t2(-\mathrm{c}_1).$$

Rational Linear Reparameterization 2

$\omega(t) = 2[(1-t)c_0 - tc_1]$ means in terms of patterns

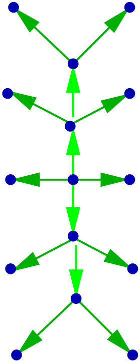


Rational Linear Reparameterization 3



3–3 edge

Rational Linear Reparameterization 3



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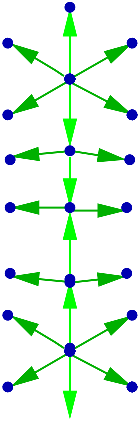
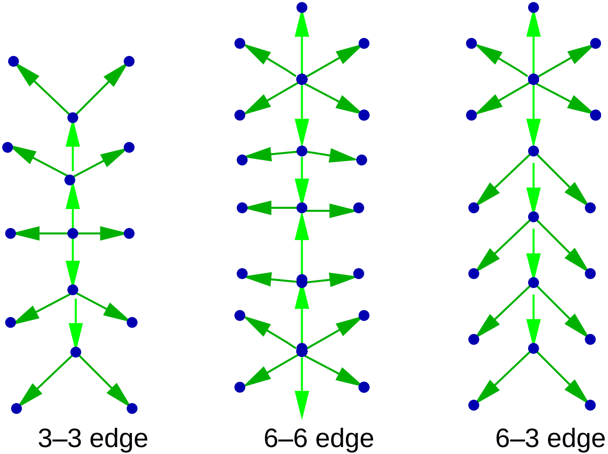
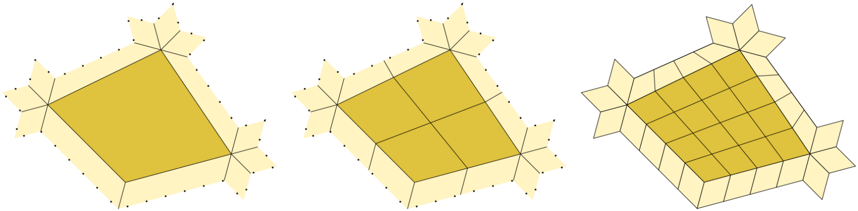


6–6 edge

Rational Linear Reparameterization 3

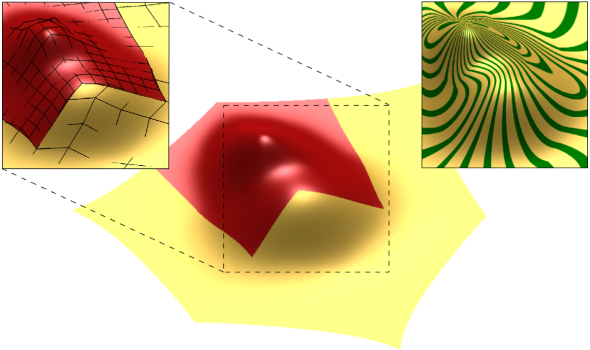


Refinement



(a) a 3-6 quad (b) level $\ell = 1$ (c) level $\ell = 2$

Combining levels



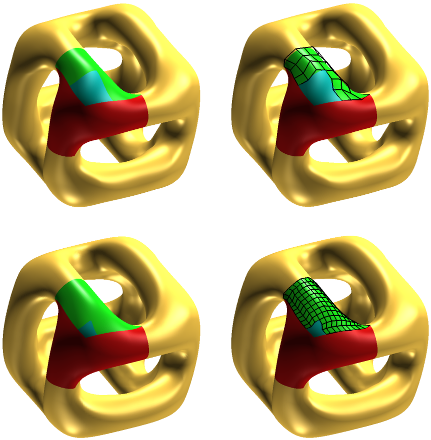
Combining level $\ell = 1$ and $\ell = 2$ along a 3-6 edge $\rightarrow$ again smooth

Multiresolution

48 [3,6,6,6] 3-6 quads:

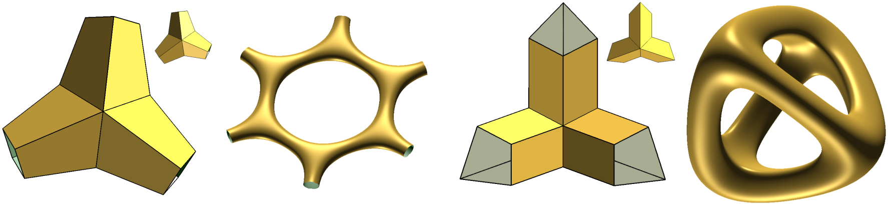
Level 1:

Level 2:

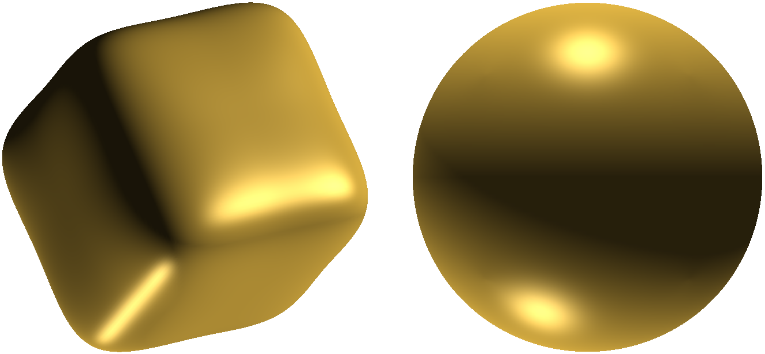


[6,6,6,6] pants

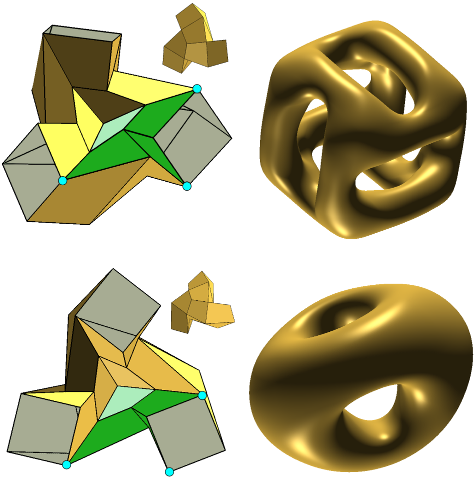
Subdivided 3-6 polyhedron: vertices = control points = degrees of freedom of the surface (G^1 construction algorithm: see paper)



[3,3,3,3] pants



[3,6,6,6] pants



cyan-colored sphere = vertex of valence 6

Conclusion

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Thank You