Generalizing bicubic splines for modelling and IGA with irregular layout

Kęstutis Karčiauskas    Thien Nguyen    Jörg Peters

Vilnius University    University of Florida

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Overview

- Irregularities in a $C^2$ bi-3 spline surface

highlight lines
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- Irregularities in a $C^2$ bi-3 spline surface
- $G^1$ completion with good distribution of highlight lines, curvature
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- generalized isogeometric analysis (gIGA) elements
Outline

1. Setup

2. $G^1$ bi-4 cap construction for $n = 5, 6, 7$

3. Obstacle Course

4. Generalized Isogeometric Analysis (gIGA) elements
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Surface cap

- Smoothly joined to form B-spline-like functions.
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Polynomial tensor-product pieces $f$ of bi-degree 3 in BB-form

$$f(u, v) := \sum_{i=0}^{d} \sum_{j=0}^{d} f_{ij} B^d_i(u) B^d_j(v), \quad (u, v) \in \Box := [0..1]^2,$$

$B^d_k(t)$ is the $k$th Bernstein-Bézier polynomial of degree $d$. 
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- Geometric Continuity – $G^1$ constraints for abutting patches $\tilde{f}$ and $f$

$$\partial_v \tilde{f}(u, 0) - a(u) \partial_v f(0, u) - b(u) \partial_u f(0, u) = 0$$
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- Minimizing distortion:
  - parameterization – minimize functionals

Characteristic parameterizations $\sigma$ when $n = 6$

\[
\mathcal{F}_k \mathbf{f} := \int_0^1 \int_0^1 \sum_{i+j=k, i, j \geq 0} \frac{k!}{i!j!} (\partial_s^i f(s, t) \partial_t^j f(s, t))^2 ds dt
\]
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- **Minimizing distortion:**
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  - geometry – sample a guide surface
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- $G^k$ constructions always $C^k$ [Peters 2014, Groisser+P 2015]
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$G^1$ continuity of the cap

\[
\begin{align*}
\dot{p}_{10} & := \left(1 - \frac{1}{c}\right)\dot{p}_{00} + \frac{\dot{p}_{01} + \dot{p}_{01}}{2c}; \\
\dot{p}_{20} & := \frac{(3c - 4)\dot{p}_{10} + 2(\dot{p}_{11} + \dot{p}_{11})}{3c}; \\
\dot{p}_{21} & := (2 - c)\dot{p}_{20} + c\dot{p}_{30} - \dot{p}_{21}; \\
\dot{p}_{30} & := \frac{2(\dot{p}_{31} + \dot{p}_{31}) - c\dot{p}_{40}}{4 - c}; \\
\dot{p}_{40} & := \frac{\dot{p}_{41} + \dot{p}_{41}}{2}.
\end{align*}
\]
$G^1$ join with the tensor-border $\mathbf{b}$

- Reparameterization across the boundary $\mathbf{b} \circ \beta$ of degree 4:

\begin{align*}
\mathbf{p}_{11} &:= \frac{(1 - 7c)\mathbf{b}_{00} + 3(\mathbf{b}_{10} + \mathbf{b}_{01}) + 9(1 - c)\mathbf{b}_{11}}{16(1 - c)}, \\
\mathbf{p}_{21} &:= \frac{(1 - 6c)\mathbf{b}_{10} + (1 + c)\mathbf{b}_{20} + 3\mathbf{b}_{11} + 3(1 - c)\mathbf{b}_{21}}{8(1 - c)}, \\
\mathbf{p}_{31} &:= \frac{(3 - 15c)\mathbf{b}_{20} + (1 + 2c)\mathbf{b}_{30} + 9\mathbf{b}_{21} + 3(1 - c)\mathbf{b}_{31}}{16(1 - c)}, \\
\mathbf{p}_{41} &:= \frac{(1 - 4c)\mathbf{b}_{30} + 3\mathbf{b}_{31}}{4(1 - c)},
\end{align*}
$G^1$ join with the tensor-border $b$

- Reparameterization across the boundary $b \circ \beta$ of degree 4:
- Interleaved $G^1$ constraints (non-linear) explicitly solved
$G^1$ bi-4 cap construction for $n = 5, 6, 7$

Construction via a bi-5 guide

- central quadratic from guide surface [KP15] (bi-5):

(a) $n = 8$
(b) Catmull-Clark
(c) KP15
Construction via a bi-5 guide

- central quadratic from guide surface [KP15] (bi-5):
- Why not use functionals?
Construction via a bi-5 guide

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bi-5 guide $g$  
sampling map
Construction via a bi-5 guide

- central quadratic from guide surface [KP15] (bi-5):
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- jets $\mathbf{J}_{s,2}^4(g \circ \sigma_5^{-1} \circ \sigma_4)$ sampled from the guide $g$
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Challenge input

http://www.cise.ufl.edu/research/SurfLab/shape_gallery.shtml
Convex input net

‘deceptively simple’

(a) $n = 6$

(b) Catmull-Clark – magnified

(c) $G^2$ bi-7

(d) this paper – magnified
Surface from design sketch

(a) input design sketch
(b) CC-net
(c) this paper
(d) BB-net
(e) mean curvature
High valence

(a) CC-net $n = 9$
(b) $G^1 \ 2 \times 2$ cap
(c) cap magnified
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Poisson’s equation on the disk

*Matched $G^k$-constructions always yield $C^k$-continuous isogeometric elements* [P14, GP 15]

$L^2$ convergence: higher than $O(h^3)$  
$L^\infty$ convergence: higher than $O(h^2)$.  

(a) $n = 5$  
(b) $n = 7$
Koiter’s shell model, 4th order PDE

Kirchhoff-Love assumptions that lines normal to the ‘middle surface’ in the original configuration.

(a) Octant of a spherical shell  
(b) Scordelis-Lo roof thin shell
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Thank You & Questions?