IGA Across Irregularities

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Overview

- Irregularities, Geometric Continuity, Iso-geometric elements
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- Matched $G^k$-constructions always yield $C^k$-continuous isogeometric elements
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- Solving second-order and fourth-order PDEs in 2d and 3d
Irregularities, Geometric Continuity, Iso-geometric elements

2 Matched $G^k$-constructions always yield $C^k$-continuous isogeometric elements

3 Isogeometric elements from B-spline-like functions

4 Solving second-order and fourth-order PDEs in 2d and 3d
Irregularities in 2d and 3d manifolds

- **Irregularity** = not regular tensor-product spline lattice
  for example: \( n = 3 \) or \( n > 4 \) quadrilateral pieces (patches) come together

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(a) input  (b) surface pieces  (c) capped bi-2
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- Subdivision splines [Cirak 2000, Barendrecht 2013, Nguyen 2013]

(a) polar subdivision

(b) polar element
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- \( G^k \) constructed surfaces – compatible with the industrial NURBS exchange standard
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- Subdivision splines [Cirak 2000, Barendrecht 2013, Nguyen 2013]
- $G^k$ constructed surfaces – compatible with the industrial NURBS exchange standard
- $G^1$ filling of multi-sided holes in bi-quadratic spline complexes [KP 2014]

(a) rocker arm
(b) fan disk
(c) frog
Geometric Continuity

survey: [Handbook of Computer Aided Geometric Design: 193-229]

- $G^k$ (geometrically continuous surface) constructions: to create surfaces that are smooth also at irregularities
**Geometric Continuity**

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- $k$-jet = $j^k_s f$ equivalence class under $\sim^k_s$

\[
(f_1, \mathcal{N}_1) \sim^k_s (f_2, \mathcal{N}_2) \quad \text{if} \quad \partial^i f_1(s) = \partial^i f_2(s) \quad \text{for all } i \text{ with } |i| \leq k,
\]

$\mathcal{N}$ is an $\mathbb{R}^m$–open neighborhood of $s$ and $f : \mathcal{N} \rightarrow \mathbb{R}^d$ is $C^k$
Geometric Continuity

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- $G^k$ (geometrically continuous surface) constructions: to create surfaces that are smooth also at irregularities
- $k$-jet $= j_s^k f$ equivalence class under $\sim_s^k$
- $x_1$ joins $x_2$ $G^k$ with reparameterization $\rho$ along $E$ if for every $s \in \bar{E}_1$ we have

$$j_s^k x_1 = j_s^k (x_2 \circ \rho)$$
Matched $G^k$-constructions always yield $C^k$-continuous isogeometric elements

Outline

1. Irregularities, Geometric Continuity, Iso-geometric elements

2. Matched $G^k$-constructions always yield $C^k$-continuous isogeometric elements

3. Isogeometric elements from B-spline-like functions

4. Solving second-order and fourth-order PDEs in 2d and 3d
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Iso-geometric elements

- **Isogeometric elements**: to unify the representation of geometry and of engineering analysis
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- **physical domain** tessellated into pieces $\mathbf{x}_\alpha(\Box_\alpha)$:

$$
\Omega := \bigcup_{\alpha=1}^{n_f} \mathbf{x}_\alpha(\Box_\alpha) \subset \mathbb{R}^d, \quad \mathbf{x}_\alpha : \Box_\alpha \subset \mathbb{R}^m \to \mathbb{R}^d. \quad (1)
$$
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- **Isogeometric element** $= u_\alpha \circ \mathbf{x}_\alpha^{-1} : \Omega \rightarrow \mathbb{R}$ where $u_\alpha$ and $\mathbf{x}_\alpha$ are tensor-product spline functions on $\Box_\alpha$. 

![Diagram](image)
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Model $G^k \rightarrow C^k$ gIGA elements

- Geometrically continuous ($G^k$) constructions naturally yield families of finite (gIGA) elements for isogeometric analysis (IGA) that are $C^k$ also for non-tensor-product layout (irregularities)

(a) Smoothness across irregularities  
(b) Geodesics on a generalized spline surface via gIGA elements
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- Generalized IGA (gIGA) elements are an alternative to subdivision surface elements [Cirak 2000, Barendrecht 2013, NKP 2013] and $C^0$ elements (2nd order PDEs) [Scott-T-E 14, Hughes 12, Sangalli-T-V 15, Zhang 13]
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gIGA used computationally [NKP 2013 appeared Jan 2014]
Theory: [Pet14] talks at Curves&Surfaces, Dagstuhl, Icosahom14
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Formal framework for isogeometric elements based on geometric continuity [Groisser and Peters CAGD March 2015]
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- Geometrically continuous ($G^k$) constructions *naturally yield* families of finite (gIGA) elements for isogeometric analysis (IGA) that are $C^k$ also for non-tensor-product layout (*irregularities*)

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  *planar,linear:* [Bercovier-M 15], *two-patch* [Kapl-V-J 15, Colin-S-T 15]
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Formal framework for gIGA

[Groisser and Peters CAGD March 2015]

- $C^k$ atlas from $G^k$ continuity
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**Formal framework for gIGA**

[Groisser and Peters CAGD March 2015]

- $C^k$ atlas from $G^k$ continuity
- $C^k$ map:

Theorem 1 (Matched $G^k$ constructions yield $C^k$ isogeometric elements)

For $i = 1, 2$, consider $C^k$ maps $u_i : \square_i \to \mathbb{R}^N$ and $x_i : \square_i \to \mathbb{R}^d$ and assume that each $x_i$ is injective and the images $x_1(\square_1)$, $x_2(\square_2)$ are disjoint. Assume that $x_1$ joins $x_2$ $G^k$ along $E := x_1(E_1) = x_2(E_2)$ with reparameterization $\rho$ (defined in (4)) and that the analysis functions $u_i$ match the setup in that $u_1$ joins $u_2$ $G^k$ with the same reparameterization $\rho$. Let $\tilde{u}_{12} : \check{X}_{12} \to \mathbb{R}^N$ (with $\check{X}_{12}$ defined in (9)) be the piecewise isogeometric element defined by

$$
\tilde{u}_{12}(y) := \begin{cases} 
  u_1 \circ x_1^{-1}(y) & \text{if } y \in x_1(\square_1), \\
  u_2 \circ x_2^{-1}(y) & \text{if } y \in x_2(\square_2 \cup E_2).
\end{cases}
$$

(15)

Then

- $\tilde{u}_{12}$ is $C^k$ and
- $\tilde{u}_{12}$ is the unique function $\check{X}_{12} \to \mathbb{R}^N$ that restricts to $u_i \circ x_i^{-1}$ on $x_i(\square_i \cup E_i)$ for $i = 1, 2$. 

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B-spline-like Isogeometric elements

- Smooth multi-sided blending of biquadratic splines [KP 2014]: B-spline-like basis functions over irregular control nets

(a) DS-net

(b) DS-mesh
B-spline-like Isogeometric elements

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- DS-mesh nodes = degrees of freedom
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- Smooth multi-sided blending of biquadratic splines [KP 2014]: B-spline-like basis functions over irregular control nets
- DS-mesh nodes = degrees of freedom
- Tabulated B-spline-like functions of low (least) polynomial degree
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- Increased flexibility and smoothness at irregularity
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- Smooth multi-sided blending of \textit{biquadratic} splines [KP 2014]: B-spline-like basis functions over \textit{irregular control nets}
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- Increased flexibility and smoothness at irregularity
- Majority of irregularities are of valence 3 and 5
  fill = patches of degree bi-3.
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  fill = patches of degree bi-3.
- A type of $C^1$ glGA elements
Isogeometric elements from B-spline-like functions

\[ b_i(x_\alpha) := N_{\alpha,i} \circ x^{-1}_\alpha \]

(a) xy of DS-mesh-points

(b) \( n = 3 \)

(c) \( n = 3 \)

(d) \( n = 3, 5 \)

(e) \( n = 5, 3 \)

(f) \( n = 4, \text{boundary} \)
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Poisson’s equation on the unit disk

- DS-mesh (to be refined by Doo-Sabin subdivision)

(a) symmetric $n = 3$

(b) $n = 3$

(c) $n = 3$ and $n = 5$
Poisson’s equation on the unit disk

- DS-mesh (to be refined by Doo-Sabin subdivision)
- Error at subdivision level $\ell = 5$
  \[ \begin{align*}
  L^\infty & : O(h^3) \text{ when symmetric} \\
  L^2 & : O(h^3)
  \end{align*} \]

(a) Layout a: \( \times 10^{-6} \)
(b) Layout b: \( \times 10^{-7} \)
(c) Layout c: \( \times 10^{-6} \)
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- Tensored: generalizes to trivariate cylinder.
Poisson’s equation on the volumetric turbine blade model

(a) A cross-section  (b) Turbine blade (solid)  (c) Computed solution
Solving second-order and fourth-order PDEs in 2d and 3d

Bi-harmonic equation on the unit disk

Requires $C^1$ elements

(a) Layout a: $\times 10^{-4}$

(b) Layout b: $\times 10^{-4}$

(c) Layout c: $\times 10^{-3}$

convergence $O(h^2)$ for $L^2$, $L^\infty$ and $H^1$ error
The heat equation and geodesics on surfaces

(a) CAD model
(b) fertility figurine
(c) rockerarm
Koiter’s thin-shell analysis – 4th order PDE

- Three different layouts for the Scordelis-Lo roof

(a) Mesh 1  
(b) Mesh 2  
(c) Mesh 3
Koiter’s thin-shell analysis – 4th order PDE

- Three different layouts for the Scordelis-Lo roof
- glIGA convergence of the displacement

(a) Scordelis-Lo roof
(b) glIGA convergence
Summary

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Thank You & Questions?
Solving second-order and fourth-order PDEs in 2d and 3d elastic plate with a circular hole
Poisson’s equation on a trivariate cylinder

\[ L^2 : O(h^3) \]

(a) Exact solution  (b) Error at \( \ell = 2, \times 10^{-4} \)  (c) Error at \( \ell = 3, \times 10^{-5} \)
Analysis?

- traditional: based on flexibility.
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- traditional: based on **flexibility**.
- obstacle course: “poor analysis on complex surfaces (of high quality) vs rich analysis on simple surfaces”
- We know how to get arbitrary reproduction – but expensive