IGA Across Irregularities

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Overview

- Irregularities, Geometric Continuity, Iso-geometric elements

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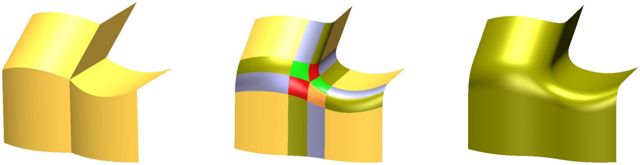
- Isogeometric elements from B-spline-like functions covering irregularities
- Solving second-order and fourth-order PDEs in 2d and 3d

Outline

1. Irregularities, Geometric Continuity, Iso-geometric elements
2. Matched G^k-constructions always yield C^k-continuous isogeometric elements
3. Isogeometric elements from B-spline-like functions
4. Solving second-order and fourth-order PDEs in 2d and 3d

Irregularities in 2d and 3d manifolds

- Irregularity = not regular tensor-product spline lattice
  for example: $n = 3$ or $n > 4$ quadrilateral pieces (patches) come together



(a) input (b) surface pieces (c) capped bi-2

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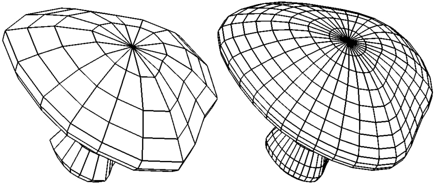
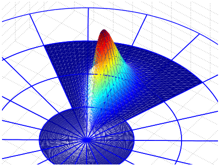


(a) polar subdivision



(b) polar element

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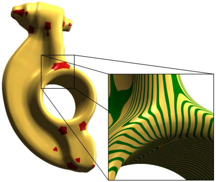
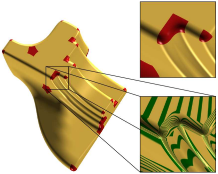
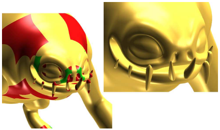
- G^1 filling of multi-sided holes in bi-quadratic spline complexes [KP 2014]



(a) rocker arm



(b) fan disk



(c) frog

Geometric Continuity

survey: [Handbook of Computer Aided Geometric Design: 193-229]

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- k-jet $= \mathbf{j}^k_{\mathbf{s}} f$ equivalence class under $\sim^k_{\mathbf{s}}$

$$(f_1, \mathcal{N}_1) \sim^k_{\mathbf{s}} (f_2, \mathcal{N}_2) \quad \text{if} \quad \partial_{\mathbf{i}} f_1(\mathbf{s}) = \partial_{\mathbf{i}} f_2(\mathbf{s}) \text{ for all } \mathbf{i} \text{ with } |\mathbf{i}| \leq k,$$

$$\mathcal{N} \text{ is an } \mathbb{R}^m\text{−open neighborhood of } \mathbf{s} \text{ and } f : \mathcal{N} \to \mathbb{R}^d \text{ is } C^k$$

Geometric Continuity

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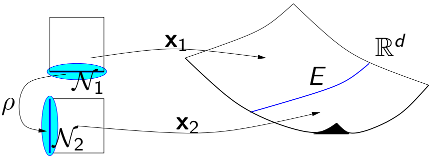
- k-jet $= \mathbf{j}_{\mathbf{s}}^k f$ equivalence class under $\sim_{\mathbf{s}}^k$
- $\mathbf{x}_1$ joins $\mathbf{x}_2$ G^k with reparameterization ρ along E if for every $\mathbf{s} \in \mathring{E}_1$ we have

$$\mathbf{j}_{\mathbf{s}}^k \mathbf{x}_1 = \mathbf{j}_{\mathbf{s}}^k (\mathbf{x}_2 \circ \rho)$$



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- Isogeometric elements: to unify the representation of geometry and of engineering analysis

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- physical domain tessellated into pieces $\mathbf{x}_\alpha(\square_\alpha)$:

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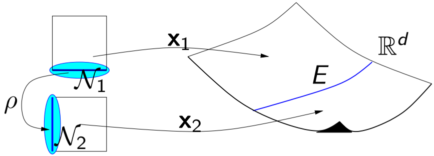

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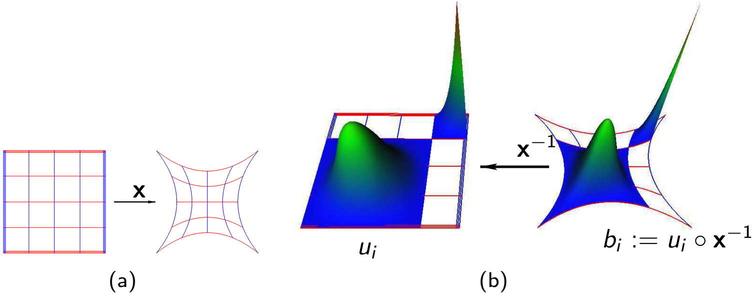

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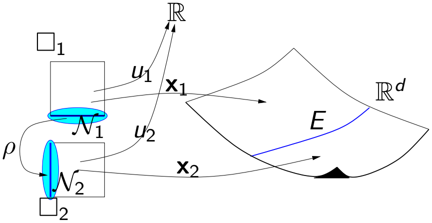
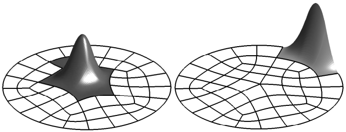
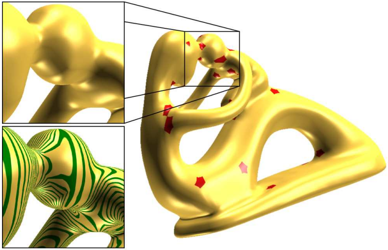


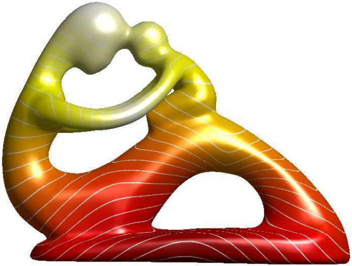


Model $G^k \to C^k$ gIGA elements

- Geometrically continuous (G^k) constructions *naturally yield* families of finite (gIGA) elements for isogeometric analysis (IGA) that are C^k also for non-tensor-product layout (irregularities)



(a) Smothness across irregularities



(b) Geodesics on a generalized spline surface via gIGA elements

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- generalized IGA (gIGA) elements are an alternative to subdivision surface elements [Cirak 2000, Barendrecht 2013, NKP 2013] and C^0 elements (2nd order PDEs) [Scott-T-E 14, Hughes 12, Sangalli-T-V 15, Zhang 13]

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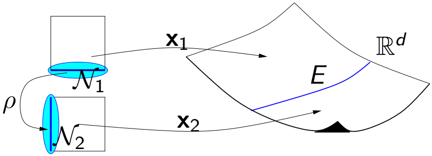
  planar,linear: [Bercovier-M 15], two-patch [Kapl-V-J 15, Colin-S-T 15]

Formal framework for gIGA

[Groisser and Peters CAGD March 2015]

- C^k atlas from G^k continuity



Formal framework for gIGA

[Groisser and Peters CAGD March 2015]

- C^k atlas from G^k continuity
- C^k map:

Theorem 1 (Matched G^k constructions yield C^k isogeometric elements)
For $i = 1, 2$, consider C^k maps $u_i : \square_i \to \mathbb{R}^N$ and $\mathbf{x}_i : \square_i \to \mathbb{R}^d$ and assume that each $\mathbf{x}_i$ is injective and the images $\mathbf{x}_1(\mathring{\square}_1), \mathbf{x}_2(\mathring{\square}_2)$ are disjoint. Assume that $\mathbf{x}_1$ joins $\mathbf{x}_2$ G^k along $E := \mathbf{x}_1(E_1) = \mathbf{x}_2(E_2)$ with reparameterization ρ (defined in (4)) *and that the analysis functions u_i match the setup in that u_1 joins u_2 G^k with* the same *reparameterization ρ. Let $\tilde{u}_{12} : \mathring{X}_{12} \to \mathbb{R}^N$ (with $\mathring{X}_{12}$ defined in* (9)) *be the piecewise isogeometric element defined by*

$$\tilde{u}_{12}(\mathbf{y}) := \begin{cases} u_1 \circ \mathbf{x}_1^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \mathbf{x}_1(\mathring{\square}_1), \\ u_2 \circ \mathbf{x}_2^{-1}(\mathbf{y}) & \text{if } \mathbf{y} \in \mathbf{x}_2(\mathring{\square}_2 \cup \mathring{E}_2). \end{cases} \tag{15}$$

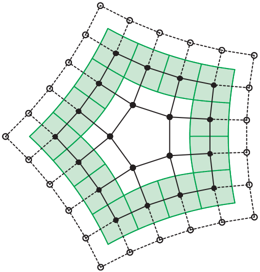
Then

- *$\tilde{u}_{12}$ is C^k and*
- *$\tilde{u}_{12}$ is the unique function $\mathring{X}_{12} \to \mathbb{R}^N$ that restricts to $u_i \circ \mathbf{x}_i^{-1}$ on $\mathbf{x}_i(\mathring{\square}_i \cup \mathring{E}_i)$ for $i = 1, 2$.*

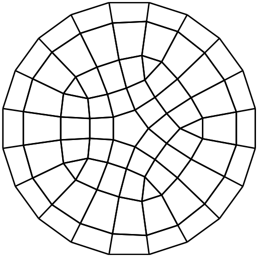
Outline

B-spline-like Isogeometric elements

- Smooth multi-sided blending of biquadratic splines [KP 2014]: B-spline-like basis functions over irregular control nets



(a) DS-net



(b) DS-mesh

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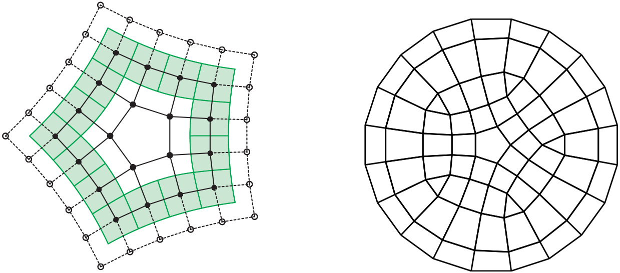

(a) DS-net (b) DS-mesh

- DS-mesh nodes = degrees of freedom

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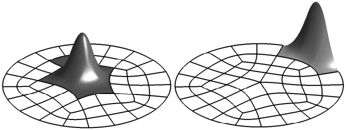
- tabulated B-spline-like functions of low (least) polynomial degree



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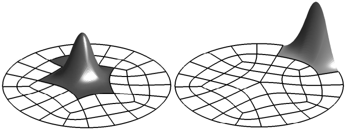

- increased flexibility and smoothness at irregularity

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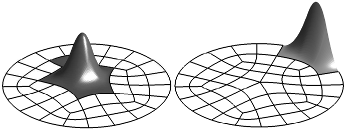

- increased flexibility and smoothness at irregularity
- majority of irregularities are of valence 3 and 5
  fill = patches of degree bi-3.

B-spline-like Isogeometric elements

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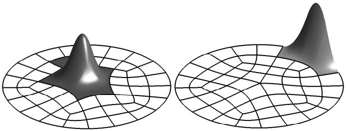

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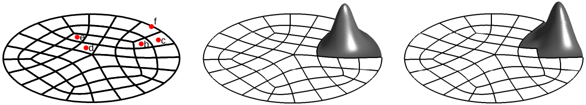
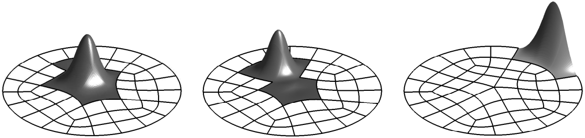
- a type of C^1 gIGA elements

gIGA elements

$$b_i(\mathbf{x}_\alpha) := N_{\alpha,i} \circ \mathbf{x}_\alpha^{-1}$$



(a) *xy* of DS-mesh-points (b) $n = 3$ (c) $n = 3$

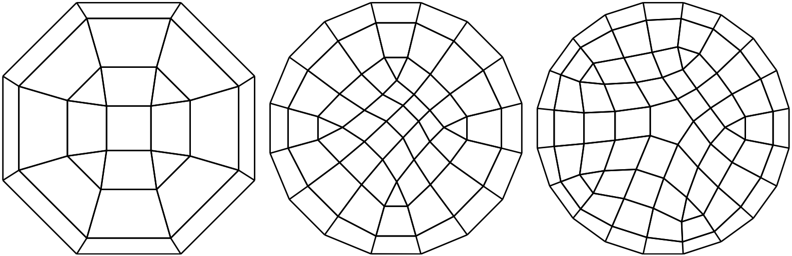


(d) $n = 3, 5$ (e) $n = 5, 3$ (f) $n = 4$, boundary

Outline

Poisson's equation on the unit disk

- DS-mesh (to be refined by Doo-Sabin subdivision)



(a) symmetric $n = 3$ (b) $n = 3$ (c) $n = 3$ and $n = 5$

Poisson's equation on the unit disk

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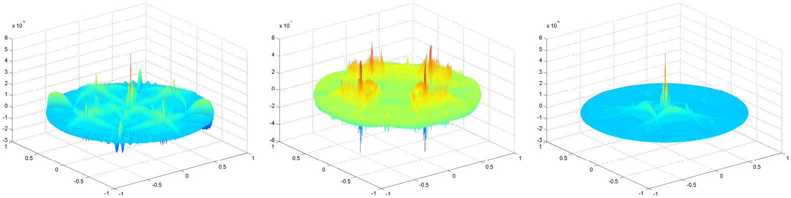
- Error at subdivision level $\ell = 5$
  L^∞: $O(h^3)$ when symmetric $\quad L^2 : O(h^3)$



(a) Layout a: $\times 10^{-6}$ (b) Layout b: $\times 10^{-7}$ (c) Layout c: $\times 10^{-6}$

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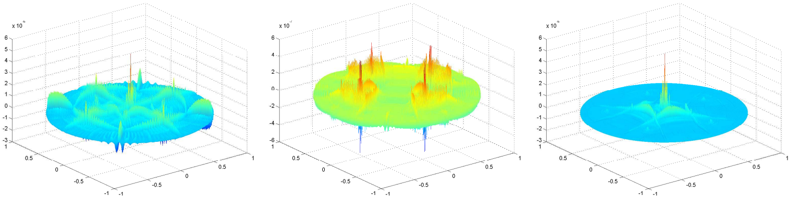
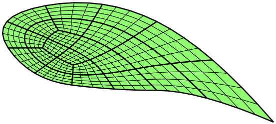


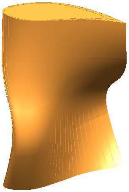
(a) Layout a: $\times 10^{-6}$ (b) Layout b: $\times 10^{-7}$ (c) Layout c: $\times 10^{-6}$

- Tensored: generalizes to trivariate cylinder.

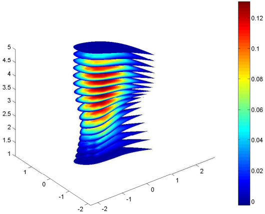
Poisson's equation on the volumetric turbine blade model



(a) A cross-section



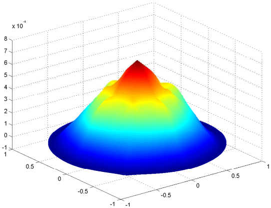
(b) Turbine blade (solid)



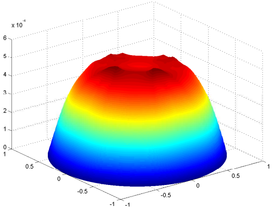
(c) Computed solution

Bi-harmonic equation on the unit disk

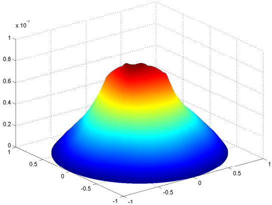
Requires C^1 elements



(a) Layout a: $\times 10^{-4}$



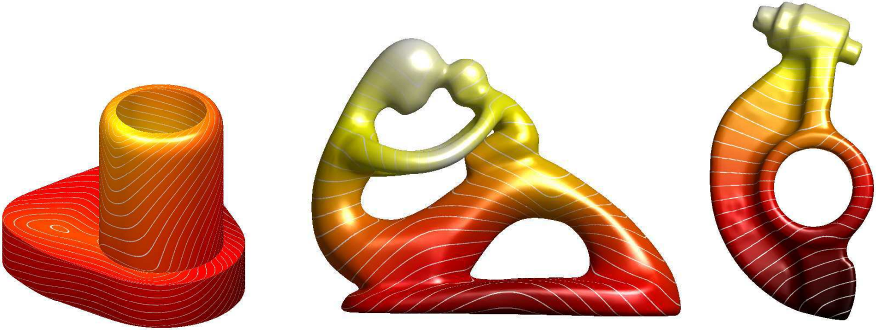
(b) Layout b: $\times 10^{-4}$



(c) Layout c: $\times 10^{-3}$

convergence $O(h^2)$ for L^2, L^∞ and H^1 error

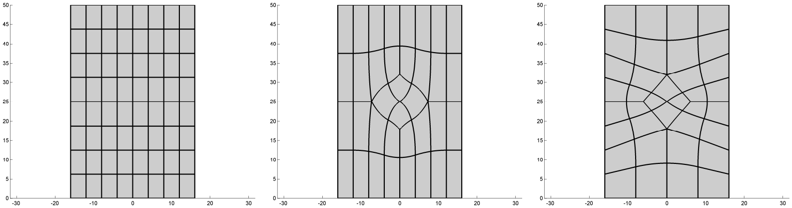
The heat equation and geodesics on surfaces



(a) CAD model (b) fertility figurine (c) rockerarm

Koiter's thin-shell analysis – 4th order PDE

- Three different layouts for the Scordelis-Lo roof



(a) Mesh 1 (b) Mesh 2 (c) Mesh 3

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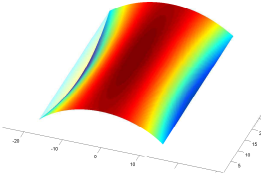
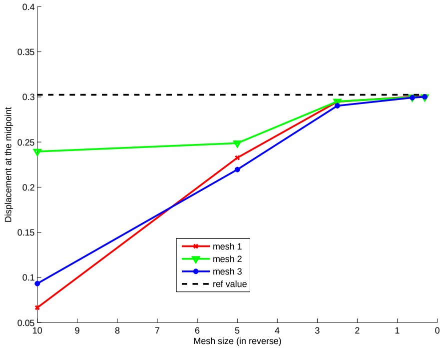
- gIGA convergence of the displacement



(a) Scordelis-Lo roof



(b) gIGA convergence

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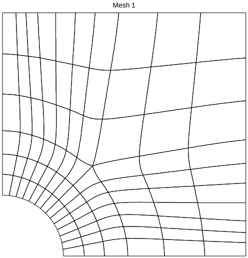
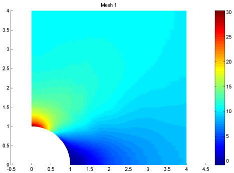
Thank You & Questions?

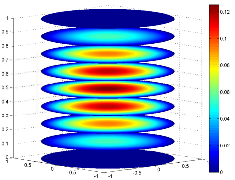


elastic plate with a circular hole

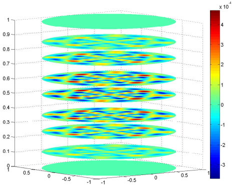


Poisson's equation on a trivariate cylinder

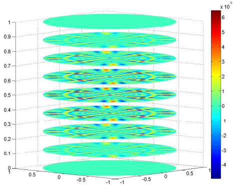
$L^2 : O(h^3)$



(a) Exact solution



(b) Error at $\ell = 2$, $\times 10^{-4}$



(c) Error at $\ell = 3$, $\times 10^{-5}$

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- We know how to get arbitrary reproduction – but expensive