Smooth Multi-Sided Blending of bi-2 Splines

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Quad models converted to CAD-compatible splines

gold = $C^1$ bi-2 splines;  red = $G^1$ bi-3;

- Continuity of normals often suffices (+ highlight lines well-distributed)
- Low degree preferable (fewer oscillations, lower downstream cost, ...)
- Matched $G^k$ constructions yield $C^k$ iso-geometric elements [P2013]
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(a) rocker arm  
(b) fan disk

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Joining and capping a collection of bi-2 spline surfaces
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Outline

1. Why not classical 1980s, 1990s solutions?
2. Multi-sided blends: unified input, geometric continuity
3. Construction Highlights
4. Bi-3 caps when $n = 3, 5$
5. More examples and comparisons
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1980s, 1990s solutions

(a) input

(b) Doo-Sabin (DS)
1980s, 1990s solutions

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(b) Doo-Sabin (DS)

(c) input

(d) Gregory-Zhou

(e) our cap
Why not classical 1980s, 1990s solutions?

1980s, 1990s solutions

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(b) Doo-Sabin (DS)
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singular constructions, rational (normalized) constructions, simplex splines, manifold splines, 3-sided patches, . . . not adopted by industry
New ingredients – 1990’s vs 2014: parameterization

quad mesh with nodes of valences 3,4,6,8

SMI 2014
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Unified Input

(a) CC-net (primal)  (b) DS-net (dual)
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(a) CC-net (primal)  (b) DS-net (dual)

(c) virtual refinement  (d) tensor-border \( b \)

Border = ring of position and derivative data in BB-form.
Geometric continuity

\[ G^1: f_v(u, 0) + g_v(u, 0) = b(u)f_u(u, 0) \]
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(1990’s) \( b(u) := 2 \cos \frac{2\pi}{n} (1 - u)^2 \Rightarrow \)
input Hermite data is matched directly (\( C^1 \)) \( \Rightarrow \) low quality.
Geometric continuity

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- bi-4 capping
  - (2014) \( b(u) := 2 \cos \frac{2\pi}{n} (1 - u) \Rightarrow \) input reparameterized to make green compatible with inter-sector.
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1. The positive effect of border reparameterization

(a) input a,b,c
(b) \(b, C^1, \text{bi-4}\)
(c) our cap
1. The positive effect of border reparameterization

(a) input a,b,c

(b) b, $C^1$, bi-4

(c) our cap

(d) Catmull-Clark
2. Curvature continuity at the extraordinary point

\[ n = 7 \text{ CC-net} \]
2. Curvature continuity at the extraordinary point

$n = 7$ CC-net

bi-4, $G^1$ eop

bi-4, $G^2$ eop
3. Functionals – but only after careful parameterization!

\[ \mathcal{F}_m f := \int_0^1 \int_0^1 \sum_{i,j \geq 0} \frac{m!}{i!j!} (\partial_s^i f \partial_t^j f)^2, \quad m\text{-jet} \]

\[ \mathcal{F}_\kappa f := \int_0^1 \int_0^1 (\partial_s^\kappa f)^2 + (\partial_t^\kappa f)^2 \]
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\[ F_m f := \int_0^1 \int_0^1 \sum_{i,j \geq 0} \frac{m!}{i!j!} (\partial_s^i f \partial_t^j f)^2 , \quad \text{m-jet} \quad F_\kappa^* f := \int_0^1 \int_0^1 (\partial_s^\kappa f)^2 + (\partial_t^\kappa f)^2 \]

Fix central point then minimize. Low \( n < 7 \rightarrow F_3 \), High \( n > 6 \rightarrow F_4 \).
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Implementation via generating functions

Tabulate 4 generating functions (3 if primal)

Assemble patch covering sector $s$

\[
\text{patch}^s_{ij} := \sum_{k=0}^{n-1} \sum_{m=1}^{4} \text{table}^{k,m}_{ij} \text{net}^{s-k}_m.
\]
Implementation via generating functions

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Assemble patch covering sector $s$

$$\text{patch}_{ij}^s := \sum_{k=0}^{n-1} \sum_{m=1}^{4} \text{table}_{ij}^{k,m} \text{net}_{m}^{s-k}. \quad (1)$$
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Bi-3 cap when $n = 3$

**Theorem** For $n = 3$ any smooth piecewise polynomial cap, satisfying symmetric $G^1$ constraints is **curvature continuous** at the central point.
Bi-3 cap when $n = 5$

One patch per sector!

(a) $n = 5$

(b) $\mathcal{F}_3$

(c) $\mathcal{F}_5$ to set central point

(d) $n = 5$

(e) Gregory-Zhou

(f) our cap
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Beams joining and subdivision

DS-net

DS
augmented DS

CC-net

CC
bi-4
Modeling with multi-sided patches

quad mesh, n=3,4,5,6  regular bi-2 + caps
Modeling with multi-sided patches

- quad mesh, n=3,4,5,6
- regular bi-2 + caps
- 'rotation'

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Modeling with multi-sided patches

quad mesh, \( n = 3, 4, 5, 6 \)  
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5-sided + bi-3 surface  
4-sided faces
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- quad mesh, n=3,4,5,6
- regular bi-2 + caps
- 'rotation'
- 5-sided + 4-sided faces
- bi-3 surface
- modification

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Multi-patch caps naturally fill a bi-2 $C^1$ complex

Mean curvature

$n = 5$
$n = 6$
$n = 7$
Conclusion

- $G^1$ with well-distributed highlight lines – sufficient for inner surfaces or mechanical parts.
- Degree bi-4 (default); bi-3 when $n = 3, 5$.
  (Alternatively bi-3 for all $n$ using a $2 \times 2$ split.)
- Immediate boundary reparameterization!
- Curvature continuity at the extraordinary point.
- Minimize functionals – but only after careful parameterization!
- $\Rightarrow$ cap behaves like one patch:
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Questions?