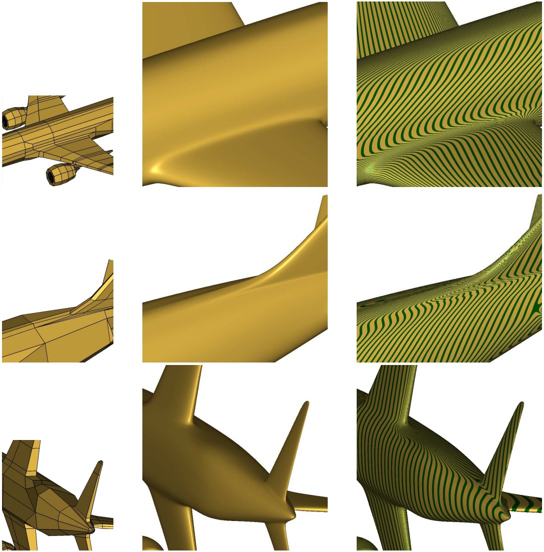
# Splines on Surfaces?

Jörg Peters
University of Florida

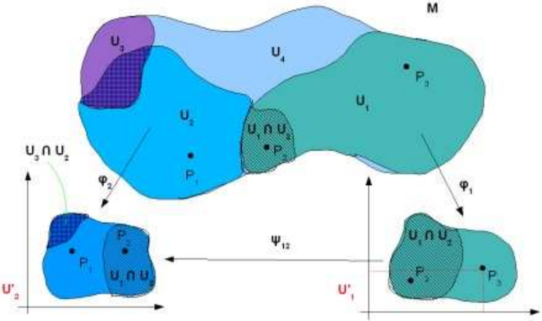
Banff 2010

# Spline-based Surface Construction



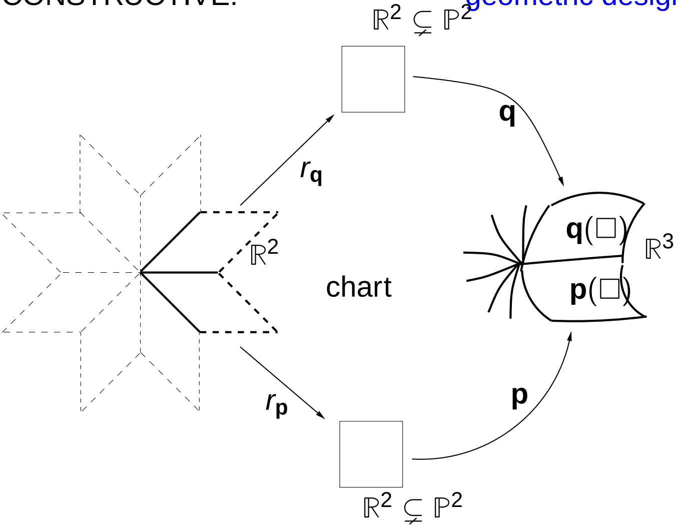
# Charts and Geometric Continuity

ANALYTIC: differential geometry



# Charts and Geometric Continuity

CONSTRUCTIVE: geometric design



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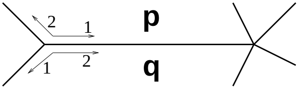


Want to construct $C^s$ surfaces of genus $g > 0$:

$$\partial^i \mathbf{p}(t,0) = \partial^i (\mathbf{q} \circ \underbrace{r_{\mathbf{q}} \circ r_{\mathbf{p}}^{-1}}_{\rho})(t,0), \quad i = 0, \ldots, s.$$

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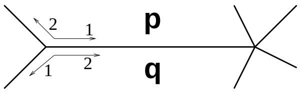


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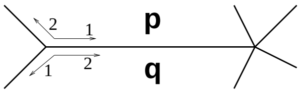


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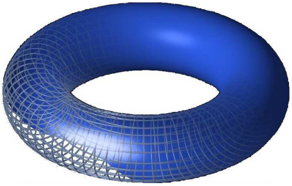
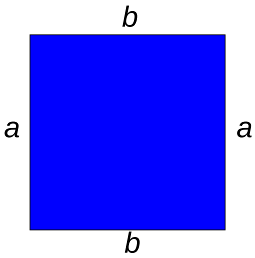
Constraint: all constructions for general meshes must employ some non-linear $\partial_2 \rho^{[2]}(t,0)$. [P,Fan '09]:

# Splines on quad-only, valence-four meshes

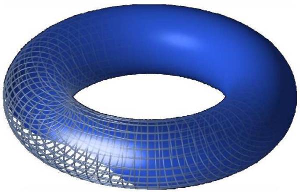





(Periodic) spline trivial: map from grid-partitioned quad, identifying edges

eg:
tensor-product splines

# Splines on more general quad-only meshes



single global domain
affine atlas
parametric continuity
shift-invariant spaces

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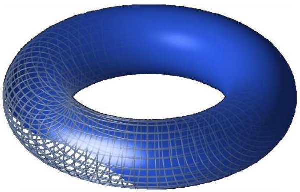
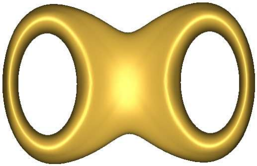
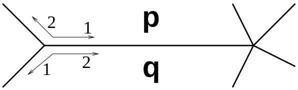




single global domain
affine atlas
parametric continuity
shift-invariant spaces

local domains
no affine atlas
geometric continuity
shift-invariant spaces?

# Chart and Geometric Continuity

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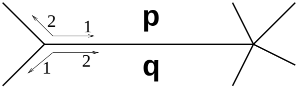


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$$\partial^i \mathbf{p}(t,0) = \partial^i (\mathbf{q} \circ \underbrace{r_{\mathbf{q}} \circ r_{\mathbf{p}}^{-1}}_{\rho})(t,0).$$

If $\rho$ is projective (rational) linear
then $\mathbf{q}$ and $\mathbf{q}(\rho)$ have the same degree.

# Fractional linear maps

Try real rational linear map $\rho$

$$\rho(u,v) := \begin{bmatrix}\rho^{[1]}\\ \rho^{[2]}\end{bmatrix}(u,v) := \begin{bmatrix}\frac{a_1+b_1u+c_1v}{d_1+e_1u+f_1v}\\ \frac{a_2+b_2u+c_2v}{d_2+e_2u+f_2v}\end{bmatrix}$$

where $a_i, b_i, \ldots, f_i$ are real scalars.

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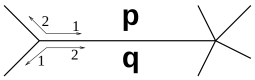


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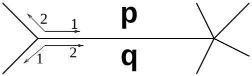


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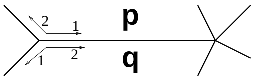


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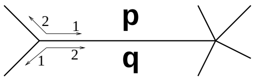


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Sufficient for special layout of quadrilaterals.

# Necessity: Constraints on the transition map for $G^2$ continuity

Find $\rho : \square \subsetneq \mathbb{R}^2 \to \mathbb{R}^2 : (u,v) \to \begin{bmatrix} \frac{a_1+b_1u+c_1v}{d_1+e_1u+f_1v} \\ \frac{a_2+b_2u+c_2v}{d_2+e_2u+f_2v} \end{bmatrix}$ so that patches $\mathbf{p},\mathbf{q} : \square \subsetneq \mathbb{R}^2 \to \mathbb{R}^d$ join smoothly across the common boundary:

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$$\mathbf{q}(t,0) = \mathbf{p}(0,t) \Longrightarrow \rho(t,0) = (0,t)$$
$$\Longrightarrow a_1 = b_1 = 0, \quad a_2 = e_2 = 0, b_2 = d_2.$$

Therefore $\rho(u,v) := \begin{bmatrix} \frac{c_1v}{d_1+e_1u+f_1v} \\ \frac{d_2u+c_2v}{d_2+f_2v} \end{bmatrix}$.

To $\rho(u,v) := \begin{bmatrix} \frac{c_1 v}{d_1+e_1 u+f_1 v} \\ \frac{d_2 u+c_2 v}{d_2+f_2 v} \end{bmatrix}$ add the $G^1$ constraints

$$\begin{aligned}(\partial_2 \mathbf{q})(t,0) &= (\partial_2(\mathbf{p}\circ\rho))(t,0) \\ &= (\partial_1 \mathbf{p})(0,t)\partial_2\rho^{[1]}(t,0) + (\partial_2 \mathbf{p})(0,t)\partial_2\rho^{[2]}(t,0)\end{aligned}$$

and require no bias for **p** over **q** and vice versa:

$$\begin{aligned}(\partial_2\rho^{[1]})(t,0) = -1 \text{ and } \tau &:= \partial_2\rho^{[2]}(0,0) = 2\cos\frac{2\pi}{n} \\ \implies e_1 = 0, d_1 = -c_1 &\text{ and } \frac{c_2}{d_2} = \tau.\end{aligned}$$

This implies

$$\rho(u,v) := \begin{bmatrix} \frac{-d_1 v}{d_1+f_1 v} \\ \frac{u+\tau v}{1+v f_2/d_2} \end{bmatrix}.$$

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$$\begin{aligned}
(\partial_2^2 \mathbf{q})(t, 0) &= (\partial_2^2 (\mathbf{p} \circ \rho))(t, 0) \\
&= (\partial_1^2 \mathbf{p})(0, t) \\
&- 2(\partial_1 \partial_2 \mathbf{p})(0, t) \partial_2 \rho^{[2]}(t, 0) \\
&+ (\partial_2^2 \mathbf{p})(0, t)(\partial_2 \rho^{[2]})^2(t, 0) \\
&+ (\partial_1 \mathbf{p})(0, t)(\partial_2^2 \rho^{[1]})(t, 0) \\
&+ (\partial_2 \mathbf{p})(0, t)(\partial_2^2 \rho^{[2]})(t, 0)
\end{aligned}$$

and since we rule out singular constructions,

$$\tau \partial_1 \partial_2 \rho^{[2]} - \partial_2^2 \rho^{[2]} = \frac{\tau}{2} \partial_2^2 \rho^{[1]}$$

# Necessity: The projective linear reparametrization

### Theorem

*The map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ for the $G^2$ construction of a $C^2$ surface is unique up to the values of $\tau := \partial_2 \rho^{[2]}(0,0)$ and $\sigma := \partial_2 \rho^{[2]}(1,0)$:*

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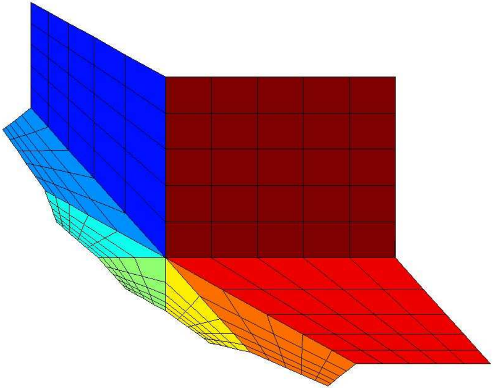
only assumption: $\mathbf{p}, \mathbf{q}$ sufficiently smooth.
Neither valence, nor polynomiality, nor the number of boundary edges of the surface pieces matters!

# The map is a root of 1: $\rho^8(\square) = \rho \circ \ldots \circ \rho = \square$



Euclidean projections of $\rho^i(\square)$.

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- $\partial_2 \rho^{[2]}(t,0)$ and $\partial_2^2 \rho^{[2]}(t,0)$ are linear functions
- $\rho = r_{\mathbf{q}} \circ r_{\mathbf{p}}^{-1}$: $\quad \eta \in \mathbb{R}$,

$$r_{\mathbf{p}}(u,v) := \frac{1}{s+\eta v} \begin{bmatrix} -v \\ su+cv \end{bmatrix}, \quad c := \cos\frac{2\pi}{n},$$

$$r_{\mathbf{q}}(u,v) := \frac{1}{s-\eta v} \begin{bmatrix} v \\ su-cv \end{bmatrix}, \quad s := \sin\frac{2\pi}{n}.$$

# Sufficiency: Does $\rho$ allow for spline constructions?

Constraint Lemma 4 [P,Fan '09]:
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~~[Hahn&Gregory 1988,9], [Ye 1997], [Prautzsch 1997], [Prautzsch&Umlauf 2000], [Reif 98], [Gregory&Zhou 1999], [Peters 2002], [Loop et al 2004,8], [Karciauskas& Peters 2004,6], . . . [Loop& DeRose 1995] [Grimm 1997], [Cotrina et al 2000, 2007], [Ying 2004],. . .~~

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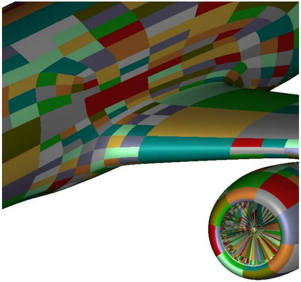
- Are there any such quadrilateral-partitions for surfaces of genus $> 1$?
- Are there corresponding spline constructions?

# Restricted Connectivity
# not restriced shape, not restricted topology

Not general connectivity meshes.



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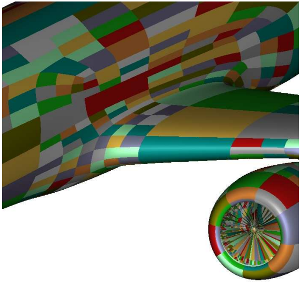
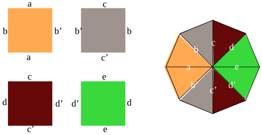
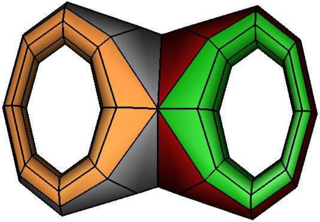


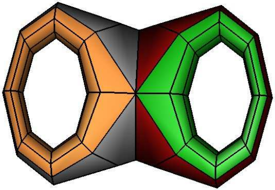
Rather



# Euler's formula & quad-only meshes

Euler's formula for polyhedra
$v - e + f = \chi = 2 - 2g$
topological genus $g > 0$:
rectangular grid
except for $-\chi = 2g - 2$
isolated vertices of valence 8.



4-valent vertex $\sim 1 - 4/2 + 4/4 = 0$
. 4 half-edges, 4 quarter-quads
8-valent $\sim 1 - 8/2 + 8/4 = -1$.

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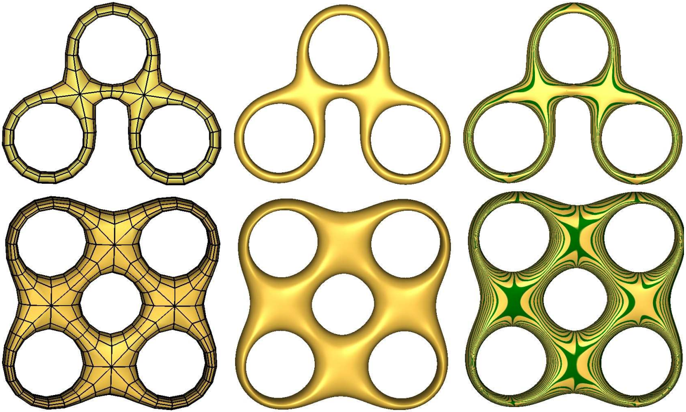

Higher-genus Surfaces (potential sampling domains). All vertices can be moved freely.

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Unique lowest-degree (= only projectively linear) bi-variate reparameterization for constructing $C^s$ manifolds of any non-zero genus.

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