Splines on Surfaces?

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Spline-based Surface Construction

Motivation: global splines
Charts and Geometric Continuity
Necessity: $G^s$ continuity
Rational Linear map & Spline Constructions
Summary
Charts and Geometric Continuity

ANALYTIC: differential geometry
Charts and Geometric Continuity

CONSTRUCTIVE:

\[ \mathbb{R}^2 \subset \mathbb{P}^2 \]

Motivation: global splines

Charts and Geometric Continuity

Fractional linear maps

Necessity: \( G^s \) continuity

Rational Linear map & Spline Constructions

Summary
Want to construct $C^s$ surfaces of genus $g > 0$:

$$\partial^i p(t, 0) = \partial^i (q \circ r_q \circ r_p^{-1})(t, 0), \quad i = 0, \ldots, s.$$
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$s = 1$:  
$$\partial_2 p(t, 0) + \partial_1 q(0, t) = \partial_1 p(t, 0) \partial_2 \rho[2](t, 0).$$
Want to construct $C^s$ surfaces of genus $g > 0$:

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\partial^i p(t, 0) = \partial^i (q \circ r_q \circ r_p^{-1})(t, 0), \quad i = 0, \ldots, s.
\]

$s = 1$: \[
\partial_2 p(t, 0) + \partial_1 q(0, t) = \partial_1 p(t, 0) \partial_2 \rho^{[2]}(t, 0).
\]

**Constraint:** all constructions for general meshes must employ some non-linear $\partial_2 \rho^{[2]}(t, 0)$. [P,Fan ‘09]:
Splines on quad-only, valence-four meshes

(Periodic) spline trivial: map from grid-partitioned quad, identifying edges

eg: tensor-product splines
Splines on more general quad-only meshes

- single global domain
- affine atlas
- parametric continuity
- shift-invariant spaces
Splines on more general quad-only meshes

- Single global domain
- Affine atlas
- Parametric continuity
- Shift-invariant spaces

- Local domains
- No affine atlas
- Geometric continuity
- Shift-invariant spaces?
Chart and Geometric Continuity

Want for $i = 0, \ldots, s$

$$\partial^i p(t, 0) = \partial^i (q \circ \underbrace{r_q \circ r_p^{-1}}_{\rho})(t, 0).$$
Chart and Geometric Continuity

Want for $i = 0, \ldots, s$

\[
\partial^i p(t, 0) = \partial^i (q \circ r_q \circ r_p^{-1})(t, 0) \cdot \rho.
\]

If $\rho$ is projective (rational) linear then $q$ and $q(\rho)$ have the same degree.
Fractional linear maps

Try real rational linear map $\rho$

$$\rho(u, v) := \begin{bmatrix} \rho[1] \\ \rho[2] \end{bmatrix} (u, v) := \begin{bmatrix} a_1 + b_1 u + c_1 v \\ d_1 + e_1 u + f_1 v \\ a_2 + b_2 u + c_2 v \\ d_2 + e_2 u + f_2 v \end{bmatrix}$$

where $a_i, b_i, \ldots, f_i$ are real scalars.
Fractional linear maps

Want for $i = 0, \ldots, s$

$\partial^i p(t, 0) = \partial^i (q \circ \rho)(t, 0)$.

$\rho := \begin{bmatrix} a_1 + b_1 u + c_1 v \\ d_1 + e_1 u + f_1 v \\ a_2 + b_2 u + c_2 v \\ d_2 + e_2 u + f_2 v \end{bmatrix}$

What are necessary and sufficient conditions for constructions using $\rho$?
Fractional linear maps

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$\partial^i \mathbf{p}(t, 0) = \partial^i (\mathbf{q} \circ \mathbf{\rho})(t, 0)$.

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What are necessary and sufficient conditions for constructions using $\mathbf{\rho}$?

**Necessary** $\mathbf{\rho}$ is projective linear (i.e. in $\mathbb{P}^2$);
Fractional linear maps

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What are necessary and sufficient conditions for constructions using $\rho$?

Necessary $\rho$ is projective linear (i.e. in $\mathbb{P}^2$); $\rho$ is unique.
Fractional linear maps

Want for $i = 0, \ldots, s$

$$\partial^i p(t, 0) = \partial^i (q \circ \rho)(t, 0).$$

$$\rho := \begin{bmatrix} a_1 + b_1 u + c_1 v \\ d_1 + e_1 u + f_1 v \\ a_2 + b_2 u + c_2 v \\ d_2 + e_2 u + f_2 v \end{bmatrix}$$

What are necessary and sufficient conditions for constructions using $\rho$?

**Necessary** $\rho$ is projective linear (i.e. in $\mathbb{P}^2$); $\rho$ is unique.

**Sufficient** for special layout of quadrilaterals.
Necessity: Constraints on the transition map for $G^2$ continuity

Find $\rho : \square \subset \mathbb{R}^2 \to \mathbb{R}^2 : (u, v) \to \begin{bmatrix} a_1 + b_1 u + c_1 v \\ d_1 + e_1 u + f_1 v \\ a_2 + b_2 u + c_2 v \\ d_2 + e_2 u + f_2 v \end{bmatrix}$ so that patches $p, q : \square \subset \mathbb{R}^2 \to \mathbb{R}^d$ join smoothly across the common boundary:

$q(t, 0) = p(0, t)$
Necessity: Constraints on the transition map for $G^2$ continuity

Find $\rho : \Box \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (u, v) \rightarrow \left[ \begin{array}{c} a_1 + b_1 u + c_1 v \\ d_1 + e_1 u + f_1 v \\ a_2 + b_2 u + c_2 v \\ d_2 + e_2 u + f_2 v \end{array} \right]$ so that patches $p, q : \Box \subset \mathbb{R}^2 \rightarrow \mathbb{R}^d$ join smoothly across the common boundary:

$q(t, 0) = p(0, t) \implies \rho(t, 0) = (0, t) \implies a_1 = b_1 = 0, \ a_2 = e_2 = 0, \ b_2 = d_2.$

Therefore $\rho(u, v) := \left[ \begin{array}{c} c_1 v \\ d_1 + e_1 u + f_1 v \\ d_2 u + c_2 v \\ d_2 + f_2 v \end{array} \right].$
To $\rho(u, v) := \begin{bmatrix} \frac{c_1 v}{d_1 + e_1 u + f_1 v} \\ \frac{d_2 u + c_2 v}{d_2 + f_2 v} \end{bmatrix}$ add the $G^1$ constraints

$$(\partial_2 q)(t, 0) = (\partial_2 (p \circ \rho))(t, 0)$$

$$= (\partial_1 p)(0, t) \partial_2 \rho^{[1]}(t, 0) + (\partial_2 p)(0, t) \partial_2 \rho^{[2]}(t, 0)$$

and require no bias for $p$ over $q$ and vice versa:

$$(\partial_2 \rho^{[1]})(t, 0) = -1$$

and $\tau := \partial_2 \rho^{[2]}(0, 0) = 2 \cos \frac{2\pi}{n}$

$$\implies e_1 = 0, d_1 = -c_1$$

and $\frac{c_2}{d_2} = \tau$.

This implies

$$\rho(u, v) := \begin{bmatrix} \frac{-d_1 v}{d_1 + f_1 v} \\ \frac{u + \tau v}{1 + vf_2/d_2} \end{bmatrix}. $$
To $\rho(u, v) := \begin{bmatrix} -d_1 v \\ d_1 + f_1 v \\ u + \tau v \\ 1 + v f_2 / d_2 \end{bmatrix}$ add the $G^2$ constraints

$$(\partial_2^2 q)(t, 0) = (\partial_2^2 (p \circ \rho))(t, 0)$$
$$= (\partial_1^2 p)(0, t) - 2(\partial_1 \partial_2 p)(0, t)\partial_2 \rho^{[2]}(t, 0)$$
$$+ (\partial_2^2 p)(0, t)(\partial_2 \rho^{[2]})^2(t, 0) + (\partial_1 p)(0, t)(\partial_2^2 \rho^{[1]})(t, 0) + (\partial_2 p)(0, t)(\partial_2^2 \rho^{[2]})(t, 0)$$

and since we rule out singular constructions,

$$\tau \partial_1 \partial_2 \rho^{[2]} - \partial_2^2 \rho^{[2]} = \frac{\tau}{2} \partial_2^2 \rho^{[1]}$$
**Theorem**

The map $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for the $G^2$ construction of a $C^2$ surface is unique up to the values of $\tau := \partial_2 \rho^{[2]}(0, 0)$ and $\sigma := \partial_2 \rho^{[2]}(1, 0)$:

$$
\rho(u, v) := \frac{1}{1 + v(\tau - \sigma)} \begin{bmatrix} -v \\ u + \tau v \end{bmatrix}.
$$
Necessity: The projective linear reparametrization

**Theorem**

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\]

*only assumption: \( p, q \) sufficiently smooth.*

Neither valence, nor polynomiality, nor the number of boundary edges of the surface pieces matters!
The map is a root of 1: $\rho^8(\Box) = \rho \circ \ldots \circ \rho = \Box$

Euclidean projections of $\rho^i(\Box)$. 
Necessity: The projective linear reparametrization

\[ \rho(u, v) := \frac{1}{1 + v(\tau - \sigma)} \begin{bmatrix} -v \\ u + \tau v \end{bmatrix}. \]

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- \( \partial_2 \rho^{[1]}(t, 0) \) and \( \partial_2 \rho^{[1]}(t, 0) \) are constant functions
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- \( \partial_2 \rho^{[2]}(t, 0) \) and \( \partial_2^2 \rho^{[2]}(t, 0) \) are linear functions
- \( \rho = r_q \circ r_p^{-1} : \eta \in \mathbb{R}, \)

\[
\begin{align*}
r_p(u, v) &:= \frac{1}{s + \eta v} \begin{bmatrix} -v \\ su + cv \end{bmatrix}, \quad c := \cos \frac{2\pi}{n}, \\
r_q(u, v) &:= \frac{1}{s - \eta v} \begin{bmatrix} v \\ su - cv \end{bmatrix}, \quad s := \sin \frac{2\pi}{n}.
\end{align*}
\]
Sufficiency: Does $\rho$ allow for spline constructions?

Constraint Lemma 4 [P,Fan ‘09]: all constructions for general meshes must employ some non-linear $\partial_2^2 \rho(t, 0)$. 
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Our Theorem (Necessity): $\partial_2 \rho^{[2]}(t, 0) = \tau(1 - t) + \sigma t$. 
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[Hahn&Gregory 1988,9], [Ye 1997], [Prautzsch 1997], [Prautzsch&Umlauf 2000], [Reif 98], [Gregory&Zhou 1999], [Peters 2002], [Loop et al 2004,8], [Karciauskas&Peters 2004,6], ... [Loop&DeRose 1995] [Grimm 1997], [Gotrina et al 2000, 2007], [Ying 2004], ...
Sufficiency: Does $\rho$ allow for spline constructions?

Constraint on mesh: Lemma 4 [P,Fan ‘09]: all constructions for general meshes must employ some non-linear $\partial_2 \rho[2](t,0)$.

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- Are there any such quadrilateral-partitions for surfaces of genus $\geq 1$?
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• Are there any such quadrilateral-partitions for surfaces of genus $> 1$?
• Are there corresponding spline constructions?
Restricted Connectivity
not restricted shape, not restricted topology

Not general connectivity meshes.
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Rather

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Euler’s formula & quad-only meshes

Euler’s formula for polyhedra

\[ v - e + f = \chi = 2 - 2g \]

topological genus \( g > 0 \):
rectangular grid
except for \(-\chi = 2g - 2\)
isolated vertices of valence 8.

4-valent vertex \( \sim 1 - 4/2 + 4/4 = 0 \)
4 half-edges, 4 quarter-quads
8-valent \( \sim 1 - 8/2 + 8/4 = -1 \).
Euler’s formula & quad-only meshes

Higher-genus Surfaces (potential sampling domains). All vertices can be moved freely.
Unique lowest-degree (= only projectively linear) bi-variate reparameterization for constructing $C^s$ manifolds of any non-zero genus.
Summary

Unique lowest-degree (= only projectively linear) bi-variate reparameterization for constructing $C^s$ manifolds of any non-zero genus.

Analogue of a splines on non-genus-1 domains? ($\mathbb{P}$ projective space)

$p, q \in \mathbb{P}^3, \ell \in \mathbb{P}^2, \quad \partial^i p = \partial^i (q \circ \ell)$.
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Thank you

Triangulations: Theorem and Lemmas apply unchanged!
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